APPROXIMATION OF THE ERDÉLYI–KOBER OPERATOR WITH APPLICATION TO THE TIME-FRACTIONAL POROUS MEDIUM EQUATION*

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Abstract. This paper describes a method of approximating equations with the Erdélyi–Kober fractional operator which arise in mathematical descriptions of anomalous diffusion. We prove a theorem on the exact form of the approximating series and provide an illustration by considering the fractional porous-medium equation applied to model moisture diffusion in building materials. We obtain some approximate analytical solutions of our problem which accurately fit the experimental data (better than other models found in the literature). This accuracy is also verified numerically. Since they are very quick and easy to implement, our approximations can be valuable for practitioners and experimentalists.

Key words. fractional calculus, porous-medium equation, anomalous diffusion, approximate solution

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1. Introduction. Diffusion is one of the most common and fundamental natural processes. This phenomenon is present in nearly all fields of science and engineering. Nowadays, diffusion is very well understood and can be described mathematically by a parabolic equation (either linear or nonlinear). This equation is very versatile and provides a unification of relatively diverse phenomena under only one class of equations. Despite very vigorous research throughout the last (and present) century, there are still some transport phenomena that cannot be described accurately by classical diffusion [1, 2, 3, 4, 5].

These new transport phenomena are characterized by a non-Fickian character of a intrinsic space-time scaling. In classical diffusion, the self-similar solutions are obtained by finding a function dependent on $x/t^{\frac{1}{2}}$. As many experiments have shown, this scaling does not describe the process accordingly. Rather, we should look for solutions which are functions of $x/t^{\frac{\alpha}{2}}$ for some $\alpha \in (0, 2)$. Then, the diffusion becomes anomalous and the classical mathematical description is not appropriate and therefore some other approaches have to be undertaken. A quite successful description of the anomalous diffusion can be made by use of the fractional calculus (see, for example, [6, 7, 8, 9]). According to that model, the partial differential equation is substituted for by its fractional version. We note that in the literature, solutions of nonlinear equations were mostly obtained numerically without any simple analytical form that could be readily used. When we substitute for the ordinary differential operator by the fractional derivative we obtain a correct space-time scaling but at the cost of severe complexity of the resulting equation. In this paper we try to circumvent this complication while retaining the accuracy of the description.

The fractional calculus methods are being successfully used in mathematical modeling and physical sciences [10, 11, 12, 13]. Moreover, they introduce a number of interesting topics that are investigated by theoreticians working in analysis, integral equations, and special-function theory [14, 15, 16, 17]. The fractional diffusion equation was mostly investigated in its simplest, linear case [18, 19, 20]. There, the exact form of the solution (in terms of the Wright function) can be obtained by the Fourier–Laplace method. In the nonlinear case, mostly numerical [21, 22, 23] or semianalytical (Adomian decomposition method, Homotopy analysis, [24]) solutions are feasible. Nevertheless, some exact solutions of the fractional nonlinear diffusion problems were also obtained [25, 26, 27].

In what follows, we investigate the following nonlinear fractional partial differential equation which models the anomalous diffusion

$$(1.1) \qquad \partial_t^\alpha u(x,t) = (D(u(x,t))u_x(x,t))_x, \quad 0 < \alpha < 1,$$

where we use the Riemann–Liouville fractional derivative ∂_t^α. As for the initial-boundary conditions, we choose

$$(1.2) \qquad u(0,t) = 1, \quad u(x,0) = 0, \quad x > 0, \quad t > 0,$$

which describe the situation investigated in those experiments which we try to explain mathematically (for example, [2, 6, 7]). We also take the diffusion coefficient to be proportional to the mth power of the transported substance concentration (here, water). Then, by an analogy with the classical case, (1.1) can be called a fractional porous-medium equation. The choice of this form of equation is, apart from physical reasons, verified by comparing our results with the data obtained for moisture diffusion in siliceous brick [2]. Some initial results concerning this topic ware obtained in [28]. Our approach starts by proving a theorem on the approximation of the Erdélyi–Kober fractional operator by a series of ordinary derivatives. We show, that taking only the first terms of that series is sufficient to retain most of the information about the fractional derivative. This allows us to substitute for the fractional equation by an ordinary equation and solve it approximately to obtain a solution of the anomalous diffusion problem. We then show some numerical results that confirm our approach. The obtained accuracy is very good and the fitting with the data is decent.

2. Main equation and the approximation of the fractional derivative. In this section we formulate the fractional problem which will be investigated and prove the main approximation theorem. Consider the one-dimensional nonlinear diffusion equation

$$(2.1) \qquad u_t(x,t) = (D(u)u_x(x,t))_x,$$

where $D(u)$ is the coefficient describing diffusion strength of the medium. Here, $u = u(x,t)$ represents substance concentration at point x at time t and subscripts denote differentiation with respect to the appropriate variables. As a model of diffusion in a porous media like soil or concrete, one often chooses diffusion coefficient $D(u) = D_0 u^m$, where $m \geq 0$. Nevertheless, as recent investigations showed, (2.1) does not satisfactory describe dynamics in some construction materials like siliceous or fired-clay brick. This is due to the different than usual characteristic scaling of the wetting front variable. Specifically, we observe $x/t^{\frac{\alpha}{2}}$ rather than the classical $x/\sqrt{t}$ as a master-curve scaling. This phenomenon is known as anomalous diffusion and can

successfully be modeled by fractional calculus applied to the time derivative. Following that lead, we investigate the porous medium equation in a subdiffusive case

$$(2.2) \qquad \partial_t^\alpha u(x,t) = (D_0 u^m(x,t) u_x(x,t))_x, \quad 0 < \alpha < 1,$$

where ∂_t^α denotes the Riemann–Liouville fractional derivative (with lower terminal 0)

$$(2.3) \qquad \partial_t^\alpha u(x,t) = \frac{\partial}{\partial t} I_t^{1-\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} u(x,s) ds,$$

where $I^{1-\alpha}$ is the fractional integral operator. A thorough treatment of the theory and applications of fractional calculus can be found, for example, in [29]. Note also that we consider only the so-called subdiffusive case, that is a slow diffusion with $0 < \alpha < 1$. Following [6, 7] we impose the initial-boundary conditions used to describe the experiment of measuring moisture diffusion in a specimen

$$(2.4) \qquad u(0,t) = C, \quad u(x,0) = 0.$$

This models the situation where the face of the sample $(x = 0)$ is maintained at a constant moisture level along with zero initial conditions.

An important remark is in order. From stochastic derivation and interpretation of subdiffusion one obtains an equation similar to (2.2) but with the Caputo derivative replacing the Riemann–Liouville one (for a thorough explanation of continuous time random walk models of anomalous diffusion see [30, 31]). These two interpretations of a fractional derivative are equivalent when we are dealing with zero initial conditions (see formula 2.4.10 in [29]). However, as was shown in [18] this equivalence persists also in our setting (2.4). For these specific conditions, the Riemann–Liouville and Caputo derivatives coincide. Hence, the solution does not depend on the type of derivative used. In what follows, we will utilize the Riemann–Liouville version since it requires less regularity of the differentiated function and is easier to apply in our reasoning.

