Ordinary differential equations in science and technology

Lecture notes

Łukasz Płociniczak*

January 29, 2021

Contents

*Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland

1 Introduction to differential equations

In an amazing way that many philosophers have wondered about, mathematics is the language that describes the laws of nature and the entire universe. It is truly amazing how much we have learned about the world from experiments and the physical theories that try to justify the results of these. Since the dawn of his history, man has tried to learn as much as possible about the world around him. To this end, he carried out experiments that evolved along with our development: from the "simple" striking of stones to collisions of subatomic particles in accelerators. Each time he was driven by curiosity and a desire to know.

When we look up from the book, we see that everything changes around us. The world is ruled by changes that happen both in time and in space. The hands on the clock rotate telling us that time is inexorably flowing and our room is in a mess. Huge amounts of water molecules in our mug collide with each other pulling the flavour out of the tea leaves. These leaves were once found on a tree that, thanks to photosynthesis, was able to convert the life-giving energy of the sun into nutrients that it built into its structure and grew from the seed. It was watered with water coming from the precipitation of small droplets suspended in the atmosphere torn by turbulent winds, temperature and pressure differences. The Sun, which supplies the entire planet with enormous amounts of energy of unprecedented quality, is in fact a huge fusion reactor in which, thanks to enormous temperatures, the lighter elements are combined into heavier[1]. Stars such as the Sun are formed by the collapse of interstellar material by gravity, which, though weak at small scales, dictates what the ever-evolving large-scale structure of the universe should look like. But the dynamics is not over: the universe is constantly expanding [2] And the pace of this expansion does not slow down rather accelerates! And since everything is expanding, seen „backwards" it must have contracted and one day there had to be a beginning - the Big Bang, in which space-time was created, and shortly after that the particles that we are made of. Everything changes and the **differential equations** are a natural mathematical description of these changes. We will deal with them later in the lecture. All major physical theories are formulated with differential equations, and it is very rewarding to know at least some of them.

The historical origin of differential equations lies in the works of Newton and Leibniz. Newton's results on ODEs were usually a side product of his development of Calculus. In particular, he solved several equations of the form $y' = f(x, y)$ by the use of infinite series. On the other hand, Leibniz developed some general methods for studying differential equations such as separation of variables, and an algorithm of solving linear equations. Then, the Bernoulli brothers: Jakob and Johann came and introduced further contributions such as the brachistochrone problem and formulating many problems in mechanics in terms of differential equations. Johann introduced what is today known as Bernoulli equation which we will solve in the sequel. A son

[1] Solar energy mainly comes from the fusion of hydrogen into helium. Helium has the highest binding energy among other elements, which is released during the fusion process and reaches us after some time. In other words: two hydrogen atoms weigh more than one helium atom - something has got to happen with the excess energy.

[2] More precisely: the distance between any two points increases with time.

of Johann, Daniel, developed partial differential equations and was the discoverer of the famous Bernoulli's Law in fluid mechanics (that today is widely used). One of the most fruitful results concerning differential equations were obtained further by Euler who was responsible for integrating factors, exact equations, power series solutions, equations for inviscid flow, and Euler numerical method to name only a few. After that, Lagrange made very important discoveries such as method of variation of parameters, variational calculus, general theory of linear equations, and partial differential equations. Almost any famous mathematician made some important contributions in differential equations or their applications. In the nineteenth century the focus switched to PDEs and theoretical aspects of them. Twentieth century brought further advancements in these fields along with a explosion of various numerical methods (with the advent of the computer) and dynamical systems. Today, differential equations are one of the most investigated and used fields in mathematics and its applications.

1.1 Literature

There are many very good textbooks on ordinary differential equations. We state only some of them.

- Elementary textbooks
  - Boyce, diPrima - *Elementary Differential Equations and Boundary Value Problems*
  - Tenenbaum - *Ordinary Differential Equations*
- Advanced monographs on theory
  - Hartman - *Ordinary Differential Equations*
  - Coddington, Levinson - *Theory of ordinary Differential Equations*

The lecture will be completely self-sufficient, however, it is always good to broaden the understanding by reading through different textbooks.

1.2 Basic definitions and concepts

Differential equation is any relationship that binds an independent variable, its function, and derivatives of the latter. In the following, we deal with motivations, intuitions, and the practical meaning of differential equations. There is a huge and beautiful theory of equations where problems such as the existence of solutions, their uniqueness and regularity are studied (see eg []). We will mainly focus on solving and applying differential equations to describe phenomena in nature, technology, and everyday life.

The concept of a differential equation is very broad, and it is impossible to study the entire class at once. A very natural classification is thus needed.

- According to the number of independent variables.

– *Ordinary Differential Equations (ODEs)*. They involve only one independent variable and we can write them in the form

$$F\left(x, y(x), y'(x), y''(x), ..., y^{(n)}(x)\right) = 0, \tag{1.1}$$

where F is a given scalar of vector function, $y = y(x)$ is the dependent varibale, that is the sought function of x. For example, $F(a, b, c) = ab - c$ yields $y'(x) = xy(x)$ in a compact form[3]. In applications the independent variable is usually taken to be the time (in initial value problems) and space (in boundary value problems).

– *Partial Differential Equations (PDEs)*. They involve two or more independent variables. They can be written as

$$G\left(x_1, ..., x_n, u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}, ..., \frac{\partial^2 u}{\partial x_1 \partial x_2}, ...\right) = 0, \tag{1.2}$$

where $u = u(x_1, ..., x_n)$ is the sought function. The arguments of G can include all partial derivatives of u of any order. Here, we interpret the independent variables as time t, and spatial coordinates x, y, and z. Almost all equations of mathematical physics are partial since they describe changes in space and time.

- According to the order of the highest derivative.
- According to linearity.

  – *Linear equations*. They are these equations that can be written in a form

$$Ly = f, \tag{1.3}$$

where L is a linear operator[4] acting on a space of differentiable functions, and f is a given function (representing external sources). In the case of ordinary equations the above is equivalent to

$$Ly(x) = a_n(x)y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + ... + a_1(x)y'(x) + a_0y(x). \tag{1.4}$$

Each of the points of the above classification can be combined with the others, thus we can speak of, for example, a second-order linear ordinary equation or a first-order. Very often this classification is directly related to the difficulty of equations: partial equations are usually more difficult to study than ordinary equations, nonlinear equations are more difficult than linear equations, and higher order equations are more troublesome to solve than first order equations. Of course there are exceptions to this rule. With the development of science, people began to study more and more difficult equations, because they also encountered more and more complex phenomena. Nowadays we know practically everything about linear equations - increasingly more research is being aimed towards nonlinear equations.

[3]Check whether you understand this clearly.

[4]Recall that L is linear when $L(\alpha y + \beta z) = \alpha Ly + \beta Lz$ for scalars α and β and vectors y and z.

At this point, one should also pay attention to a rather non-obvious thing for some readers - we are not able to provide a solution in the form of a combination of elementary (or even special) functions for each equation. A similar situation occurs with polynomials. We can find all the roots of polynomials of degree four or less. For higher degrees, there are no longer the general formulas. In the case of differential equations, the situation is even more complicated. However, such difficulties were and are fuel for new ideas and mathematical methods. First of all, we are often able to prove that the solution of the equation exists and is unique (often very difficult). Although we do not know the exact form of the solution, we do know that it exists. This is very valuable knowledge. Using a number of ingenious techniques, we can also find features of this unknown solution, such as asymptotic behaviour for small or large parameters, estimates, and general form. We are also able to implement numerical methods, thanks to which we will know the exact shape of the solution for parameters that interest us. Thanks to numerical solutions of differential equations, we are able to simulate a lot of phenomena ranging from vibrations of bridges and buildings, development of bee colonies, galactic collisions to the development of the structure of the universe. In this lecture, we are going to talk about basic equations that can usually be solved analytically. Thanks to this, we will develop an intuition that will be used to study more complicated phenomena.

A good news is also the fact that usually, we meet equations that have not higher order than two (however, there are important equations of third or forth order). The reasons of this are hidden in physics where we frequently use Newton's Second Law of Dynamics or similar mechanical principles.

1.3 Examples

In this section we will see many examples of equations found in various areas of life and science. These examples, of course, can be multiplied infinitely, but here are either very fundamental equations or those that illustrate the use of their particular class for modelling real phenomena. We will meet some of these equations later in the lecture and present their discussion and derivation there.

Example. (*Bacterial evolution, radioactive decay*). All these phenomena can be described using one differential equation (very often we will suppress writing the independent variable)

$$y' = ky, \tag{1.5}$$

where k is a constant. This equation describes the size of y (population size, amount of radioactive element), which changes at *rate proportional to its value*. The constant k in the case of decay is negative because y decreases. For bacteria, the constant k can be either positive (proliferation) or negative (extinction). This is a first-order ordinary linear equation. □

Example. (*Newton's Second Law*) This is the most important equation of classical mechanics. According to Newton's Second Law of Motion, the net force on a body is

proportional to the acceleration. Mathematically, we can write it as [5]

$$m\frac{d^2x}{dt^2} = F\left(t, x, \frac{dx}{dt}\right), \tag{1.6}$$

where m is the mass of the body, $x = x(t)$ is its position [6] and F is the resultant force. Note that this force can depend on both time and position as well as the speed of the particle itself. We can immediately say that the above equation is an ordinary equation of the second order. The linearity (or lack thereof) depends on the form of the resultant force F. Let us consider some well-known special cases.

- *Law of inertia.* If the net force is equal to zero, i.e.

$$m\frac{d^2x}{dt^2} = 0, \tag{1.7}$$

  we have after a simple integration

$$m\frac{dx}{dt} = p, \tag{1.8}$$

  where p is the integration constant (the momentum). Therefore, the velocity does not change. Furthermore, we can integrate once again to obtain

$$x(t) = \frac{p}{m}t + x_0, \tag{1.9}$$

  where we introduced another integration constant representing the initial position of the body. Therefore, we have proved that if there are no forces acting on the particle, it moves with a uniform velocity.

- *Constant net force.* In the case of constant F we have

$$m\frac{d^2x}{dt^2} = F, \tag{1.10}$$

  which can be integrated twice to have

$$x(t) = \frac{at^2}{2} + v_0t + x_0, \tag{1.11}$$

  where we have put $a := F/m$. Here, v_0 jest is the initial velocity while x_0 initial position. This formula states that under a constant net force the particle accelerates uniformly with acceleration F/m.

