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STRICTLY COOPERATIVE SYSTEMS WITH A FIRST INTEGRAL*

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Abstract. We consider systems of differential equations $dx_i/dt = F_i(x_1, \cdots, x_n)$ in the nonnegative orthant in the n-space satisfying the following hypotheses: i) $F(0) = 0$; ii) if $x_i < y_i$ and $x_j = y_j$ for $j \neq i$ then $F_k(x) < F_k(y)$ for $k \neq i$; iii) F possesses a first integral with positive gradient. We prove that every solution to such a system either converges to an equilibrium or eventually leaves any compact set.

Key words. equilibrium, first integral, Lyapunov function, ω-limit set, nonnegative orthant, order preserving, strictly cooperative system

AMS(MOS) subject classification. Primary 34C05

1. Introduction. The purpose of the present paper is to study the limiting behavior of solutions of systems of ordinary differential equations possessing a first integral, where the right sides of equations as well as the first integrals are subject to some monotonicity conditions. We prove that any solution to such a system either converges to an equilibrium or eventually leaves any compact set.

Particular classes of systems of differential equations $\dot{x} = F(x)$, $x \in U \subset \mathbb{R}^n$, satisfying conditions $\partial F_i/\partial x_j \geqq 0$ for $i \neq j$, were studied by many authors (see references in [1]; see also [3, Chap. III]). Recently, in [1] and [2], M. W. Hirsch initiated investigation of systems of that type (which he calls cooperative systems), using ideas taken from the dynamical systems theory. Such systems may describe, for instance, competition between biological species or chemical reactions. In cooperative systems it is natural to expect convergence of bounded solutions to an equilibrium or to a closed orbit. A decisive step toward answering this conjecture was made by M. W. Hirsch, who in [2] proved that, for systems that are cooperative and irreducible (that is, the matrices $[(\partial F_i/\partial x_j)(p)]$ are irreducible), almost all (with respect to the Lebesgue measure) points whose forward orbits are bounded approach the equilibrium set. In [11] J. Smillie has found a class of cooperative irreducible systems for which the following holds: every solution either converges to an equilibrium or eventually leaves any compact set.

To our knowledge, a general class of first integrals for cooperative systems was considered only in [2, Thm. 4.7]. However, that result is negative: if the set of equilibria is countable then every continuous invariant function is constant. On the other side, many authors (e.g. [5], [6], [7], [9], [10]) considered cooperative (or related) systems on the nonnegative orthant $\mathbb{R}^n_+$ having $\sum_i x_i$ as a first integral. For such systems it was proved that every solution converges to an equilibrium. In [9] and [10] these results were extended to the case of nonautonomous cooperative systems periodic (resp. almost periodic) in t.

The results contained in the present paper are a generalization of the theorems mentioned above to the case of not necessarily linear first integral. Methods used here are geometric, and the only nonelementary tool made use of is the Brouwer fixed-point theorem. The exposition is independent of any other work on this subject; however, the idea of a Lyapunov function L is taken from the author's previous work [8].

2. Definitions and preliminary lemmas. We define $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geqq 0\}$, $\partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n_+ : x_i = 0 \text{ for some } i\}$, $\operatorname{Int} \mathbb{R}^n_+ = \mathbb{R}^n_+ \backslash \partial\mathbb{R}^n_+$. Moreover, we denote $e_i = (0, \cdots, 0, {}_i1, 0, \cdots, 0)$—the ith vector of the standard base in $\mathbb{R}^n$, $B = \{x \in \mathbb{R}^n : 0 \leqq x_i \leqq 1$

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for each $i\}$, $B_+ = \{x \in B: x_i = 1$ for some $i\}$, $x < y$ if $x_i < y_i$ for each i, and $x <_i y$ if $x_i < y_i$ and $x_j = y_j$ for $j \neq i$. $\langle \cdot, \cdot \rangle$ will denote the usual inner product in $\mathbb{R}^n$, $\|\cdot\|$—the corresponding norm.

Let $H: \mathbb{R}^n_+ \to \mathbb{R}$ be a C^1 function. By a gradient of H at p we mean the vector $((\partial H/\partial x_1)(p), \cdots, (\partial H/\partial x_n)(p))$ (written grad $H(p)$).

Let $F: \mathbb{R}^n_+ \to \mathbb{R}^n$ be a vector field. By a *first integral* for F we understand a function $H: \mathbb{R}^n_+ \to \mathbb{R}_+$, of class C^1, such that $\operatorname{grad} H(p) \neq 0$ at each $p \in \mathbb{R}^n_+$ and $\langle \operatorname{grad} H(p), F(p) \rangle = 0$.