Let us scale equation (2.2) into a nondimensional form by taking

$$(2.5) \qquad x^* = \frac{x}{L}, \quad t^* = \frac{t}{T}, \quad u^* = \frac{u}{C}, \quad T = \left(\frac{L^2}{D_0 C^m}\right)^{\frac{1}{\alpha}},$$

where L is the typical length scale for our problem, for example, water-table level or sample size and C is a typical value of concentration. Equation (2.2) transforms into

$$(2.6) \qquad \partial_t^\alpha u(x,t) = (u^m(x,t) u_x(x,t))_x, \quad 0 < \alpha < 1,$$

where we suppressed asterisks for clarity of notation. The initial-boundary conditions (2.4) are now formulated in the following way:

$$(2.7) \qquad u(0,t) = 1, \quad u(x,0) = 0.$$

Our further considerations follow the classical case analyzed in [32, 33]. Due to assumed initial-boundary conditions (2.7) we can look for similarity solutions in the form

$$(2.8) \qquad u(x,t) = U(\eta), \quad \eta = \frac{x}{t^{\frac{\alpha}{2}}}.$$

This choice comes from the symmetry of (2.6) and is motivated by the experiments described in the literature. In the classical diffusion we would have $\alpha = 1$ and reobtain the well-known similarity solution for the heat equation. The left-hand side of (2.6) transforms as

$$
\begin{aligned}
\partial_t^\alpha u(x,t) &= \frac{\partial}{\partial t}\left(I_t^{1-\alpha}\left(U(xt^{-\frac{\alpha}{2}})\right)\right) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t (t-z)^{-\alpha}U(xz^{-\frac{\alpha}{2}})dz \\
&= \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\left(t^{-\alpha+1}\int_0^1 (1-s)^{-\alpha}U(\eta s^{-\frac{\alpha}{2}})ds\right).
\end{aligned}
$$
(2.9)

Let us introduce the Erdélyi–Kober fractional integral operator (see [34, 35])

$$
I_c^{a,b}U(\eta) = \frac{1}{\Gamma(b)}\int_0^1 (1-z)^{b-1}z^a U(\eta z^{\frac{1}{c}})dz.
$$
(2.10)

Thus (2.9) can be rewritten as

$$
\partial_t^\alpha u(x,t) = \frac{\partial}{\partial t}\left(t^{-\alpha+1}I_{-\frac{2}{\alpha}}^{0,1-\alpha}U(\eta)\right) = t^{-\alpha}\left[(1-\alpha)-\frac{\alpha}{2}\eta\frac{d}{d\eta}\right]I_{-\frac{2}{\alpha}}^{0,1-\alpha}U(\eta),
$$
(2.11)

where we used the chain rule for (2.8)

$$
\frac{\partial}{\partial t} = \frac{\partial\eta}{\partial t}\frac{d}{d\eta} = -\frac{\alpha}{2}\eta t^{-1}\frac{d}{d\eta}.
$$
(2.12)

The right-hand side of (2.6) transforms into

$$
(u^m(x,t)u_x(x,t))_x = t^{-\alpha}\frac{d}{d\eta}\left(U^m(\eta)\frac{d}{d\eta}U(\eta)\right).
$$
(2.13)

Finally, the fractional porous-medium equation (2.6) has the self-similar form

$$
\frac{d}{d\eta}\left(U^m(\eta)\frac{d}{d\eta}U(\eta)\right) = \left[(1-\alpha)-\frac{\alpha}{2}\eta\frac{d}{d\eta}\right]I_{-\frac{2}{\alpha}}^{0,1-\alpha}U(\eta),
$$
(2.14)

where t-variable has canceled. This is an ordinary integro-differential equation. The initial-boundary conditions (2.7) now become

$$
U(0) = 1, \quad U(\infty) = 0.
$$
(2.15)

We are going to find an approximate solution for (2.14), when $0 < \alpha \leq 1$. The strategy is to replace the integral operator by a series of derivatives. This will translate the integro-differential equation into an ordinary one, which will be easier to analyze. First, let us prove a general theorem about the series representation of the Erdélyi–Kober operator. A thorough and accurate analysis of an approximation of the fractional integral was conducted in [36].

THEOREM 1. *For U analytic and $a > -1$, $b > 0$, $c > 0$ the following is true:*

$$
I_c^{a,b}U(\eta) = \sum_{k=0}^\infty \lambda_k U^{(k)}(\eta)\frac{\eta^k}{k!},
$$
(2.16)

where

$$
\lambda_k = \sum_{j=0}^k \binom{k}{j}(-1)^{k-j}\frac{\Gamma\left(a+\frac{j}{c}+1\right)}{\Gamma\left(a+b+\frac{j}{c}+1\right)}.
$$
(2.17)

Moreover, the following asymptotic relation holds for $k \to \infty$,

$$(2.18) \qquad \lambda_k \sim (-1)^k \frac{c}{\Gamma(b)} \sum_{n=0}^{\infty} \binom{b-1}{n} (-1)^n \Gamma\left(c(a+n+1)\right) \left(\frac{1}{k}\right)^{c(a+n+1)}.$$

Proof. First, note the Faá di Bruno's formula for the nth derivative of a composite function

$$(2.19) \qquad \frac{d^n}{dz^n} U(g(z)) = \sum_{k=0}^{n} B_{n,k}\left(g'(z), g''(z), \ldots, g^{(n-k+1)}(z)\right) U^{(k)}(g(z)),$$

where $B_{n,k}$ is the Bell polynomial. Taking $g(z) = \eta z^{\frac{1}{c}}$, we have

$$g^{(j)}(z) = (-1)^j (-c^{-1})_j \eta z^{\frac{1}{c} - j}$$

with $(a)_j = \Gamma(a+j)/\Gamma(a)$ being the Pochhammer symbol. Combining these expressions we obtain the Taylor series for $U(\eta z^{1/c})$ at $z = 1$,

$$(2.20)$$
$$U\left(\eta z^{\frac{1}{c}}\right) = \sum_{n=0}^{\infty} \frac{d^n}{dz^n} U\left(\eta z^{\frac{1}{c}}\right)\bigg|_{z=1} \frac{(-1)^n (1-z)^n}{n!}$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} B_{n,k}(c, \eta) U^{(k)}(\eta) \frac{(-1)^n (1-z)^n}{n!},$$

where we denoted

$$(2.21) \quad B_{n,k}(c, \eta) = B_{n,k}\left(c^{-1}\eta, -c^{-1}\left(-c^{-1}+1\right)\eta, \ldots, (-1)^{n-k+1}\left(c^{-1}\right)_{n-k+1}\eta\right)$$

for compactness of the notation. Now, changing the order of summation we obtain

$$(2.22) \qquad U\left(\eta z^{\frac{1}{c}}\right) = \sum_{k=0}^{\infty} U^{(k)}(\eta) \sum_{n=k}^{n} B_{n,k}(c, \eta) \frac{(-1)^n (1-z)^n}{n!}.$$