[5] This is an equation that describes a particle (material point) moving in one dimension. Further generalizations are of course possible.

[6] Recall that if $x(t)$ is the position of the particle in time t then dx/dt is its velocity, and d^2x/dt^2 the acceleration.

Newton's equation can yield many more interesting objects to investigate and simulate and we will meet it many times. □

Example. (*Linear oscillator*) Let the position of an oscillating particle be denoted by $x = x(t)$. Then if the damping of the medium is proportional to the speed, the following equation describes the evolution

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = f, \tag{1.12}$$

where β is damping coefficient, ω is the anglar frequency of oscillations, and f is the external force. This is the archetypal equation modelling oscillations that can be found anywhere some periodic movement occurs. It is a linear ordinary differential equation of second order and constant coefficients. □

Example. (*Predator-Prey Model.*) Consider two populations, one of which we will call predators (eg foxes) and denote its number at time t by $P(t)$. The second population consists of prey (e.g. hares) with number $O(t)$. We are interested in the dynamics of the development of these two groups of animals in the case of mutual existence in a certain area. This is very important from the point of view of ecology, because knowing both numbers at some point we would like to know the composition of both populations at later times. In this way, we could predict whether populations would remain in equilibrium or whether one of them would die out. The simplest interaction model was derived by Lotka and Volterra and can be formulated using the following system of differential equations

$$\begin{cases} \dfrac{dO}{dt} = aO - bOP, \\ \dfrac{dP}{dt} = -cP + dOP, \end{cases} \tag{1.13}$$

where a, b, c, and d are positive constants. We will discuss the above model in detail later in the lecture. Note that now we have two coupled nonlinear ordinary differential equations of the first order. □

Example. (*Friedmann's equations*) In the first half of the 20th century, Friedmann developed a system of equations to describe the dynamics of the expansion of the universe that follow from General Relativity. For a long time, the concept that the geometry of the entire cosmos could change over time was unacceptable to most physicists. It was only Hubble's discovery that distant galaxies were "fleeing" from us at speeds proportional to their distance (Hubble's law) that broke the Aristotelian dogma of the unchanging universe. Friedmann's equations describing the dynamics of the universe have the form

$$\begin{cases} \left(\dfrac{1}{a}\dfrac{da}{dt}\right)^2 + \dfrac{k}{a^2} = \dfrac{8\pi G}{3}\rho, \\ \dfrac{d\rho}{dt} = \dfrac{3}{a}\dfrac{da}{dt}(\rho + p), \end{cases} \tag{1.14}$$

where $a = a(t)$ is the scale factor determining how the distances between two points change and $\rho = \rho(t)$ is the matter-energy density in the universe. The above two nonlinear ordinary equations, together with the algebraic state equation $p = p(\,rho)$, give us a well-defined system in which the constant $k = -1, 0, +1$ describes the curvature of the universe (open, flat, closed). So far, many experiments indicate that the universe is flat with a relative error of only a fraction of one percent. □

Example. (*Travelling wave equation*) Imagine a pollution coming from the factory chimney. It is extremely important to recognize the distance to which these pollutants will spread over a given time. If we assume that the diffusion phenomenon is negligible here, the contaminant density $\rho = \rho(\mathbf{x}, t)$ at a point in space $\mathbf{x} \in \mathbb{R}^3$ and time $t > 0$ is given by[7]

$$\frac{d\rho}{dt} + \operatorname{div}(\mathbf{v}\rho) = 0, \tag{1.15}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ is the wind velocity vector. It is a linear partial differential equation of the first order. □

Example. (*Potential.*) A potential $u = u(x, y, z)$ of a gravitational (electrostatic) field in some region in space is described by the *Poisson's equation*[8]

$$-\Delta u = \rho, \tag{1.16}$$

gwhere $\rho = \rho(x, y, z)$ is the density of mass (electrostatic charge) being the source of the field. In the vacuum, that is when $\rho \equiv 0$, we obtain the *Laplace's equation*

$$\Delta u = 0. \tag{1.17}$$

Laplace's equation also describes the steady-state temperature distribution or the velocity field of certain flows (for ex. water waves) . Its solutions are called *harmonic functions*. Both Laplace's and Poisson's equations are second-order linear partial equations. □

Example. (*Heat equation*) Heat conduction is some region in space is given by the following equation

$$\frac{\partial u}{\partial t} = \alpha^2 \Delta u, \tag{1.18}$$

where α is the thermal diffusivity (here assumed constant) that characterises the medium. The sought function $u = u(\mathbf{x}, t)$ is the temperature at a point $\mathbf{x} \in \mathbb{R}$ and time $t > 0$. Interestingly, the equation of the same form describes the diffusion phenomenon. Moreover, under a seemingly small modification the heat equation becomes the fundamental relation of non-relativistic quantum mechanics - *Schrödinger's equation*

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + V(\mathbf{x}, t)\Psi, \tag{1.19}$$

[7]The *divergence* of a vector $\mathbf{u} = (u_1, u_2, u_3)$ is defined as $\operatorname{div} \mathbf{u} := \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial y}$.

[8]The *Laplacian* Δ of a function $u = u(x, y, z)$ is defined by $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial z^2}$.

where $\Psi = \Psi(\mathbf{x}, t)$ is the complex wave function, which the absolute value squared describes the probability of finding a particle of mass m at a point $\mathbf{x} \in \mathbb{R}^3$ and time $t > 0$. Moreover, $\hbar$ is the reduced Planck's constant, and $V = V(\mathbf{x}, t)$ is the force potential. □

Example. (*Wave equation*) Let $u = u(x, t)$ describes a deflection at x of a guitar string at time $t > 0$. Then,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1.20}$$

where c is the wave speed. In higher dimensions we have

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u. \tag{1.21}$$

For example: in two dimensions, this equation describes the vibrations of a drum and it is also a first approximation of the behaviour of sea waves. In three dimensions, the solution to the wave equation describes acoustic or electromagnetic waves. Wave phenomena are found in almost every field of physics, therefore a careful analysis of the above equation is extremely important. □

Example. (*Navier-Stokes equations*) At the end of our review, let us take a look at one of the most important differential equations in physical mathematics, the research of which is still very active and the equation itself is used to describe the enormous variety of phenomena in nature [9]. The Navier-Stokes equation describes the flow of a fluid, understood as a continuum [10]. Fluids can be liquids, gases and solids - what matters is the scale on which we consider the movement. The Navier-Stokes equation describes the dynamics of the velocity field $\mathbf{u}(x, y, z) = (u(x, y, z), v(x, y, z), w(x, y, z))$, that is the velocity of a fluid particle at (x, y, z). For a incompressible fluid we have

$$\begin{cases} \rho \left(\dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{f}, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{1.22}$$

where ρ is the density, p the pressure, μ viscosity, and $\mathbf{f}$ describes all forces acting on the volume of the fluid (for ex. gravity))[11]. The Navier-Stokes equation has actually four components: each for a different component of the fluid velocity vector and the *continuity equation* stating that the field $\mathbf{u}$ is divergenceless. Apart from the velocity field, there is a fourth unknown in the equation - the pressure p, and having four partial differential equations makes the problem well-posed.

Solving the Navier-Stokes equation is *difficult*: both analytically and numerically. The main reason for this is the non-linear convection component $(\mathbf{u} \cdot \nabla) \mathbf{u}$, which is responsible, inter alia, for the phenomenon of accelerating the flow along with the change of the geometry of the medium. The Navier-Stokes equation is a nonlinear, partial, second order equation. □

[9] One of the Millennium Prize problems concerns the Navier-Stokes equation.

[10] Continuum is an averaging of molecular movements over volumes on the appropriate scales.

[11] The *nabla operator* (from Greek "harp") is defined as $\nabla := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$. Therefore, the gradient of f can be written as ∇f, the divergence of $\mathbf{u}$ as $\nabla \cdot \mathbf{u}$, and the rotation $\nabla \times \mathbf{u}$.

2 First order equations

In this chapter, we will familiarize ourselves with the basic types of first-order ordinary differential equations, which we can generally write as[12]

$$y'(t) = f(t, y(t)), \quad t_0 < t < T. \tag{2.1}$$

From the very beginning, we will meet examples of applications of the discussed equations for modelling real phenomena. Although we will talk about very simple models, they are often quite accurate and constitute the foundation for building more complex descriptions of reality.

2.1 Direction fields

The analysis of differential equations usually begins with looking at its form and assessing the most information that can be gathered without even solving it. This is very useful when an explicit solution is not available. There are many methods of doing that and they frequently rely on analyst's experience. The simplest of them is to utilize the direction field.

Definition 1. *Let $F(t, y) = 0$ be the implicit equation defining a curve satisfying (2.1). Then, the graph of each such curve is called the **integral curve** of the equation (2.1). Moreover, the family of all such curves is called the **direction field** (or **slope field**).*

To understand the definition we just have to look at the ODE. The equation $y' = f(t, y)$ is a rule that assigns a slope of a curve to a point in the (t, y) space. That is, $f(t, y)$ is the slope of a curve that is a solution of our ODE. Choosing different points we can draw a vector $(1, f(t, y))$ at each such point and obtain an approximation of the direction field of the ODE. This is very useful because without even solving the equation we get an overall glimpse of the shape and behaviour of the solutions. We will illustrate this concept on several examples.

Example. Let us draw the direction field for $y' = y - 1$. We immediately see that $f(t, y) = y - 1$ and hence the slope is zero for $y = 1$. That is, one of the solutions is a constant equal to 1. The slope does not depend on x and thus, the direction field is symmetric with respect to $y = 1$. For example, when $y = 2$ we have $y' = 1$ and the tangent at $y = 0$ has the slope $y' = -1$. The slope grows linearly when we move away from the critical line.

The overall drawing is presented on Fig. 1. We immediately can see that the integral curves either increase (for $y > 1$), are constant (for $y = 1$) or decrease (for $y < 1$). In the first and third case, they move away from the critical line. Note how much we can tell about solutions without solving the equation. □

Example. Now, suppose we would like to investigate $y' = t + y$. The direction field is not nontrivial for both coordinates. A good starting point of drawing the direction

[12] In almost every case the general form (1.1) can, at least in principle, be inverted to obtain a expression for y'.