A point $p \in \mathbb{R}^n_+$ for which $F(p) = 0$ is called an *equilibrium*.

Let $x: [0, T) \to \mathbb{R}^n_+$ be a nonextendible solution to the initial value problem $dx/dt = F(x)$, $x(0) = x$. We say the set $\{x(t): 0 \leqq t < T\}$ is a *forward orbit* of x.

The set $\omega(x)$ consists of those points $y \in \mathbb{R}^n_+$ for which there exists a sequence $t_k \to T$ such that $x(t_k) \to y$. This set is called an *ω-limit set* of x. The following facts are well known.

THEOREM 2.1. *If $\omega(x) = \{y\}$ then the solution x is defined on $[0, +\infty)$ and converges to an equilibrium y.*

THEOREM 2.2. *If $\omega(x) = \varnothing$ then $x(\cdot)$ eventually leaves any compact set.*

The set $A \subset \mathbb{R}^n_+$ is called *positively invariant* if for each $a \in A$ its forward orbit is contained in A.

We consider a system of ordinary differential equations in $\mathbb{R}^n_+$ defined by a C^1 vector field $F: \mathbb{R}^n_+ \to \mathbb{R}^n$,

$$\frac{dx_i}{dt} = F_i(x_1, \cdots, x_n) = F_i(x), \quad x \in \mathbb{R}^n_+, \qquad F = (F_1, \cdots, F_n), \tag{2.1}$$

satisfying:

(A1) $F(0) = 0$;

(A2) If $x <_i y$ then $F_j(x) < F_j(y)$ for $j \neq i$;

(A3) There exists a first integral H for F such that grad $H(x) > 0$ for $x \in \mathbb{R}^n_+$ and (for convenience) $H(0) = 0$. Systems satisfying (A2) will be called *strictly cooperative.*

Let M denote the least upper bound of the values of H. From (A3) it follows that H is onto $[0, M)$, where $0 < M \leqq +\infty$.

By Int $H^{-1}(h)$ we denote the set $\{x \in \operatorname{Int} \mathbb{R}^n_+: H(x) = h\}$.

LEMMA 2.1. *Let c be an equilibrium. Then $c + \mathbb{R}^n_+$ is positively invariant. Moreover, c is a unique equilibrium on $c + \partial \mathbb{R}^n_+$.*

Proof. By performing, if necessary, the change of coordinates $\tilde{x}_i = x_i - c_i$, we reduce a general case to the case $c = 0$. Let $x \in \mathbb{R}^n_+$. Then let I denote the subset of $\{1, \cdots, n\}$ such that $x_i = 0$ exactly for $i \in I$. If $x \neq 0$ then $x_j > 0$ for indices j belonging to some nonvoid subset J of $\{1, \cdots, n\}$. For convenience assume $I = \{1, \cdots, k\}$, $J = \{k+1, \cdots, n\}$. By (A2) we obtain

$$F_i(0, \cdots, 0, x_{k+1}, \cdots, x_n) > F_i(0, \cdots, 0, x_{k+2}, \cdots, x_n) > \cdots > F_i(0) = 0 \quad \text{for } i \in I.$$

Therefore on the boundary of $\mathbb{R}^n_+$ (except 0) the vector field F is directed inward, which in a standard way implies that $\mathbb{R}^n_+$ is positively invariant. Q.E.D.

LEMMA 2.2. *Let c be an equilibrium. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for each $h \in [H(c) - \delta, H(c) + \delta] \cap [0, M)$ there exists an equilibrium c_h such that $H(c_h) = h$, $c_h > c$ for $h > H(c)$ (resp. $c_h < c$ for $h < H(c)$) and $\|c_h - c\| \leqq \varepsilon$.*