In the next step we use the following identity for Bell's polynomials (see [37]):

$$(2.23) \qquad \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \frac{t^n}{n!} = \frac{1}{k!} \left(\sum_{n=1}^{\infty} x_n \frac{t^n}{n!}\right)^k,$$

where x_j are some real numbers. We transform (2.22) into

$$(2.24) \qquad U\left(\eta z^{\frac{1}{c}}\right) = \sum_{k=0}^{\infty} \left(\sum_{n=1}^{\infty} \left(-c^{-1}\right)_n \frac{(1-z)^n}{n!}\right)^k U^{(k)}(\eta) \frac{\eta^k}{k!}.$$

Using the Taylor series for $(1-x)^{-\gamma}$ we finally arrive at

$$(2.25) \qquad U\left(\eta z^{\frac{1}{c}}\right) = \sum_{k=0}^{\infty} \left(z^{\frac{1}{c}} - 1\right)^k U^{(k)}(\eta) \frac{\eta^k}{k!}.$$

Now, plugging this formula into (2.10) we obtain

$$(2.26) \qquad I_c^{a,b} U(\eta) = \sum_{k=0}^{\infty} \lambda_k U^{(k)}(\eta) \frac{\eta^k}{k!},$$

where (by using the binomial theorem and the definition of the beta function)

(2.27)

$$\lambda_k = \frac{1}{\Gamma(b)} \int_0^1 (1-z)^{b-1} z^a (z^{\frac{1}{c}} - 1)^k dz = \sum_{j=0}^k \binom{k}{j} \frac{(-1)^{k-j}}{\Gamma(b)} \int_0^1 (1-z)^{b-1} z^{a+\frac{j}{c}} dz$$

$$= \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\Gamma\left(a + \frac{j}{c} + 1\right)}{\Gamma\left(a + b + \frac{j}{c} + 1\right)}.$$

To find the asymptotic behavior of λ_k we will use the integral representation as in the first line in (2.27). With a change of the variable $s = 1 - z^{\frac{1}{c}}$ we obtain

$$\lambda_k = (-1)^k \frac{c}{\Gamma(b)} \int_0^1 (1 - (1-s)^c)^{b-1} (1-s)^{c(a+1)-1} s^k ds$$

(2.28)

$$= (-1)^k \frac{c}{\Gamma(b)} \sum_{n=0}^\infty \binom{b-1}{n} (-1)^n \int_0^1 (1-s)^{c(a+n+1)-1} s^k ds$$

$$= (-1)^k \frac{c}{\Gamma(b)} \sum_{n=0}^\infty \binom{b-1}{n} (-1)^n \frac{\Gamma(c(a+n+1)) \Gamma(k+1)}{\Gamma(c(a+n+1) + k + 1)}.$$

Now, we use the known asymptotic formula for the Pochhammer symbol for $k \to \infty$ (see [43, formula 6.1.47]) and finally obtain the asymptotic series

(2.29)

$$\lambda_k = (-1)^k \frac{c}{\Gamma(b)} \sum_{n=0}^\infty \binom{b-1}{n} (-1)^n \frac{\Gamma(c(a+n+1))}{(k+1)_{c(a+n+1)}}$$

$$\sim (-1)^k \frac{c}{\Gamma(b)} \sum_{n=0}^\infty \binom{b-1}{n} (-1)^n \Gamma(c(a+n+1)) \left(\frac{1}{k}\right)^{c(a+n+1)}.$$

This finishes the proof. ☐

We are going to use Theorem 1 to approximate the fractional integral operator in (2.14) by the first terms in (2.16). First, notice the following remarks suggesting that our approximation should be accurate. Those are rather formal, but still give us some intuition.

When $b \to 0$, only λ_0 does not vanish in (2.17). Hence we expect that approximating $I_c^{a,b}$ with only the first terms in (2.16) will be most accurate for b close to 0. Also, as can be seen from Theorem 1, by the asymptotic behavior $\lambda_k \sim C_{a,b,c} k^{-c(a+1)}$, the convergence of the series (2.16) is very fast. This gives us a basis to think that the majority of information about U is contained in the first terms of the series. This idea is also strengthened by the fact that the mass of the integral (2.10) is concentrated near $z = 1$. And mostly its neighborhood gives the main contribution to the integral. The situation here is similar to the one in the Laplace method for finding asymptotic behaviors of integrals. Moreover, due to the term η^k in (2.16) we expect that the approximation will be accurate for $\eta \to 0$ (that is for $t \to \infty$ by (2.8)). This is especially convenient for U which has a compact support containing 0. It is also easy to see that for $U(\eta) = \eta^\gamma$ with $\gamma < c(a+1)$, the $I_c^{a,b} U$ can be readily computed giving

(2.30)
$$I_c^{a,b}(\eta^\gamma) = \frac{\Gamma\left(a + \frac{\gamma}{c} + 1\right)}{\Gamma\left(a + b + \frac{\gamma}{c} + 1\right)} \eta^\gamma,$$

which together with (2.17) gives us the following identity:

$$(2.31) \qquad \frac{\Gamma\left(a + \frac{\gamma}{c} + 1\right)}{\Gamma\left(a + b + \frac{\gamma}{c} + 1\right)} = \sum_{k=0}^{\infty} \binom{\gamma}{k} \lambda_k.$$

Of course, when we choose U to be a polynomial (i.e., when γ is an integer), the series for λ_k has only a finite number of nonzero terms.

Notice also that we have proven Theorem 1 under the assumptions $a > -1$, $b > 0$, and $c > 0$ in order for integrals to be convergent. Despite this fact, the formula (2.17) for λ_k is meaningful for those parameters for which the gamma function is defined (by analytic continuation; see, for example, [38]). In what follows we will notify the reader where we will be using these generalized values of λ_k.

3. Linear case. In this section we provide an example of using approximation (2.16) in order to find a solution to the fractional equation (2.14) with $m = 0$. We choose this case; since its analytical solution is known. To start, let us analyze

$$(3.1) \qquad U'' = \left[(1 - \alpha) - \frac{\alpha}{2}\eta\frac{d}{d\eta}\right] I_{-\frac{2}{\alpha}}^{0,1-\alpha} U, \quad 0 < \alpha \leq 1,$$

where the prime denotes differentiation with respect to η. The initial-boundary conditions are as in (2.15). The solution to this problem can be written in terms of the Wright function [39], that is,

$$(3.2) \qquad U(\eta) = \phi\left(-\frac{\alpha}{2}, 1; \eta\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma\left(1 - \frac{k\alpha}{2}\right)} \frac{\eta^k}{k!}.$$

When $\alpha \to 1$, (3.2) becomes the complementary error function, which is a solution to the classical version of (3.1). For reference, let us restate (2.17) in terms of α:

$$(3.3) \qquad \lambda_k = \sum_{j=0}^{k} \binom{k}{j}(-1)^{k-j}\frac{\Gamma\left(1 - \frac{j}{2}\alpha\right)}{\Gamma\left(2 - (1 + \frac{j}{2})\alpha\right)}.$$

To see how our approximation works replace the fractional operator at the right-hand side of (3.1) by the first term in (2.16), that is, solve the following equation

$$(3.4) \qquad U_0'' = (1 - \alpha)\lambda_0 U_0 - \frac{\alpha}{2}\lambda_0 \eta U_0'.$$