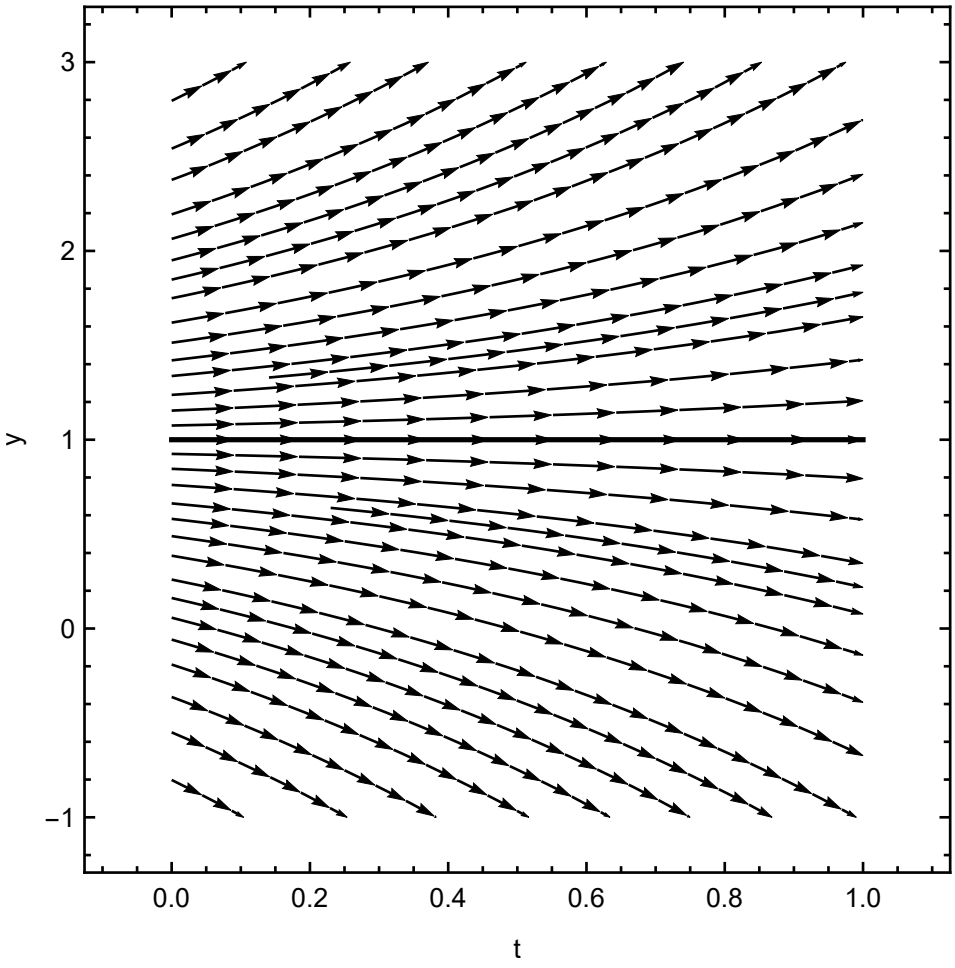

Figure 1: A direction field for $y' = y - 1$.

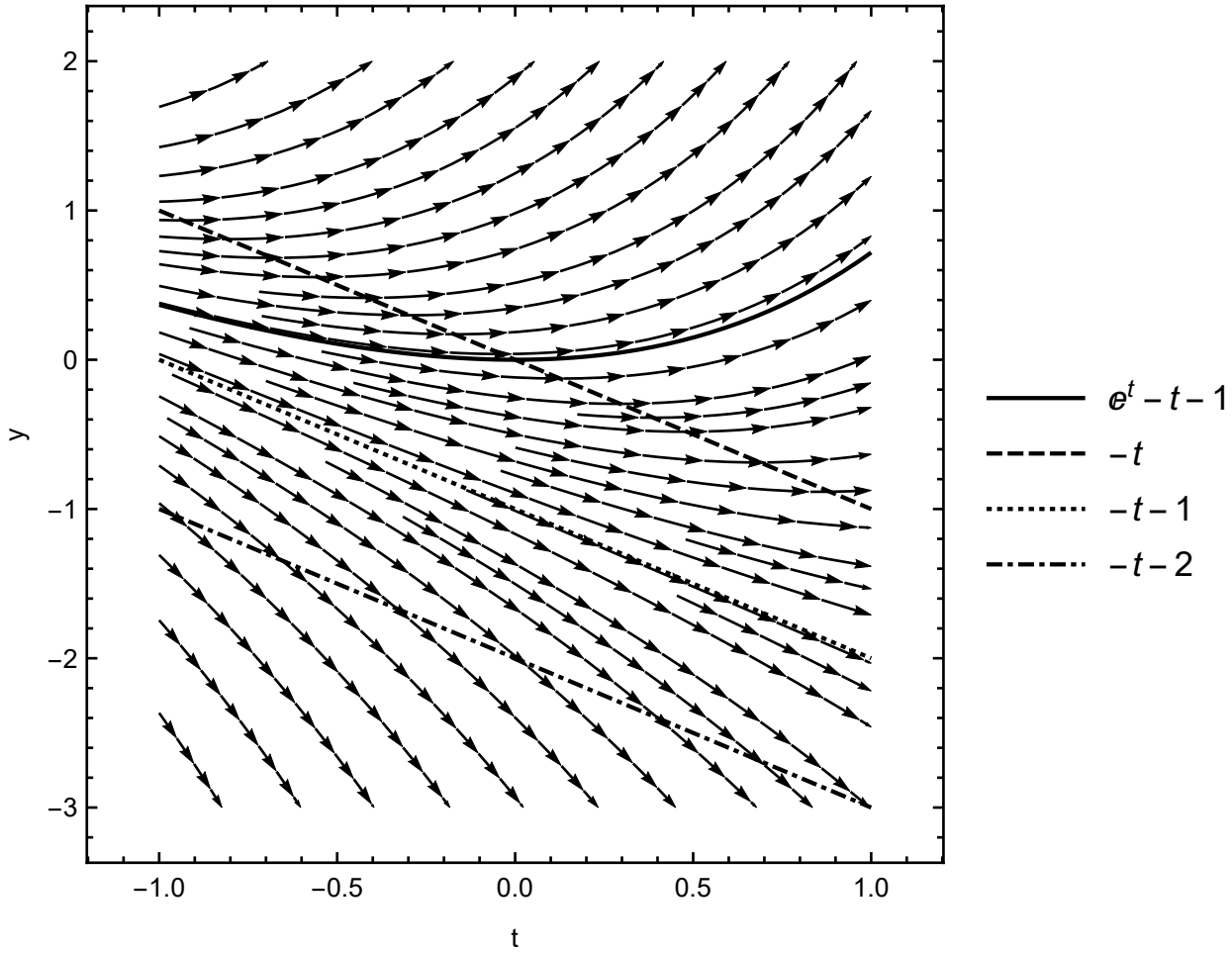

Figure 2: A direction field for $y' = y + t$.

field is finding the *isoclines*, that is curves of constant slope

$$y' = c \quad \Longleftrightarrow \quad f(t, y) = c. \tag{2.2}$$

Here, we have $y = c - t$, that is the isoclines are straight lines. Therefore, thanks to that we can quickly draw many tangents since we know that they have the same slope at these lines. This is exemplified on Fig. 2. □

We will generalize this graphical method in further studies. It becomes indispensable in analysis of systems of equations.

2.2 Separable equations

We will start our review of the most frequent first order differential equations with separable equations. These, usually are nonlinear, however, thanks to their simple structure can be integrated and studied.

Example. (*Malthus (exponential) model of population growth*) Consider a bacterial population that has an unlimited food supply and environmental opportunities to grow.

We are only interested in the dynamics of population development - we will not try to describe their spatial distribution. Let $y = y(t)$ denote the population size at time t. Note that we will build a model with continuous time dependence to take advantage of the possibilities offered by calculus. There are also discrete time models where difference equations are considered instead of differential equations. Some of them harbour rather interesting phenomena. However, the most useful ones are these in which we can use derivatives.

Let us fix a very short time Δt and consider the increase of bacteria in an interval $[t, t + \Delta t]$

$$\text{population increase between } t \text{ and } t + \Delta t = y(t + \Delta t) - y(t). \tag{2.3}$$

Since bacteria proliferate by division and their development possibilities are unlimited, their growth rate should be proportional to their number at a particular time since each member can divide itself[13]. Moreover, the longer the $[t, t + \Delta t]$ interval is, the bacteria can make more divisions in it. So the formula for an increase is

$$y(t + \Delta t) - y(t) = k\Delta t\, y(t), \tag{2.4}$$

where $k = k(t) > 0$ is a proportionality factor that can change in time and depends on the units chosen and bacterial species[14]. For example, during the increased solar or chemical activity, some bacteria can multiply faster than without these feedbacks. This could be modelled by prescribing an appropriate form of $k(t)$.

Dividing by Δt and taking the limit $\Delta t \to 0$ we get the differential equation

$$y' = k(t)y. \tag{2.5}$$

This is the Malthus model describing population dynamics (not only bacterial).

We would like to find a systematic way to solve (2.5) and similar equations. Of course, $y \equiv 0$ is one of the solutions. However, we would like to obtain something less trivial. It turns out that there is a algorithmic way to deal with a fairly wide class of first-order equations.

Definition 2. *A **separable equation** is a first order equation that can be written in a form*

$$y'(t) = f(y)g(t), \tag{2.6}$$

where f and g are given functions of a single variable.

We can see that (2.5) is a separable equation with $f(y) = y$ and $g(t) = k(t)$. There is a systematic method for solving equations of the form (2.6) that uses their structure. As the name of the equation indicates, you must first separate the variables, i.e to write

$$\frac{1}{f(y(t))} y'(t) = g(t). \tag{2.7}$$

[13]This is because we assume that the increment Δt is small. In fact, the change in the number of bacteria in the interval under consideration should be proportional to the mean value of y in $[t, t + \Delta t]$, because y may be different at each time. However, using the mean value theorem for the integrals and the continuity of y gives us the same result as our simplified assumption.