Proof. We consider the case $h > H(c)$. From (A3) it follows that $H(c + n^{-1/2} e_i) > H(c)$. Set $\delta = \min_i H(c + n^{-1/2} e_i) - H(c)$. For $x \in c + n^{-1/2} B_+$ we have $H(x) \geqq H(c) + \delta$. Let for a moment $h \in (H(c), H(c) + \delta)$ be fixed. By $\ell(x)$ we denote the straight line passing through c and x. Let the mapping $K: (c + n^{-1/2} B_+) \to H^{-1}(h) \cap (c + \mathbb{R}^n_+)$ be defined in the following way: $K(x)$ is the unique point on $\ell(x)$ at which $H(K(x))=h$. The existence and uniqueness of this point follow from (A3). Having in mind our choice of h, we infer that K is in fact onto $H^{-1}(h)\cap(c+n^{-1/2}B)=H^{-1}(h)\cap(c+\mathbb{R}^n_+)$. The definition of K implies that K is one-to-one. Moreover, K^{-1} is continuous as a "central projection" onto $(c+n^{-1/2}B_+)$. Therefore, from the compactness of the domain of K, K is a homeomorphism. Thus $H^{-1}(h)\cap(c+\mathbb{R}^n_+)$ is homeomorphic to the $(n-1)$-dimensional disk. This set, as an intersection of positively invariant sets, is positively invariant. The Brouwer fixed-point theorem tells us that there is an equilibrium $c_h\in H^{-1}(h)\cap(c+\mathbb{R}^n_+)$. From Lemma 2.1 it follows that $(c_h)_i>c_i$. Moreover, for every $y\in c+n^{-1/2}B$ we have $\|y-c\|\leqq\varepsilon$, so $\|c_h-c\|\leqq\varepsilon$. The case $h<H(c)$ is treated in an analogous way. Q.E.D.

PROPOSITION 2.1. *The set S of equilibria is linearly ordered by $<$.*

Proof. Suppose there exist $c, d\in S$ such that $c_i=d_i$ for $i\in I$, $c_j<d_j$ for $j\in J$, $c_l>d_l$ for $l\in L$ and at least two of these sets are nonempty. Consider the point z, $z_k=\max\{c_k, d_k\}$ for $1\leqq k\leqq n$. Proceeding as in the proof of Lemma 2.1 we obtain $F_i(z)>F_i(d)=0$ for $i\in I$, $F_j(z)>F_j(d)=0$ for $j\in J$ and $F_l(z)>F_l(c)=0$ for $l\in L$. But from this it follows that $\langle F(z), \operatorname{grad} H(z)\rangle>0$, a contradiction. Q.E.D.

COROLLARY. *For each $h\in[0, M)$ there is at most one equilibrium on $H^{-1}(h)$.*

PROPOSITION 2.2. *There exists U, $0<U\leqq M$, such that there is an order-preserving homeomorphism between $[0, U)$ and the set S of equilibria of F.*

Proof. The function $H|S$ is continuous. From the corollary and Proposition 2.1 it follows that $H|S$ is one-to-one, so Z, the function inverse to $H|S$, exists. Lemma 2.2 implies that Z is continuous and preserves order. Therefore Z is a homeomorphism. The statement on the domain of Z also follows from Lemma 2.2. Q.E.D.

Let Z denote, as in the above proof, the function inverse to $H|S$. We define the function $L:\mathbb{R}^n_+\to\mathbb{R}_+$, $L(x)=\min\{Z_i^{-1}(x_i): Z_i^{-1}(x_i) \text{ is defined}\}$, where Z_i^{-1} denotes the function inverse to the ith coordinate of Z.

LEMMA 2.3. *L is well defined and continuous.*

Proof. Suppose that for each i, $Z_i(h)\to a_i$ as $h\to U$. Then $a=(a_1,\cdots, a_n)$ would be an equilibrium, and $H(a)=U$. But by Lemma 2.2 there would exist an equilibrium $b>a$ and $H(b)>U$. Thus for some i, $Z_i(h)\to\infty$ as $h\to U$, so L is well defined.

In order to prove the continuity of L notice that, by Lemma 2.2, the image of Z_i is a right-open interval. Therefore if in the definition of $L(x)$ the minimum is realized exactly by indices $i\in I$ then for y belonging to some sufficiently small neighborhood of x the minimum is realized by indices from some subset of I. The minimum of a finite family of continuous functions is continuous, so we have obtained the desired result. Q.E.D.

Remark. As we have said in the Introduction, L has a simple geometric interpretation. Namely, the level surface corresponding to h is equal to the set $Z(h)+\partial\mathbb{R}^n_+$.

LEMMA 2.4. *For $h\in[0, U)$, $\sup\{L(x): x\in H^{-1}(h)\}$ is attained only at $Z(h)$. For $h\in[U, M)$, $\sup\{L(x): x\in H^{-1}(h)\}$ is attained nowhere.*

Proof. First assume $h\in[0, U)$. It is easy to check out that $L(Z(h))=h$. Suppose that for some $x\neq Z(h)$, $x\in H^{-1}(h)$, we have $L(x)\geqq h$. From the definition of L it follows that $x_i\geqq Z_i(h)$ for every i. But for some j, $x_j\neq Z_j(h)$. (A3) implies that $H(x)>h$, a contradiction.