Notice that $\lambda_0 = 1/\Gamma(2 - \alpha)$ for our case of $a = 0$, $b = 1 - \alpha$, and $c = -2/\alpha$. Let us make a substitution

$$(3.5) \qquad U_0 = e^{-z^2} y, \quad z = \frac{1}{2}\sqrt{\frac{\alpha}{\Gamma(2 - \alpha)}}\eta,$$

which transforms (3.4) into a Hermite differential equation

$$(3.6) \qquad y'' - 2zy' + 2\left(1 - \frac{2}{\alpha}\right)y = 0,$$

which subjected to conditions (2.7) has a solution ([38, chapter 10])

$$(3.7) \qquad y(z) = \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\sqrt{\pi}2^{1-\frac{2}{\alpha}}}H_{1-\frac{2}{\alpha}}(z),$$

where $H_{1-\frac{2}{\alpha}}$ is the Hermite function of order $1-\frac{2}{\alpha}$. Finally, going back to the original variables (3.5) we obtain

$$(3.8) \qquad U_0(\eta) = \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\sqrt{\pi}2^{1-\frac{2}{\alpha}}} e^{-\frac{\alpha}{4\Gamma(2-\alpha)}\eta^2} H_{1-\frac{2}{\alpha}}\left(\frac{1}{2}\sqrt{\frac{\alpha}{\Gamma(2-\alpha)}}\eta\right).$$

Observe that when $\alpha \to 1$ the function U_0 reduces to $2/\sqrt{\pi}e^{-\eta^2/4}H_{-1}(\eta) = 2/\sqrt{\pi}e^{-\eta^2/4}$ Erfc$(\eta/2)$.

We want to compare the solutions of (3.1) and (3.4). In order to accomplish this, we will derive a series representation of $U_0(z)$, then substitute as in (3.5), and finally compare different terms with (3.2). To this end, write

$$(3.9) \qquad U_0(z) = \frac{\Gamma(\frac{1}{\alpha})}{\sqrt{\pi}2^{1-\frac{2}{\alpha}}} e^{-z^2} H_{1-\frac{2}{\alpha}}(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n} \cdot \sum_{n=0}^{\infty} b_n z^n,$$

where $a_{2n} = (-1)^n/n!$ and by the Taylor series of a Hermite function we have

$$(3.10) \qquad b_n = \frac{(-1)^n 2^n}{n!}\frac{\Gamma\left(\frac{n-1}{2}+\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}-\frac{1}{2}\right)}.$$

Now, we collect the powers of z in (3.9) to obtain

$$(3.11) \qquad U_0(z) = \sum_{n=0}^{\infty} \underbrace{\left(\sum_{k=0}^{n} a_{2n-2k}b_{2k}\right)}_{c_{2n}} z^{2n} + \underbrace{\left(\sum_{k=0}^{n} a_{2n-2k}b_{2k+1}\right)}_{c_{2n+1}} z^{2n+1}.$$

To go further, notice that $2^{2k}/(2k)! = \left(k!\left(\frac{1}{2}\right)_k\right)^{-1}$ and

$$(3.12) \qquad c_{2n} = \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!}\frac{2^{2k}}{(2k)!}\frac{\Gamma\left(k+\frac{1}{\alpha}-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{\alpha}-\frac{1}{2}\right)} = \frac{(-1)^n}{n!}\sum_{k=0}^{n}\frac{(-n)_k\left(\frac{1}{\alpha}-\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_k k!},$$

where we again used the Pochhammer symbol to denote the rising factorials. The last sum in (3.12) can be evaluated in terms of the hypergeometric function
(3.13)
$$c_{2n} = \frac{(-1)^n}{n!} \, {}_2F_1\left(-n, \frac{1}{\alpha}-\frac{1}{2}; \frac{1}{2}; 1\right) = \frac{(-1)^n}{n!}\frac{\left(1-\frac{1}{\alpha}\right)_n}{\left(\frac{1}{2}\right)_n} = \frac{(-1)^n 2^{2n}\left(1-\frac{1}{\alpha}\right)_n}{(2n)!}.$$

Similarly, we calculate the odd-numbered coefficients
(3.14)
$$c_{2n+1} = -\sum_{k=0}^{n}\frac{(-1)^{n-k}}{(n-k)!}\frac{2^{2k+1}}{(2k+1)!}\frac{\Gamma\left(k+\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha}-\frac{1}{2}\right)} = -\frac{2(-1)^n\Gamma\left(\frac{1}{\alpha}\right)}{n!\Gamma\left(\frac{1}{\alpha}-\frac{1}{2}\right)}\sum_{k=0}^{n}\frac{(-n)_k\left(\frac{1}{\alpha}\right)_k}{\left(\frac{3}{2}\right)_k k!}.$$

Again, using the hypergeometric function we have

$$(3.15) \qquad c_{2n+1} = -\frac{2(-1)^n\Gamma\left(\frac{1}{\alpha}\right)}{n!\Gamma\left(\frac{1}{\alpha}-\frac{1}{2}\right)} \, {}_2F_1\left(-n, \frac{1}{\alpha}; \frac{3}{2}; 1\right) = -\frac{2^{2n+1}}{(2n+1)!}\frac{\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(-n+\frac{1}{\alpha}-\frac{1}{2}\right)},$$
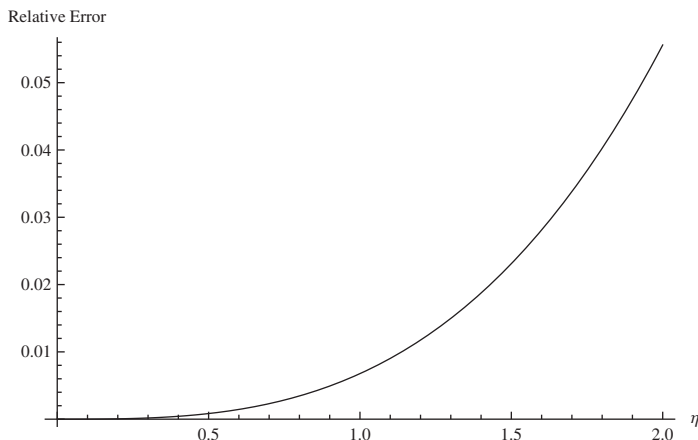
Fig. 1. *A comparison between Wright function U (3.2) and the approximate solution of the linear anomalous diffusion problem (3.16) for $\alpha = 0.9$.*

where we used the fundamental identity for the gamma function $\Gamma(1 + z) = z\Gamma(z)$. Finally, substituting $z = \frac{1}{2}\sqrt{\alpha/\Gamma(2 - \alpha)}\eta$ we obtain

$$(3.16) \qquad \begin{aligned} U_0(\eta) = & \sum_{n=0}^{\infty} (-1)^n \left(1 - \frac{1}{\alpha}\right)_n \left(\frac{\alpha}{\Gamma(2 - \alpha)}\right)^n \frac{z^{2n}}{(2n)!} \\ & - \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(-n + \frac{1}{\alpha} - \frac{1}{2}\right)} \left(\frac{\alpha}{\Gamma(2 - \alpha)}\right)^{n+\frac{1}{2}} \frac{z^{2n+1}}{(2n+1)!}. \end{aligned}$$

Comparing coefficients in (3.16) and (3.2) we see that

$$(3.17) \qquad U(\eta) - U_0(\eta) = \left(\frac{\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{1}{\alpha} - \frac{1}{2}\right)} \sqrt{\frac{\alpha}{\Gamma(2 - \alpha)}} - \frac{1}{\Gamma\left(1 - \frac{\alpha}{2}\right)}\right) \eta + O(\eta^3).$$

The comparison of U with U_0 is depicted on Figure 1. We see that the accuracy is very good, especially for $\eta \to 0$ (which describes the asymptotic behavior for $t \to \infty$). The difference between U and U_0 is smaller that 1% for $\eta \leq 1$. The relative error grows when $\eta \to \infty$, which is a consequence of the form of the approximating series (2.16). We will later see that this issue is not relevant when U has a compact support, which is also more appealing physically.