[14]We have made the tacit assumption here that bacteria do not die over time. If it were the case, we would add a negative term into the increment.

Note that the case when f can vanish has to be treated independently. Next, integrate with respect to t

$$\int \frac{1}{f(y(t))} y'(t)dt = \int g(t)dt. \tag{2.8}$$

We can now see that after a substitution $z = y(t)$, we have $dz = y'(t)dt$ and finally

$$\int \frac{dz}{f(z)} = \int g(t)dt. \tag{2.9}$$

We have thus obtained an algebraic equation (usually implicit)that determines the functional relationship between y and t. In some cases, we are able to calculate both of the integrals in (2.9) and find $y = y(t)$ or $t = t(y)$. Note that, it is very important to always remember that here we are dealing with indefinite integrals and thus they are families of primitive functions differing by a constant.

Mnemonic
In practice, separable equations are solved in a little modified and easy to remember way. First, we use Leibniz notation for the derivative to rewrite (2.6) as

$$\frac{dy}{dt} = f(y)g(t). \tag{2.10}$$

Then, we separate the variables (note that we treat dy and dt as numbers!)

$$\frac{dy}{f(y)} = g(t)dt. \tag{2.11}$$

And finally, add integrals

$$\int \frac{dy}{f(y)} = \int g(t)dt. \tag{2.12}$$

Note that treating dy/dt as a quotient of two numbers has to be understood as a change of variables inside the integral as we did before. This *useful fiction* is easy to remember and quickens the calculations.

Example. (*Malthus model cont'd*) Let us write (2.5) in a form

$$\frac{dy}{y} = k(t)dt, \tag{2.13}$$

which immediately can be integrated to obtain

$$\int \frac{dy}{y} = \int k(t)dt. \tag{2.14}$$

That is, since y is positive

$$\ln y(t) = \int k(t)dt. \tag{2.15}$$

or

$$y(t) = e^{\int k(t)dt}. \tag{2.16}$$

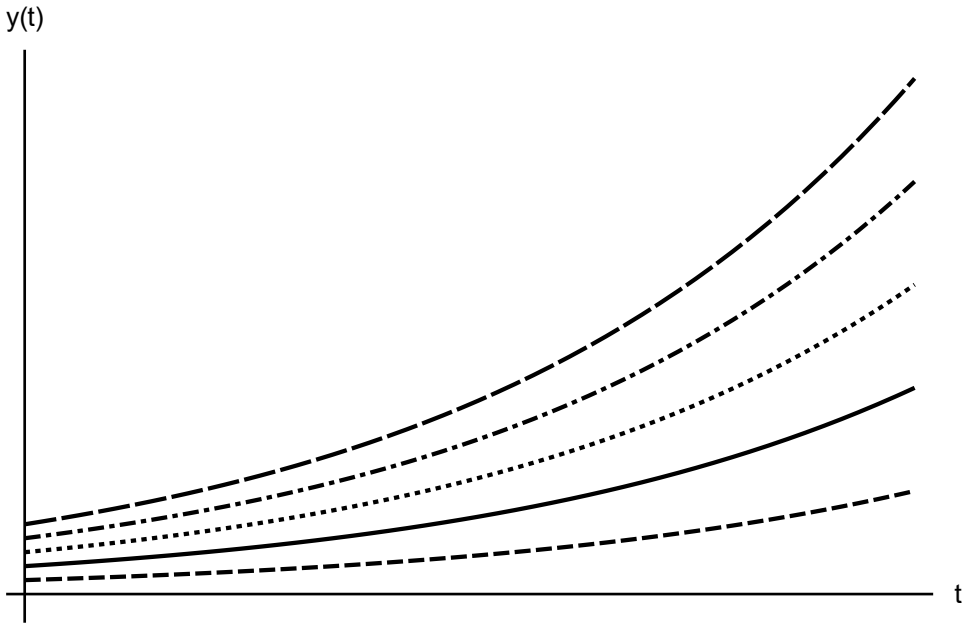

Figure 3: Different curves satisfying (2.17).

For example, in the important case of constant k we have $\int k(t)dt = kt + C$ and, hence,

$$y(t) = De^{kt}, \tag{2.17}$$

where we have denoted $D := e^C$. This is the famous exponential model of population dynamics which was pioneered by Thomas Malthus in the end of eighteenth century. We see that bacteria (or any other living species) grow exponentially in the complete abundance of needed resources. This model is also very accurate for describing people growth in initial stages after wars or other cataclysms.

The obtained *family* of curves (2.17) is indexed by the unknown integration constant D. Therefore, the general solution of our equation constitute an infinite set of curves. We can choose a particular one that satisfy a given condition. To this end, suppose that initially we have measured that $y(0) = y_0$. Then,

$$y_0 = y(0) = De^{0\cdot k} = D, \tag{2.18}$$

and finally,

$$y(t) = y_0 e^{kt}. \tag{2.19}$$

The exemplary plot of this solution is depicted on Fig. 3. □

Definite integration
The integrals in (2.8) can also be definite and many times this is convenient

$$\int_{t_0}^{t} \frac{1}{f(y(\tau))} y'(\tau) d\tau = \int_{t_0}^{t} g(\tau) d\tau, \tag{2.20}$$

which, after substitution $z = y(t)$ leads to

$$\int_{y(t_0)}^{y(t)} \frac{1}{f(z)} dz = \int_{t_0}^{t} g(\tau) d\tau. \tag{2.21}$$

Now, the integration constant is incorporated into the integral. For example, the integral in the Malthus model for constant k would be

$$\int_{y_0}^{y(t)} \frac{dy}{y} = \int_{0}^{t} k dt \quad \rightarrow \quad \ln \frac{y(t)}{y_0} = kt, \tag{2.22}$$

which mediately gives (2.17).

Since we have learned about solutions to some differential equations, it is time to define them rigorously.

Definition 3. *A solution of a differential equation is any sufficiently smooth curve* $\{(t, y(t)) : t \in [t_0, T]\}$ *identically satisfying it.*

Note that the above definition is much more general than stating that a function is a solution. Here, the solution can be implicitly or parametrically given family of curves. If additional data is given we can define our basic problem to solve.

Definition 4. *The* ***initial value problem*** *(or* ***Cauchy problem****) is a differential equation along with a given* ***initial condition***

$$\begin{cases} y' = f(t, y), \\ y(t_0) = y_0, \end{cases} \qquad t_0 \leq t \leq T, \tag{2.23}$$

for a given function f *and some fixed constants* t_0, T, *and* y_0.

Thus, solving the initial value problem consists of finding a family of curves satisfying a given equation and then selecting the one that passes through a predetermined point. The initial condition is called as such because in many problems time is the physical interpretation of the independent variable. In this sense, the differential equation gives the dynamics of y with y_0 at the beginning of the time evolution usually taken to be at $t_0 = 0$. The theoretical importance of the initial value problem is its uniqueness (under some suitable assumptions). That is, we have to be certain that equation has only one solution. If not, we have to be alerted since this may have some important consequences for our model.

Example. (*Skydiver and terminal velocity*) A classical problem in mechanics concerns finding a velocity of a particle falling in a medium with a drag force. This model is a first approximation of a skydiver jump or falling raindrop.

Consider a skydiver of mass m with position $x = x(t)$ and velocity $v = dx/dt$ that is being subjected to a constant gravity $F_g = mg$ acting in x-direction and drag force

$$F_d = \frac{1}{2} C_d A \rho v^2 = \gamma v^2, \tag{2.24}$$

where C_d is the shape dependent constant, A its cross sectional area projected onto the direction of flight, and ρ the density of air. Here, we assume that all of these parameters are constant, however, much more realistic model would be to assume that ρ depends on height. Moreover, both C_d and A change when the parachute is being opened and, thus, we assume that we model the situation when it is closed.

The balance of forces along with Newton's law gives

$$m \frac{d^2x}{dt^2} = mg - \gamma \left(\frac{dx}{dt} \right)^2. \tag{2.25}$$

This is a second order equation, however, there is no x in it, and thus we can use the velocity

$$\frac{dv}{dt} = g - \frac{\gamma}{m} v^2, \tag{2.26}$$

which is a separable equation for the velocity alone. As an initial condition we take

$$v(0) = 0, \tag{2.27}$$

since the skydiver stepped out of a plane with zero initial velocity. We can separate variables, and integrate to obtain

$$\int \frac{dv}{g - \frac{\gamma}{m} v^2} = \int dt. \tag{2.28}$$

Only one integral is not trivial and can be dealt with by expanding into simple fractions

$$\frac{1}{g - \frac{\gamma}{m} v^2} = \frac{1}{2} \sqrt{\frac{m}{\gamma g}} \left(\frac{1}{\sqrt{\frac{mg}{\gamma}} + v} + \frac{1}{\sqrt{\frac{mg}{\gamma}} - v} \right). \tag{2.29}$$

We can now perform the integration

$$\int \frac{dv}{g - \frac{\gamma}{m} v^2} = \frac{1}{2} \sqrt{\frac{m}{\gamma g}} \ln \frac{\sqrt{\frac{mg}{\gamma}} + v}{\sqrt{\frac{mg}{\gamma}} - v} + C, \tag{2.30}$$

with C the integration constant. Therefore,

$$\frac{1}{2} \sqrt{\frac{m}{\gamma g}} \ln \frac{\sqrt{\frac{mg}{\gamma}} + v}{\sqrt{\frac{mg}{\gamma}} - v} + C = t. \tag{2.31}$$

It is convenient to determine the integration constant right now, for $v(0) = 0$ gives

$$\frac{1}{2}\sqrt{\frac{m}{\gamma g}}\ln\frac{\sqrt{\frac{mg}{\gamma}}}{\sqrt{\frac{mg}{\gamma}}} + C = 0, \tag{2.32}$$

that is $C = 0$. Inverting, we obtain

$$\frac{\sqrt{\frac{mg}{\gamma}} + v}{\sqrt{\frac{mg}{\gamma}} - v} = e^{2\sqrt{\frac{\gamma g}{m}}t}. \tag{2.33}$$

This algebraic equation can easily be solved for v to yield

$$v(t) = \sqrt{\frac{mg}{\gamma}}\frac{e^{\sqrt{\frac{\gamma g}{m}}t} - e^{-\sqrt{\frac{\gamma g}{m}}t}}{e^{\sqrt{\frac{\gamma g}{m}}t} + e^{-\sqrt{\frac{\gamma g}{m}}t}} = \sqrt{\frac{mg}{\gamma}}\tanh\sqrt{\frac{\gamma g}{m}}t, \tag{2.34}$$

where we have utilized the known formula for hyperbolic tangent. A plot of the above is presented on Fig. 4. Most importantly, observe that the velocity approaches the so-called *terminal velocity*

$$v_t = \sqrt{\frac{mg}{\gamma}}, \tag{2.35}$$

in which gravity balances the drag. This could have been found from the ODE itself by setting $dv/dt = 0$ in (2.26). That is, in the situation of equal forces. However, the analytical solution (2.34) gives us the temporal evolution of the velocity. In particular, the characteristic time scale

$$\tau = \sqrt{\frac{m}{\gamma g}}, \tag{2.36}$$

on which the velocity changes significantly. For example

$$v(\tau) = 0.76\,v_t, \quad v(2\tau) = 0.96\,v_t, \tag{2.37}$$

that is, after one characteristic time the velocity approaches 76% of the terminal velocity, while for two such time scales it is greater than 96% of the final velocity.