Now we assume $h\in[U, M)$. Let x be any point of $H^{-1}(h)$. Denote $L(x)$ by g. This means that $x\in Z(g)+\partial\mathbb{R}^n_+$ (of course $g<h$). Let I stand for the set of indices i for which $x_i=Z_i(g)$. Define the point $\tilde{x}$ in the following way: $\tilde{x}_i=x_i+1$ for $i\in I$, $\tilde{x}_j=x_j$ for $j\notin I$. From (A3) it follows that $H(\tilde{x})>h$. Considering, as in the proof of Lemma 2.2, the straight line passing through $Z(g)$ and x, we can find a point $y\in Z(g)+\operatorname{Int}\mathbb{R}^n_+$

such that $H(y)=h$. But the fact that $y\in Z(g)+\operatorname{Int}\mathbb{R}^n_+$ means that $H(y)>h$, a contradiction. Q.E.D.

3. The main result.

THEOREM 3.1. *For $x\in\mathbb{R}^n_+$ we have either $\omega(x)=\{Z(H(x))\}$ or $\omega(x)=\varnothing$.*

Proof. Let I denote the set of indices realizing the minimum in the definition of $L(x(t))$. Analogously, as in the proof of Lemma 2.1 we can show that $\dot{x}_i(t)>0$ for $i\in I$. Hence L is strictly increasing along the forward orbits of (2.1), except for the constant solutions. Let $x\in\mathbb{R}^n_+$ be fixed, and let E denote the set $H^{-1}(H(x))$. H is a first integral, so the forward orbit of x is contained in E. Furthermore, from the closedness of E it follows that $\omega(x)\subset E$. Now it suffices to show that $\omega(x)\subset\{y\in E\colon L(y)=\sup_{z\in E}L(z)\}$. Suppose not. Then there exists $y\in\omega(x)$ such that $L(y)<\sup_{z\in E}L(z)$. From Lemma 2.4, y is not an equilibrium, so L strictly increases along the orbit of y. Let v be any point on the forward orbit of y, distinct from y. Obviously $L(v)>L(y)$. Choose neighborhoods V of v, Y of y, such that $\inf_{z\in V}L(z)>\sup_{z\in Y}L(z)$. From the very definition of the ω-limit set we can find moments $t_1<t_2$ such that $x(t_1)\in V$, $x(t_2)\in Y$. From this we deduce that $L(x(t_1))<L(x(t_2))$, which contradicts our choice of V and Y. Q.E.D.

The following example shows that the case $\omega(x)=\varnothing$ is possible.

Example. Consider the nonnegative quadrant $\mathbb{R}^2_+$. As a first integral choose a function $H(x_1,x_2)=G_1(x_1)+G_2(x_2)$, where $G_2(x_2)=x_2$, and $G_1\colon\mathbb{R}_+\to\mathbb{R}_+$ is a function of class C^2 such that $G_1(0)=0$, G_1' is positive and $\lim_{x_1\to\infty}G_1(x_1)=1$. Let $W\colon\mathbb{R}_+\to\mathbb{R}_+$ be a function of class C^1 such that $W(0)=0$, W' is positive and $\lim_{x\to\infty}W(x)=1$. As the equilibrium set S we choose the graph of W, i.e., $S=\{(x_1,x_2)\colon x_2=W(x_1),\ x_1\in\mathbb{R}_+\}$. We define

$$F_2(x_1,x_2)=W(x_1)-x_2,\ F_1(x_1,x_2)=-(F_2(x_1,x_2))/(G_1'(x_1))=(x_2-W(x_1))/(G_1'(x_1)).$$

It is easy to check out that F is C^1 and that H is a first integral for our system. Moreover

$$(\partial F_2/\partial x_1)(x_1,x_2)=W'(x_1)>0,\qquad (\partial F_1/\partial x_2)(x_1,x_2)=[G_1'(x_1)]^{-1}>0,$$

so that the system is strictly cooperative. From the choice of W it follows that $\sup_{x\in S}H(x)=2$. Hence the level surface $H^{-1}(3)$ does not intersect S, so for each $x\in H^{-1}(3)$, $\omega(x)=\varnothing$.

Remark. Professor Morris W. Hirsch has informed the author that he obtained the same result under slightly different hypotheses [4].

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