4. Nonlinear case. The case of $m \geq 1$ in (2.14) is treated in this section. This problem is the most interesting from the applications' point of view. As we shall see, the solution we obtain will have a compact support meaning that the wetting front has a constant speed of propagation. This phenomenon is widely known in literature (see, for example, [40]) but the analytical solutions of this problem are not obtainable even in the classical case of $\alpha = 1$. However, many authors proposed some techniques to deal with this issue (see [33, 41, 42]). After applying our approximation of the fractional operator, we follow the reasoning proposed in [32].

It is motivating to expect that the (weak) solution of (2.14) with conditions (2.7) will have a compact support, that is $U(\eta) = 0$ for $\eta \geq \eta^*$ with some η^*. This point characterizes the speed and position of the wetting front. The free boundary η^* has to be determined as a part of the solution. We are going to transform (2.14) to obtain

an initial-value problem and solve it by finding Taylor series coefficients. We will demonstrate how our approximation of the fractional Erdélyi–Kober operator (2.16) works in this case. Due to the compactly supported U, the issues of accuracy loss for large η which emerged in the linear case, will not be present in this more important nonlinear version.

We use the expression (2.16) with (2.14) to produce

(4.1)
$$(U^m U')' = (1-\alpha)\lambda_0 U + \left(\left(1 - \frac{3\alpha}{2}\right)\lambda_1 - \frac{\alpha}{2}\lambda_0\right)\eta U' - \frac{\alpha}{2}\lambda_1\eta^2 U'' + R\left(\lambda_k, \eta^k, U^{(k)}\right),$$

where the λ_k are as in (2.17) and $k \geq 2$. We stated explicitly the λ_0 and λ_1 terms only while the rest are combined under R. At first we note that retaining higher than second derivatives would require additional conditions for U apart from (2.15). For instance, we would require $U^{(k)}(0)$ for $k \geq 2$ and these can be obtained, for example, by solving an approximate form of (4.1) with only the λ_0 term present. From that we could determine an approximation of the first derivative $U'(0)$. Finally, differentiating (4.1), setting $\eta = 0$, and using this approximation would give us further derivatives to be used as a initial conditions. This is a rather complicated way of reasoning which would not necessarily give us decent accuracy of approximation. Moreover, additional higher order terms would not improve the method. As we noted before, although Theorem 1 is valid for $a > -1$, $b > 0$, and $c > 0$, the formulas for coefficients (2.17) can be analytically continued into a larger domain. Gamma functions, which arise in coefficients λ_k, become unbounded for their arguments approaching nonpositive integer values and thus doing numerical calculations with λ_k for $k > 0$ could produce unwanted errors. Also, as can be suggested by (2.18) for this analytically continued case, coefficients in (2.16) would not decay sufficiently fast. Very delicate cancellations between different coefficients of (2.16) could therefore be destroyed unless we take a large number of λ_k terms. Moreover, by an analogy with the classical case we cannot expect that U will be smooth (hence will not be analytic). In that case, constructing an approximation with a large number of derivatives could introduce some serious numerical errors. As was noted before, retaining only the λ_0 term in (2.16) is analogous to the Laplace method for finding the leading order contribution of the integral. If U is bounded (which clearly we assume) the integrand in (2.10) is concentrated near $z = 1$ and hence it is sensible to approximate $U(z^{1/c}\eta)$ by $U(\eta)$ and thus to obtain the first term in (2.17). This does not require any severe assumptions on a, b, c, and U and as we shall see the resulting free-boundary problem can be treated analytically. This will yield a leading order approximation.

Having previous remarks in mind we proceed to analyze (4.1) with only the λ_0 term retained deferring treatment of higher approximations to the numerical analysis. We thus obtain

(4.2)
$$(U^m U')' = \frac{1}{\Gamma(1-\alpha)}U - \frac{\alpha}{2\Gamma(2-\alpha)}\eta U',$$

where have taken only the first term in series (2.16). Observe that by the following substitution

(4.3)
$$U(\eta) = \left(m(\eta^*)^2 y(z)\right)^{\frac{1}{m}}, \quad z = 1 - \frac{\eta}{\eta^*},$$

(4.2) becomes

(4.4)
$$\frac{1}{m}y'^2 + yy'' = \frac{1}{\Gamma(1-\alpha)}y + \frac{1}{m}\frac{\alpha}{2\Gamma(2-\alpha)}(1-z)y'.$$

Notice also, that from (4.3) and (2.7) we immediately know the wetting front position η^*,

$$(4.5) \qquad \eta^* = \frac{1}{\sqrt{my(1)}}.$$

As for initial conditions we have

$$(4.6) \qquad y(0) = 0, \quad y'(0) = a_1,$$

where a_1 has to be determined from the structure of (4.4). First, we look for a solution in terms of the Taylor series

$$(4.7) \qquad y(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots.$$

By plugging (4.7) into (4.4), equating coefficients, and doing some algebraic manipulations we obtain

$$(4.8) \qquad \begin{aligned} a_1 &= \frac{\alpha}{2\Gamma(2-\alpha)}, \quad a_2 = \frac{\frac{m}{\Gamma(1-\alpha)} - \frac{\alpha}{2\Gamma(2-\alpha)}}{2(1+m)}, \\ a_3 &= \frac{m\left(\frac{\alpha}{2\Gamma(2-\alpha)} - \frac{m}{\Gamma(1-\alpha)}\right)\left(\frac{\alpha}{2\Gamma(2-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\right)}{6(1+m)^2(1+2m)}, \end{aligned}$$

and so forth. The wetting front position (4.5) can be then approximated by

$$(4.9) \qquad \eta_i^* = \frac{1}{\sqrt{m(a_1 + a_2 + \cdots + a_i)}}.$$

The Taylor series combined with the substitution (4.3) yields successive approximations of U:

$$(4.10) \qquad \begin{aligned} U_1(\eta) &= (1 - \eta/\eta_1^*)^{\frac{1}{m}}, \\ U_2(\eta) &= ((1 - \eta/\eta_2^*)(1 - ma_2\eta_2^*\eta))^{\frac{1}{m}}, \\ U_3(\eta) &= ((1 - \eta/\eta_3^*)(1 - m(a_2 + 2a_3)\eta_3^*\eta + ma_3\eta^2))^{\frac{1}{m}}, \end{aligned}$$

where $a_{1,2,3}$ are as in (4.8) and we have used the definition of η_i^* (4.9). Additionally, we can calculate the cumulative moisture intake which is an important empirical characteristic of the diffusion,

$$(4.11) \qquad I(t) := \int_0^\infty u(x,t)dx = \int_0^\infty U\left(\frac{x}{t^{\frac{\alpha}{2}}}\right)dx = t^{\frac{\alpha}{2}} \int_0^{\eta^*} U(\eta)d\eta.$$