Further integration of $dx/dt = v$ with $x(0) = x_0$ can let us obtain the explicit form for the distance travelled

$$x(t) = x_0 + v_t\int_0^t \tanh\frac{s}{\tau}ds = x_0 + \frac{m}{\gamma}\log\cosh\frac{t}{\tau}. \tag{2.38}$$

We can see that for large t with respect to τ the hyperbolic cosine is approximately equal to an exponential which gives us an almost uniform motion. □

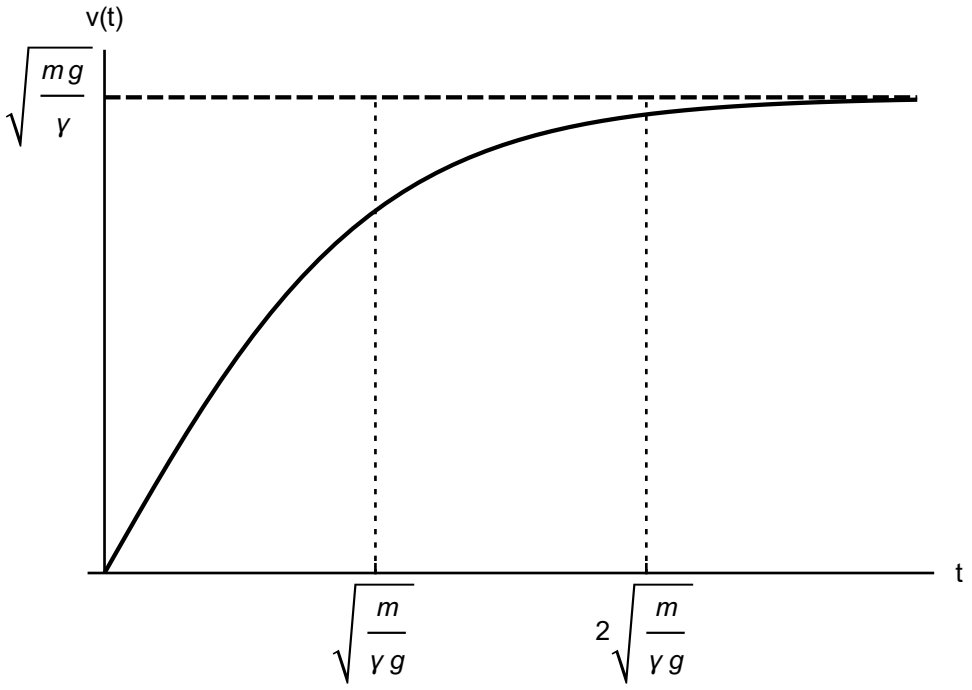

Figure 4: A plot of the solution of (2.26). Vertical lines denote characteristic times.

2.3 Autonomous equations

We will now investigate an important, although seethingly simple, class of first order equations.

Definition 5. *An **autonomous equation** is an ODE in the form*

$$y' = f(y), \tag{2.39}$$

that is, the right-hand side of the above does not depend on the independent variable.

One of the most important consequences of the above definition is the fact that there exists a powerful geometrical method of inquiring information about the behaviour of the solution without solving the autonomous equation. It is based on a very simple observation that when y is such that $f(y) = 0$ then, of course, $y' = 0$ at that point. Therefore, the solution does not change.

Definition 6. *A **stationary point** (or **critical point**) of the autonomous equation (2.39) is y_c satisfying $f(y_c) = 0$. Furthermore, if $f'(y_c) < 0$ the point is **stable**, if $f'(y_c) = 0$ the critical point is **neutrally stable**, and otherwise it is **unstable**.*

The stability of the critical point has a great significance in analysis since it helps us to draw general conclusions about the behaviour of the solution. To see this, suppose that y_c is a critical point and perturb it by adding a new function φ. We can think to this as a noise acting on a steady-state system described by an ODE. Then,

$$(y_c + \varphi)' = f(y_c + \varphi). \tag{2.40}$$

By expanding the right-hand side into Taylor series at $y = y_c$ we have

$$\varphi' = f(y_c) + f'(y_c)\varphi + \frac{1}{2}f''(\zeta_c)\varphi^2. \tag{2.41}$$

First, since y_c is the critical point we have $f(y_c) = 0$. Next, if we assume that φ is small we can truncate the quadratic term. This is done because we would like to investigate only **local** properties of the solution near the critical point[15]. We thus obtain

$$\varphi' = f'(y_c)\varphi, \tag{2.42}$$

which is a separable equation that we have met before. The solution is

$$\varphi(t) = Ce^{f'(y_c)t}, \tag{2.43}$$

and it tells us everything. Since φ is a perturbation we see that it decays to zero in an exponential pace when $f'(y_c) < 0$, that is, if the critical point is stable. Similarly, the solution moves away the critical point for unstable case. We will shortly see that this is almost everything we should need to know in order to obtain an accurate information of the solution. Note also, that the above analysis is only local. The solution can move away from one critical point and be captured by the other. A complete analysis of all stationary points is thus needed.

Everything we have said about critical points and stability readily generalizes to systems of ODEs where it has its most important applications (we will do this in later sections). However, we will see that this first step serves a lot of modelling in population dynamics. For example, we have met the Malthus growth model $y' = ky$ which falls in this class. Another, very important one, is the generalization of it to cope with limited supplies.

Example. (*Verhulst (logistic) model of population growth*) Note the major drawback of the Malthus model of population growth was the assumption of unlimited environmental supplies. Verhulst fixed that in early nineteenth century by introducing a limited carrying capacity. Suppose we want to extend Malthus model in the following form

$$y' = g(y)y, \tag{2.44}$$

that is, when $g(y) = k$ we obtain the exponential dynamics. Here, we would like to limit the growth rate of the population by appropriately choosing g. The main assumption is the premise that if the population grows too much, both food and place to live should become more and more sparse limiting the ability to proliferate. Hence, for y near some *carrying capacity* K the growth should slow down, while for small y the Malthusian dynamics should be appropriate. The simplest f that satisfies these requirements is a linear function $g(y) = r(1 - y/K)$. Therefore,

$$y' = ry\left(1 - \frac{y}{K}\right), \tag{2.45}$$

which is called *logistic equation*. Here, $r > 0$ is the growth rate, and $K > 0$. We see that when y is small compared with K the parenthesis is essentially equal to 1 and we obtain

[15]Everything here can be made rigorous by, for example, the use of Grönwall's Lemma.

the Malthus model. For $y \approx K$ the parenthesis is close to zero making the derivative very small. This simple model is a great tool in modelling a number of real-life growth phenomena: bacterial, plants, animals, and even people! □

Although (2.45) can readily be solved using separation of variables, we will show how to use geometrical analysis in order to analyse it. First, we look for critical points

$$0 = ry\left(1 - \frac{y}{K}\right) = f(y). \tag{2.46}$$

The solution of the above is

$$y_{c1} = 0, \quad y_{c2} = K. \tag{2.47}$$

Therefore, when we start the evolution at either of these points (that is, we choose the initial condition to be y_c) we obtain a constant solution. This makes sense, because when there are no individuals to replicate the population will not grow. Similarly, when the carrying capacity is reached, there will be essentially no food or volume to produce offspring. Further, we compute the derivative to investigate the stability

$$f'(y) = r\left(1 - 2\frac{y}{K}\right). \tag{2.48}$$

Hence, $f'(y_{c1}) = r > 0$ and $f'(y_{c2}) = -r < 0$. Therefore, the zero stationary point is unstable while the other is stable. This is a good information since we know that the nontrivial system state, i.e. $y = K$, will attract nearby solutions. Moreover, $f'(y) < 0$ for $y > K$ and $f'(y) > 0$ for $y < K$. The function $y = y(t)$ will thus increase up to $y_{c2} = K$ from initial conditions $y(0) \in (0, K)$ while decrease to the same limit for $y(0) > K$. The convexity can be calculated in the usual way

$$y'' = (f(y))' = f'(y)y' = f'(y)f(y), \tag{2.49}$$

where the primes are understood. The graph $y = y(t)$ is thus convex for f' and f having the same sign and this happens when $y \in (0, K/2)$ and $y > K$. The inflection point occurs at $y = K/2$. Note how much information about the solution we have been able to extract from just studying the given ODE.

An invaluable tool in visualising the overall dynamics is to draw a y' versus y diagram. Since $y' = f(y)$ the just have to sketch the graph of the function $f = f(y)$. This is done on 5. The interpretation is clear: initial conditions that start in the interval $(0, K)$ increase and approach $y_{c2} = K$. If the population occupies initially a niche than the environment can provide it will decrease. Having all of this in mind we can also sketch the temporal evolution $y = y(t)$. Having investigated the convexity we can even draw the overall shape of the curve rather accurately. Note the clearly visible inflection point and initial exponential dynamics for $y_0 \in (0, K/2)$.