The first approximation can be readily computed:

$$(4.12) \qquad \begin{aligned} I_1(t) &= t^{\frac{\alpha}{2}} \int_0^{\eta_1^*} U_1(\eta)d\eta = t^{\frac{\alpha}{2}} \int_0^{\eta_1^*} \left(1 - \frac{\eta}{\eta_1^*}\right)^{\frac{1}{m}} d\eta \\ &= \frac{m}{m+1}\eta_1^* t^{\frac{\alpha}{2}}. \end{aligned}$$

To evaluate the second approximation we use the integral representation for the hypergeometric function [43]

(4.13)
$$
\begin{aligned}
I_2(t) &= t^{\frac{\alpha}{2}} \int_0^{\eta_2^*} U_1(\eta) d\eta = t^{\frac{\alpha}{2}} \int_0^{\eta_2^*} \left((1 - \eta/\eta_2^*) \left(1 - ma_2 \eta_2^* \eta \right) \right)^{\frac{1}{m}} d\eta \\
&= \eta_2^* t^{\frac{\alpha}{2}} \int_0^1 \left((1 - s) \left(1 - ma_2 \left(\eta_2^* \right)^2 s \right) \right)^{\frac{1}{m}} ds \\
&= \frac{\Gamma \left(1 + \frac{1}{m} \right)}{\Gamma \left(2 + \frac{1}{m} \right)} \eta_2^* \, {}_2F_1 \left(-\frac{1}{m}, 1; 2 + \frac{1}{m}; ma_2 \left(\eta_2^* \right)^2 \right) t^{\frac{\alpha}{2}} \\
&= \frac{m}{m+1} \eta_2^* \, {}_2F_1 \left(-\frac{1}{m}, 1; 2 + \frac{1}{m}; \frac{a_2}{a_1 + a_2} \right) t^{\frac{\alpha}{2}},
\end{aligned}
$$

where in the last equality we used the definition of η_2^* (4.9). Unfortunately, integrals of higher order approximations cannot be evaluated explicitly and we have to rely only on numerical results.

5. Numerical analysis and application to the real experimental data.
In this section we provide a numerical illustration of our theoretical results and fit the approximate solutions to the real data obtained by the experimenters (we use the dataset described in [2]). This is to demonstrate a possible use of the considered model in the applications. Further investigations can validate this claim.

For a numerical verification of Theorem 1 we used the scientific environment MATHEMATICA. On Figures 2 and 3 we can see the of application of $I_c^{a,b}$ to the two functions: $\sin(\pi x)$ and $\log(1 + x)$. We can see that taking only terms with $k = 0, 1$ gives very good results.

Figure 4 shows the plot of $\lambda_k / \lambda_k^{\mathrm{asym}}$, where λ_k is as in (2.17) and $\lambda_k^{\mathrm{asym}} = (-1)^k c\Gamma(c(a+1))/\Gamma(b)k^{-c(a+1)}$ as in (2.18). We can see that the asymptotic form for λ_k becomes dominant very quickly as $\lambda_k \geq 0.95 \, \lambda_k^{\mathrm{asym}}$ for $k \geq 1$ and $\lambda_k \geq 0.99 \, \lambda_k^{\mathrm{asym}}$ for $k \geq 10$. This shows that even the first term in the asymptotic series is very close to the exact value of λ_k.
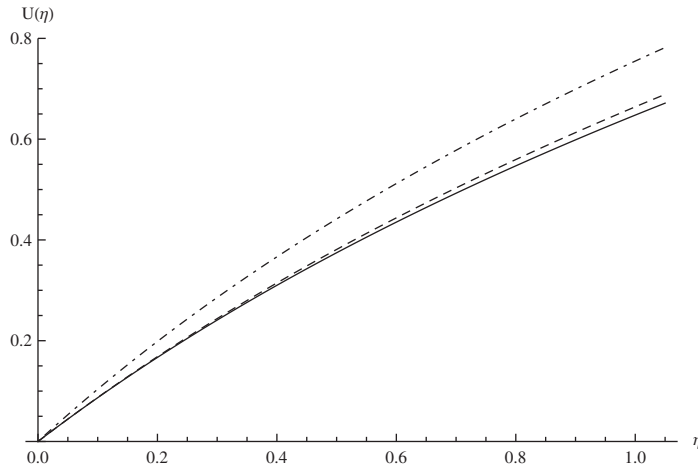

FIG. 2. *Approximating series* (2.17) *for the operator* $I_c^{a,b}$ *with* $a = 0$, $b = 0.2$, $c = 1$, *and* $U(\eta) = \log(1 + x)$. *Solid line:* $I_c^{a,b}U$; *dot-dashed line: one term approximation and dashed line: two term approximation.*
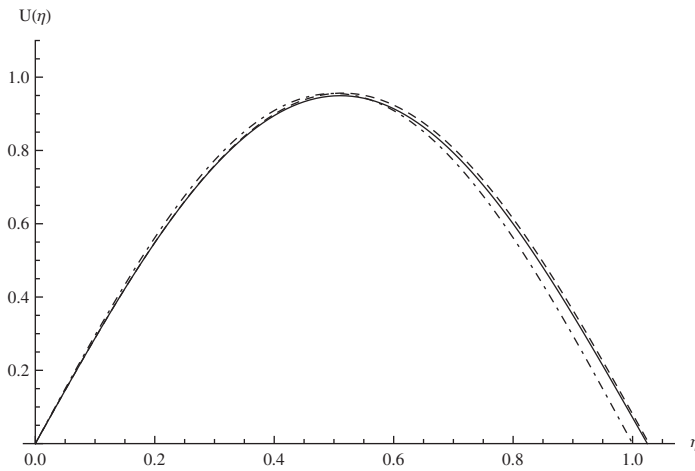
FIG. 3. *Approximating series (2.17) for the operator $I_c^{a,b}$ with $a = 1$, $b = 0.1$, $c = 2$, and $U(\eta) = \sin(\pi x)$. Solid line: $I_c^{a,b}U$; dot-dashed line: one term approximation and dashed line: two term approximation.*
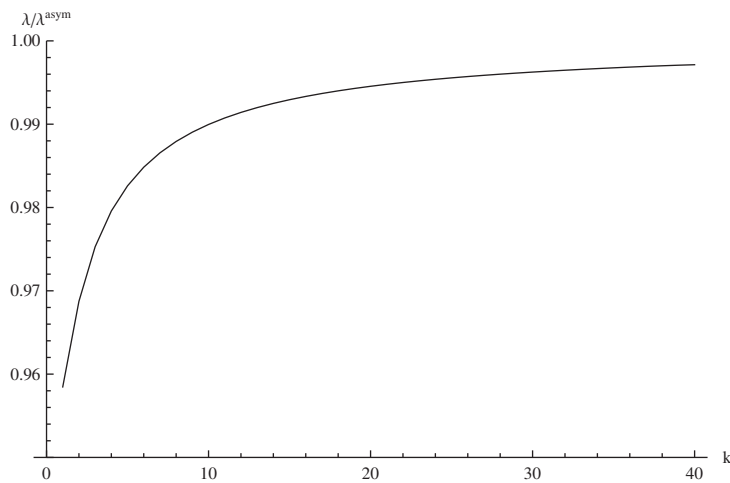

FIG. 4. $\lambda_k/\lambda_k^{asym}$ *for $a = -0.9$, $b = 0.1$, and $c = 2$.*

Now, we present a comparison between our approximate solutions U_i (4.10) and the numerical solution of (2.6)–(2.7). During the computations we used the MATLAB scientific environment. Numerical realizations of the Riemann–Liouville fractional derivative are calculated via the rectangular rule for quadratures of the corresponding integral (2.3). Introduce the (x_j, t_i)-grid: $x_j = jk$ and $t_i = ih$ with $k = X/M$ and $h = T/N$ for $M \times N$ number of grid points. If we denote $u_j^i = u(x_j, t_i)$ we can write

$$(5.1) \qquad (\partial_t^\alpha) u_j^i \approx \frac{h^{-\alpha}}{\Gamma(2 - \alpha)} \left(u_j^{i+1} + \sum_{k=1}^{i} a_{k,i} u_j^k \right)$$

with weights

$$(5.2) \qquad a_{k,i} = (i - k + 2)^{1-\alpha} - 2(i - k + 1)^{1-\alpha} + (i - k)^{1-\alpha}.$$

The numerical scheme we use in our computation is a weighted by $0 \leq \theta \leq 1$ explicit-implicit method

(5.3)
$$u_j^{i+1} - (1 - \theta) \frac{h^\alpha}{k^2 \Gamma(2 - \alpha)} \left(D_{j-1/2}^{i+1} u_{j-1}^{i+1} + \left(D_{j-1/2}^{i+1} + D_{j+1/2}^{i+1} \right) u_j^{i+1} + D_{j+1/2}^{i+1} u_{j+1}^{i+1} \right)$$
$$= - \sum_{k=1}^{i} a_{k,i} u_j^k + \theta \frac{h^\alpha}{k^2 \Gamma(2 - \alpha)} \left(D_{j-1/2}^{i} u_{j-1}^{i} + \left(D_{j-1/2}^{i} + D_{j+1/2}^{i} \right) u_j^{i} + D_{j+1/2}^{i} u_{j+1}^{i} \right),$$

where $D_{j \pm 1/2}^{i}$ is the average value of the diffusion coefficient

(5.4)
$$D_{j \pm 1/2}^{i} = \frac{1}{2} \left((u^m)_j^i + (u^m)_{j \pm 1}^i \right).$$

Note that (5.3) is a system of nonlinear equations for advancing in time—diffusion coefficient in step $i + 1$ depends on the concentration u_j^{i+1}. To linearize we adopt a method from [44] obtained by applying a Taylor series in t

(5.5)
$$(u^m)_j^{i+1} = (u^m)_j^i + m(u^{m-1})_j^i \left(u_j^i - u_j^{i-1} \right) + O(h^2)$$

and hence we can approximate the diffusion coefficient by

(5.6)
$$D_{j \pm 1/2}^{i+1} \approx \frac{1}{2} \left((u^m)_j^i + m(u^{m-1})_j^i \left(u_j^i - u_j^{i-1} \right) \right.$$
$$\left. + (u^m)_{j \pm 1/2}^i + m(u^{m-1})_{j \pm 1/2}^i \left(u_{j \pm 1/2}^i - u_{j \pm 1/2}^{i-1} \right) \right),$$

where concentrations were taken only at $i - 1$ and i time steps. Note also that we obtain explicit, Crank–Nicolson, and implicit numerical schemes by taking $\theta = 1$, 0.5, 0, respectively. For a more detailed and thorough analysis of finite-difference methods for fractional differential equations, see [45]. Figure 5 shows the numerical calculations of the diffusion evolution. We can see that, according to our previous remarks, the accuracy of our approximation is good especially for small values of x, where curves are almost indistinguishable. Apart from U_3 a numerical solution of (4.1) is pictured and it shows discrepancies in the wetting front region. This is in accord with our previous remarks that it would require many higher order terms in (4.1) to provide a sensible accuracy. This suggests that the leading order approximation with U_3 is the optimal choice. The wetting front and the cumulative moisture also approximated very well what can be learned from Figures 6 and 7.

Finally, we show the fitting results of our time-fractional diffusion model to the real data obtained and described in [2]. We use the approximate equation U_3 as in (4.10) with the rescaled variables (2.8). All points in the dataset form a self-similar shape for $\alpha = 0.885$. From the best fit condition we obtained $C = 0.71$ m^3/m^3, $m = 6.98$, $D_0 = 5.36$ mm/s$^{0.855}$ which are very close to the parameters obtained in [9] by a numerical model supplied with the same form of diffusivity. On Figures 8 and 9 we can see the results of numerical fitting for both the self-similar form and the time evolution of the diffusion process. The accuracy is acceptable and the goodness of fit is better than in the analysis proposed in [6]. This tentatively demonstrates that our model could constitute a candidate for a mathematical description of subdiffusion. But nevertheless, to decisively verify whether such a model is correct, a further theoretical and experimental study in this topic is required.
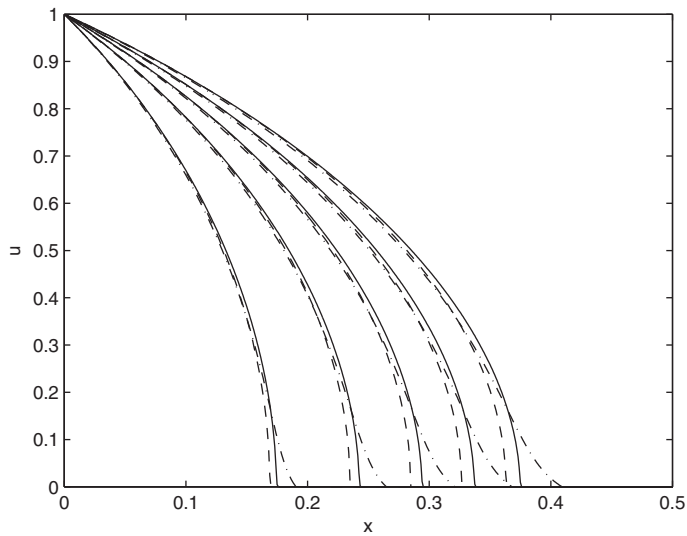
FIG. 5. *Comparison between the approximate solutions* (4.10), (4.1) *and numerical solution of* (2.6) *with* $\alpha = 0.95$ *and* $m = 2$ *for* $t = 0.02$, 0.04, 0.06, 0.08, $t = 0.1$ *(solid line),* U_3 *(dashed line), and a numerical solution of* (4.1) *(dot-dashed line).*
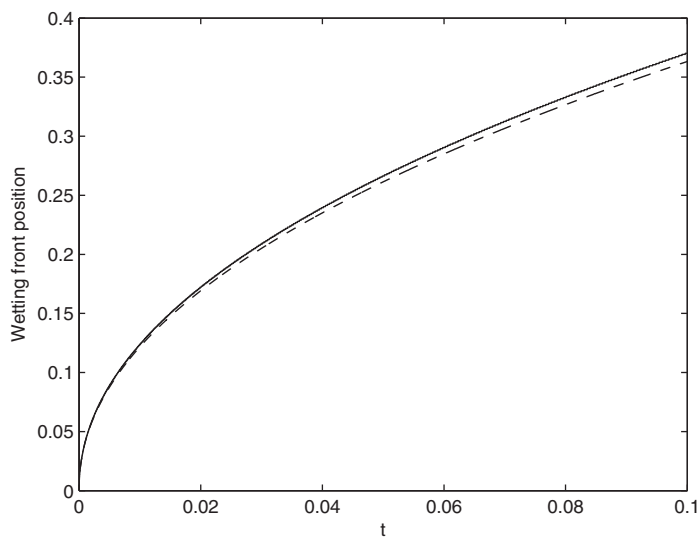