Finally, as an exercise in separable equation we can easily solve (2.45) with $y(0) = y_0$ to obtain

$$y(t) = \frac{1}{\frac{1}{K} + \left(\frac{1}{y_0} - \frac{1}{K}\right)e^{-rt}}, \tag{2.50}$$

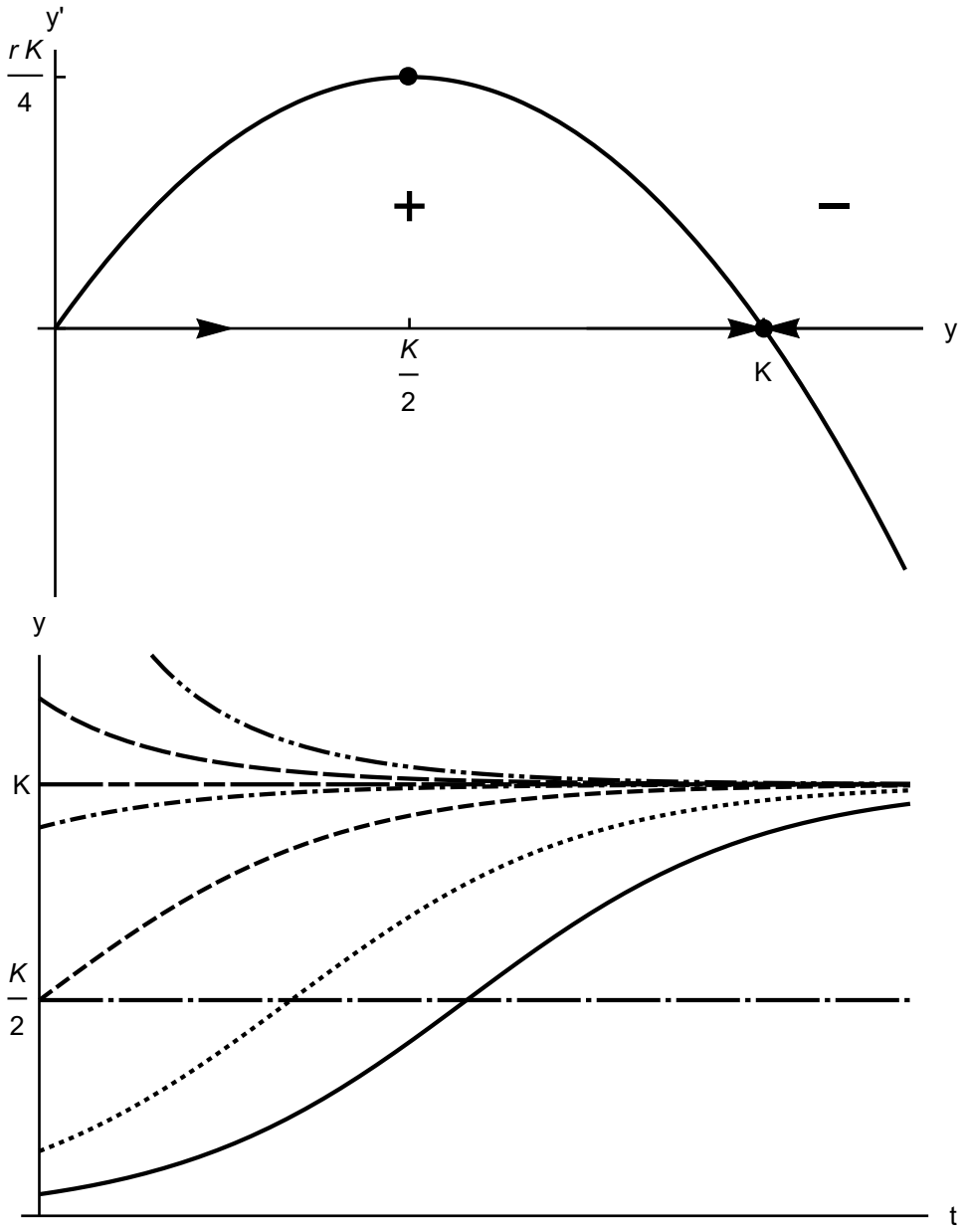

Figure 5: A geometrical visualisation of the dynamics of logistic equation (2.45) (top) and the temporal evolution for different initial conditions (bottom).

which is known as the *logistic curve*[16] and it has an enormous field of applications in: ecology (population dynamics), medicine (growth of tumours, modelling pandemic), chemistry (reaction models), agriculture (crop growth), fishery, economy, and others. Essentially everything that grows with limited supplies can be well-modelled with logistic model as a first approximation. The exact solution (2.50) behaves exactly as we anticipated by our geometric reasoning (see Fig. 5). For instance, it is easy to compute the limit for $t \to \infty$ of (2.50) and obtain either 0 (for $y_0 \leq 0$ (unphysical)) or K (for $y_0 > 0$).

Example. (*Pandemic modelling*)[17] We will fit the logistic curve to the data representing daily new cases of COVID-19 in Hubei, China from 22 January to 31 March 2020[18]. Note that this is a very naïve approach since the model does not assume anything about pandemic dynamics, characteristics of displease spreading or safety measures. However, it serves as a nice illustration of ubiquitous nature of the logistic curve and, since China immediately applied quarantine, might be of some forecast value. The virus initially attacked and spread exponentially infecting many people. However, thanks to the rigorous quarantine it started to slow down due to a lack of access to new susceptible individuals. The data is presented on Fig. 6. We clearly see the logistic S-shape of the curve.

There are many ways of finding the correct parameters of our model (2.50), that is y_0, K, and r, in order to provide a good fit with the data. Remember that in reality we do not hope that the curve will intersect every data point and, hence, we are left with an overdetermined system of equations: three parameters of the logistic model versus many more data points. A simple approach would be to choose y_0 to be the first while K to be the last value

$$y_0 = 444, \quad K = 67801. \tag{2.51}$$

The growth rate can now be found by solving an equation $y(t_1) = y_1$ where (t_1, y_1) is an arbitrarily chosen data point

$$y(t_1) = y_1 \quad \to \quad r = \frac{1}{t} \ln \frac{(K - y_0)y_1}{(K - y_1)y_0}, \tag{2.52}$$

For example, we may choose the middle of February $t_1 = 11$ and $y_1 = 7153$ to obtain

$$r = 0.26 \text{ day}^{-1}. \tag{2.53}$$

Although, this approach is simple and very quick it has a large space for arbitrariness. In practice we use some more advanced fitting algorithms such as *nonlinear least squares*. In simplest terms, the method chooses the unknown parameters such that the sum of squares of errors between the model and the data is the smallest (hence the name).

[16] It is not completely clear why Verhulst chose the name "logistic". Presumably, in order to make a contrast with Malthusian exponential model. The latter historically was called "logarithmic".

[17] Based on M. Bahrami and B. Wood, *Logistic Growth Model for COVID-19*, Wolfram COVID-19 Data and Resources

[18] Some may say that we are doing "Data Science" and there are people who claim to be data scientists having only fitted some simple models to the data. Be aware of them - Data Science is much more than logistic curve fitting.

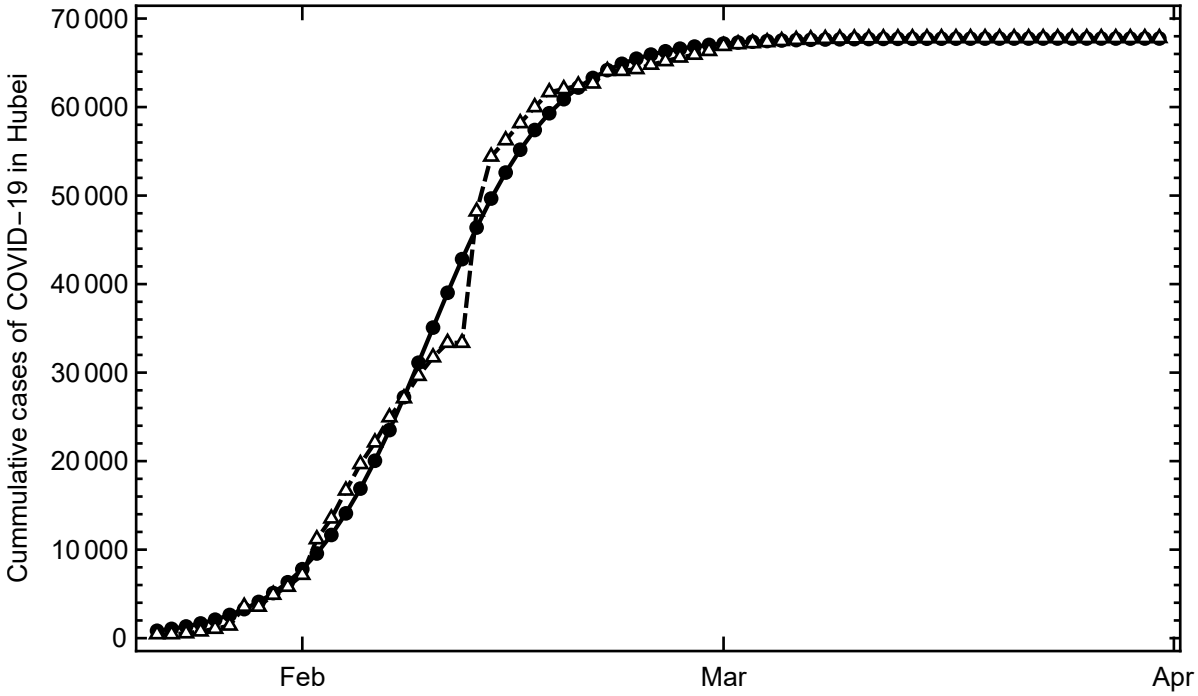

Figure 6: A fitting of the logistic model (2.50) with parameters (2.54) (solid line with circles) to the data of cumulative cases of COVID-19 in Hubei, China (dashed line with triangles).

This makes the fitting unique in the linear model cases and, at least, well-defined in nonlinear ones. Using least squares gives

$$y_0^{lsq} = 659, \quad K^{lsq} = 67743, \quad r^{lsq} = 0.234, \tag{2.54}$$

which is close to our simple approach. The least-squares model fit is presented on Fig. 6. Notice the very good accuracy with a relative error of 3%. □

Example. (*Logistic growth with a threshold*) There are situation where either a population or a disturbance in the climate system will not be allowed to grow until a certain *threshold* $0 < T < K$ is met. For example, the number of individuals in a population can be too small to gather sufficiently many supplies in order to sustain a thriving herd and stand up to predators more efficiently. This phenomenon is called *Allee effect* after Americal ecologist Warder Allee who observed that goldfish grow more rapidly when their number is large. Since then, they have more chances to survive due to cooperation.