FIG. 6. *Wetting front position* $\eta^*(t)$ *(solid line) and approximation* $\eta_3^*(t) = \eta_3^* t^{\alpha/2}$ *(dashed line). Here* $\alpha = 0.95$ *and* $m = 2$.

6. Conclusion. In this paper we presented an analysis of the anomalous (sub)diffusion equation. This model was based on the Riemann–Liouvile time-fractional diffusion equation with a nonlinear diffusivity. We proposed a method of obtaining an approximation of the exact solution to the corresponding problem. Apart from the numerical analysis, finding approximate solutions of nonlinear equations is usually the only way of extracting any useful information about the process. When looking for a solution in a self-similar form, the equation describing diffusion transforms into an ordinary integro-differential along with a free-boundary problem. We proved a
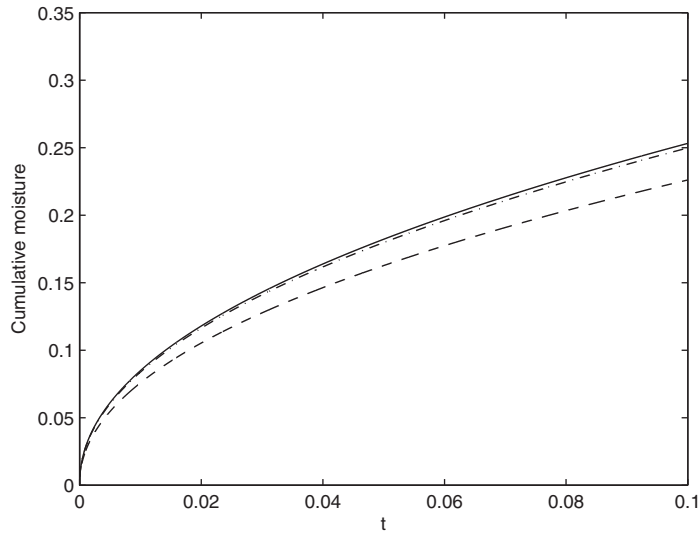
FIG. 7. *Cumulative infiltration (solid line) and its approximations I_1 (dashed line) (4.12) and I_2 (dot-dashed line) (4.13). Here $\alpha = 0.95$ and $m = 2$.*
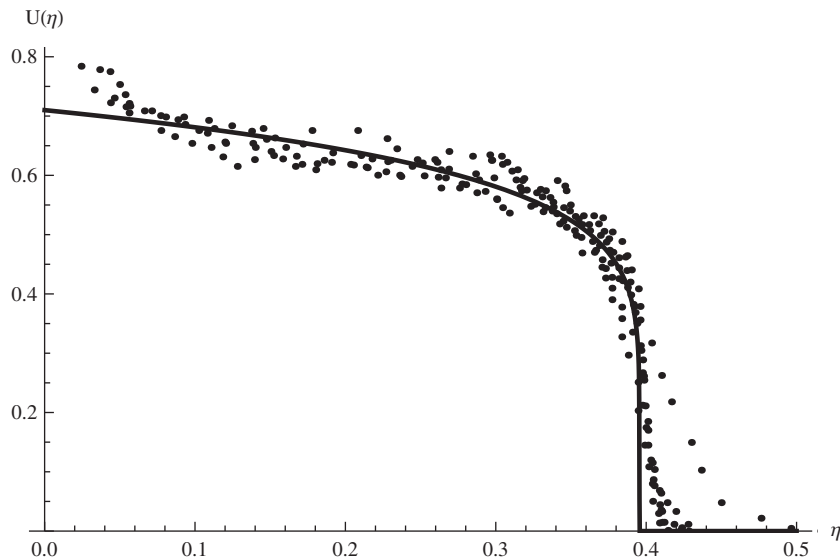

FIG. 8. *Fitting a self-similar profile of $U_3(\eta)$ as in (4.10) with experimental data [2]. Here $\alpha = 0.855$, $C = 0.71$ m^3/m^3, $m = 6.98$, $D_0 = 5.36$ $mm/s^{0.855}$.*

theorem that approximates the so-called Erdélyi–Kober fractional operator and simplified our equation into an ordinary differential one. To illustrate the accuracy of our approach we considered the linear case for which the exact solution is known. In the nonlinear version we were only able to obtain an approximate solution which, as the numerical analysis showed, was very accurate. Fitting with real experimental data of siliceous brick obtained and described in [2] gave good results tentatively suggesting that our mathematical model for subdiffusion in building materials can be sensible.
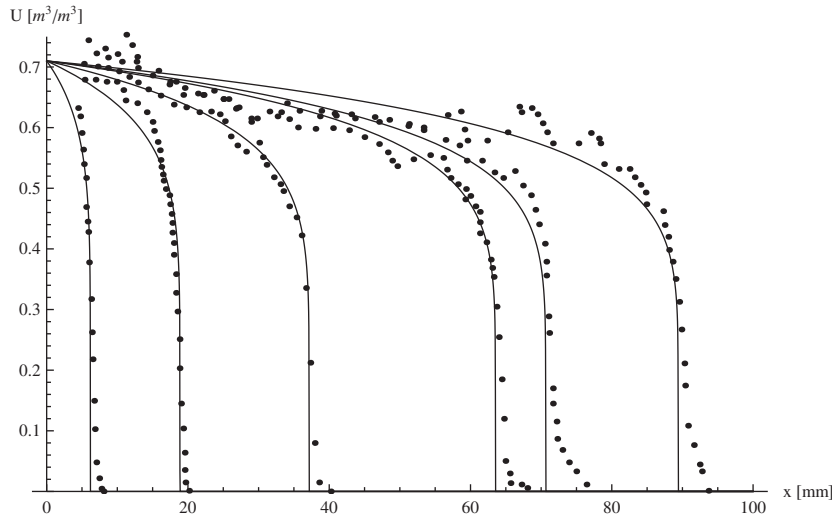
Fig. 9. *Fitting time evolution of $u_3(x,t)$ with experimental data* [2]. *Here $\alpha = 0.855$, $C = 0.71$ m^3/m^3, $m = 6.98$, $D_0 = 5.36$ $mm/s^{0.855}$.*

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