To model the Allee effect we have to include some asymmetry in the logistic model. The simplest choice is to modify the growth rate once again by a linear term

$$y' = ry\left(1 - \frac{y}{K}\right)\left(\frac{y}{T} - 1\right) = f(y). \tag{2.55}$$

Note that the last parenthesis is responsible for slowing down the growth for $y < T$ and promoting it for $y > T$. There are three critical points: y_{c1}, y_{c2} as in the logistic model, and the new one

$$y_{c3} = T. \tag{2.56}$$

Their stability is now changed

$$f'(0) = -r < 0, \quad f'(K) = r\left(1 - \frac{K}{T}\right) < 0, \quad f'(T) = r\left(1 - \frac{T}{K}\right) > 0. \tag{2.57}$$

Therefore, the zero stationary point acquired stability while the new critical point $y_{c3} = T$ is unstable. The solution will thus move away from the threshold either to the complete annihilation of the population or to the carrying capacity. This is depicted on the $y'-y$ graph (Fig. 7). There is a clear asymmetry which is the most important feature of Allee effect - the growth is more rapid for large population numbers. Moreover, we see that the solution will increase only when $y \in (T, K)$ and decrease otherwise. The calculation of the inflection points is leaf as an exercise.

This model is not only useful for modelling but also for educative purposes. As opposed to the logistic model, here we cannot obtain an exact for of the solution in the form $y = y(t)$ but rather $t = t(y)$

$$rt(y) + C = \frac{1}{r(K-T)}\left(-(K-T)\ln y + K\ln|y-T| - T\ln(K-y)\right), \tag{2.58}$$

which is much more cluttered and not very much useful. However, we know that much of its behaviour that we may not need it to invert it. □

2.4 Linear equations

Linear phenomena are usually associated with first approximations to the more complex nonlinear ones near some special situations. This, very frequently, helps to understand the very structure of the problem. In linearisation we usually assume that a certain quantity changes only in a small fraction of the whole. For example, acoustics is a linearisation of the Euler equations of fluid dynamics where the pressure changes caused by an acoustic wave are small compared with the ambient pressure. Linear theory of acoustics is extremely accurate.

The most profound practical aspect of linearity of a given problem is the superposition principle - we can combine various solutions in order to obtain more complex one. This has a fundamental meaning in PDEs, linear wave theory, and quantum mechanics since then we can split a difficult problem into an infinity of easy tasks, solve them, and form the solution of the original problem. Here, we will focus on linear ODEs leaving further developments to the PDE course.

A linear differential equation has been defined in (1.4) as a ODE in the form $Ly = f$ where f is given and L is a linear operator. For the first order case we can write a general linear equation in the form

$$y' + p(t)y = q(t), \quad y(0) = y_0, \tag{2.59}$$

where p and q are assumed to be given continuous functions. These equations arise very frequently in many applications and before we tackle a derivation a general solution we will see two examples.

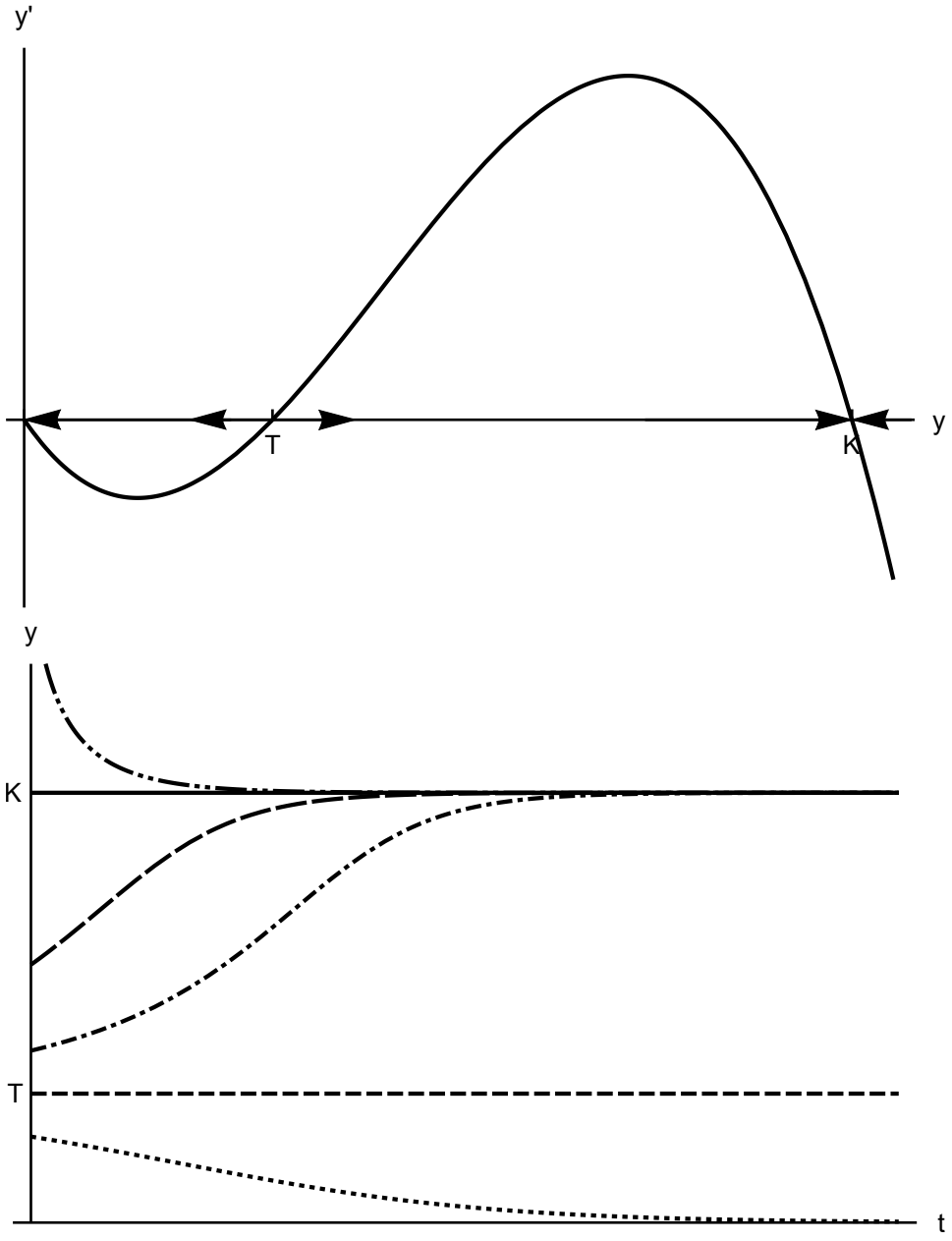

Figure 7: The geometrical representation of the dynamics of the threshold model (2.55) with Allee effect (top) and temporal evolution of $y = y(t)$ for different initial conditions (bottom).

Example. (*Newton's Law of Cooling*) Suppose we are working with CSI and have a task of determining victims death. One of the methods that is used is to take advantage of physics. *Newton's Law of Cooling* states that a body looses its heat proportional to the difference of its temperature and the surroundings. If $T = T(t)$ is the temperature of the body and $A = A(t)$ is a given ambient temperature, this law can be written as

$$T' = -k(T - A(t)), \tag{2.60}$$

where $k > 0$ is a proportionality constant dependent on the specific heat of the body and its density. The minus sign is conventional because usually the body cools down in a colder environment. The equation is linear and can be written is a standard form $T' + kT = kA(t)$.

We can use (2.60) to determine the time of death by measuring the temperature of the corpse at two points, say t_0 and t_1. Then, as we know, the general solution of (2.60) will be a two parameter family of curves with parameters k and the integration constant. We can determine these by our two measurements T_0 and T_1. Finally, we can trace the curve back to a time of death t_d, i.e. when $T(t_d) = 36.6°C$ - when the victim was just killed. This strategy is very useful and is a basis of the actual techniques used by CSI. □

Example. (*Mixing*) Assume that at $t = 0$ a tank with a volume V contains Q_0 of salt. Further, there is a influx of $q(t)$ kilograms per litre of salt that flows inside the tank with a rate $r(t)$ litres per minute. Inside the tank, the salt is being mixed to produce a brine. Finally, the whole solution exists the container at a rate $r(t)$. Find the amount of salt inside the tank at time $t > 0$.

This is a typical problem in chemical engineering. In order to solve it we have to write a mass balance in terms of an ODE. This is needed since every parameter here changes continuously with time. If $Q = Q(t)$ is the amount of salt at time $t > 0$ we have

$$Q' = \text{influx} - \text{outflux}, \tag{2.61}$$

that is, the change in the mass of the salt is equal to the mass that enters the container minus the amount leaving it. We know that a concentration of $q(t)$ of salt per litre of brine enters the tank at a rate $r(t)$ litres per minute, hence

$$\text{inlfux} = q(t)r(t). \tag{2.62}$$

Further, the whole solution is being mixed and leaves the container. However, the concentration of salt is now $Q(t)/V$ and therefore,

$$\text{outflux} = \frac{Q}{V}r(t). \tag{2.63}$$

Our mass balance equation can now be written in the form

$$Q' + \frac{r(t)}{V}Q = q(t)r(t), \tag{2.64}$$

which is a linear ODE. □

Having learned about some real-world examples of linear equations we would like to be able to find their solution. Note that, in general, they are not autonomous. However, there are at least two methods that can always yield an exact solution to any linear first order ODE: integrating factor and variation of parameters. Here, we will describe only the former and later return to the latter when studying higher order equations.

Usually applying the integrating factor quickly leads to a solution. The idea is to transform (2.59) into a form that can be integrated

$$(\mu(t)y)' = Q(t), \tag{2.65}$$

for some new functions μ and Q. We see that the above yields an exact form of the solution after integration. The most important fact is that we can *always* transform a general linear first order ODE into this special form. To see this, fix an arbitrary function $\mu = \mu(t)$ and multiply (2.59) by it

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)q(t). \tag{2.66}$$

Now, we would like to choose μ such that the above is a derivative of a product with y

$$(\mu(t)y)' - \mu'(t)y + \mu(t)p(t)y = \mu(t)q(t). \tag{2.67}$$

We see that this can be achieved when the excess vanishes, that is μ is a solution of the separable ODE

$$\mu' = p(t)\mu(t), \tag{2.68}$$

with a solution

$$\mu(t) = e^{\int_0^t p(s)ds} \tag{2.69}$$

which is called the *integrating factor*. We can always take the integration constant in the above formula to be equal to 0 since otherwise we would obtain an integrating factor $C\mu(t)$. After multiplication with linear ODE the constant C would cancel. This is equivalent to integrating from 0 to t. After integrating (2.67) with μ chosen according to (2.69) we obtain the following result.

Theorem 1 (Leibniz)**.** *The general solution of the linear first order ODE (2.59) is given by the formula*

$$y(t) = e^{-\int_0^t p(s)ds}\left(y_0 + \int_0^t q(s)e^{\int_0^s p(u)du}ds\right). \tag{2.70}$$

Therefore, a linear initial value problem always has a solution.

Integrating factor
When solving linear ODEs one usually does not need to remember the general formula for an exact solution (2.70). Instead, it is only needed to do the following steps.

- Multiply the linear first order ODE by $\mu(t)$ given by the formula (2.69).
- Identify the derivative of a product.
- Integrate both sides of the ODE.

Remember that it is *easiest* to use the integrating factor when the equation is in the standard form (2.59). Many students fail to do so and incorrectly identify $p(t)$ and $q(t)$.

We now can return to our examples.

Example. (*Newton's Law of Cooling*) The ODE derived previously has the form

$$T' + kT = kA(t), \tag{2.71}$$

with $A(t)$ and k given. The integrating factor is

$$\mu(t) = e^{\int_0^t k ds} = e^{kt}, \tag{2.72}$$

which transforms the ODE into

$$e^{kt}T' + e^{kt}kT = ke^{kt}A(t), \tag{2.73}$$

where we immediately recognize the derivative of a product

$$(e^{kt}T)' = ke^{kt}A(t). \tag{2.74}$$

After integration we have

$$T(t) = e^{-kt}\left(T_0 + k\int_0^t e^{ks}A(s)ds\right). \tag{2.75}$$

This is a closed formula for an exact solution of our problem. However, to illustrate the matters more vividly we assume that the ambient temperature is constant[19]. Then, we can carry out the integral

$$T(t) = e^{-kt}\left(T_0 + kA\int_0^t e^{ks}ds\right) = A + (T_0 - A)e^{-kt}. \tag{2.76}$$

Note that $T(t) \to A$ for large times. Now, if we measure the temperature at some second time $t_1 > 0$ we can determine k by solving $T(t_1) = T_1$ resulting in

$$k = \frac{1}{t_1}\ln\frac{T_0 - A}{T_1 - A}. \tag{2.77}$$

[19] A good model for daily change of the temperature is a trigonometric function.

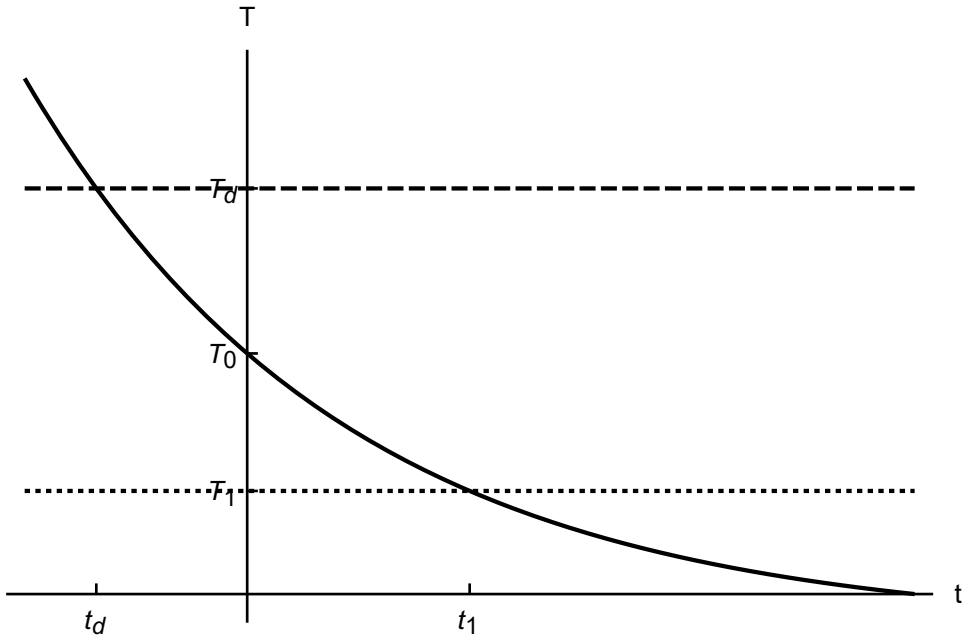

Figure 8: A graphical representation of the determination of the time of death.

Therefore, our model is calibrated. Now, we are able to follow the curve back to the time of death $T(t_d) = T_d = 36.6°C$

$$T_d = A + (T_0 - A)e^{-kt_d} \quad \rightarrow \quad t_d = -t_1 \frac{\ln \frac{T_d - A}{T_0 - A}}{\ln \frac{T_0 - A}{T_1 - A}}. \tag{2.78}$$

Notice that correctly, $t_d < 0$ since the death occurred in the past by our choice of coordinates. The procedure is illustrated on Fig. 8. Note also that finding a past value of a fitted curve is usually consider ill-posed and in real situations some more sophisticated algorithms are necessary to obtain a viable result. □

Example. (*Mixing*) For the mixing equation we have

$$Q' + \frac{r(t)}{V} Q = q(t)r(t), \tag{2.79}$$

for which the integrating factor (2.69) is

$$\mu(t) = e^{\frac{1}{V}\int_0^t r(s)ds}. \tag{2.80}$$

After multiplication, the ODE can be written as

$$(\mu(t)Q)' = \mu(t)q(t)r(t), \tag{2.81}$$

which gives

$$Q(t) = e^{-\frac{1}{V}\int_0^t r(s)ds} \left(Q_0 + \int_0^t e^{\frac{1}{V}\int_0^s r(u)du} q(s)r(s)ds \right). \tag{2.82}$$

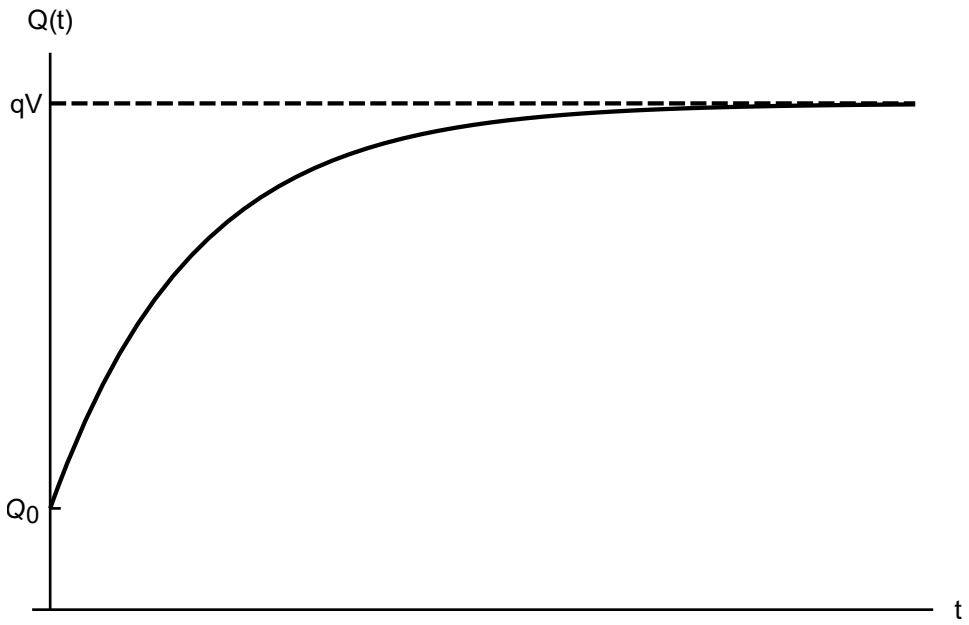

Figure 9: Salt amount versus time.

For simplicity, if q and r are constant we have

$$Q(t) = qV - (qV - Q_0)\, e^{-\frac{q}{V}t}, \tag{2.83}$$

and the total mass of salt approaches steady-state $Q_s = qV$ for large times (Fig. 9). □

In the sequel we will see how linearity plays the main role in forming the structure of higher order equations.

2.5 Other integrable equations (optional)

In this section we will briefly describe some other integrable classes of equations. Maybe they are not as frequent as separable or linear, but can appear in many important situations.

2.5.1 Bernoulli equations

Bernoulli equations are a class of nonlinear ODEs that can be transformed into a linear ones with a suitable change of variables. They have the form

$$y' + p(t)y = q(t)y^n, \quad n > 0, \tag{2.84}$$

and have been investigate by Jakob Bernoulli at the end of seventeenth century. Of course, when $n = 0$ or $n = 1$ the equation is linear and hence we assume otherwise. Bernoulli found that when we divide by y^n we obtain

$$y^{-n}y' + p(t)y^{1-n} = q(t), \tag{2.85}$$

which can be simplified by a change of the dependent variable

$$u = y^{1-n}. \tag{2.86}$$

This is because $u' = (1-n)y^{-n}y'$ which is the leading term. Therefore we obtain a linear equation for u

$$u' + (1-n)^{-1}p(t)u = (1-n)^{-1}q(t). \tag{2.87}$$

We can now use the integrating factor to solve it and then return to the substitution $y = u^{1/(1-n)}$.

Example. (*Logistic growth with variable capacity*) An important example of emergence of Bernoulli equation is a population that grows according to the logistic law where the carrying capacity vary with time, that is $K = K(t)$. This situation can arise when, for example, winter conditions are harsh enough to significantly reduce the amount of food or there is an introduction of predation. We have

$$y' = ry\left(1 - \frac{y}{K(t)}\right), \tag{2.88}$$

which can be written as

$$y' - ry = -\frac{r}{K(t)}y^2. \tag{2.89}$$

We identify that $n = 2$ and substitute

$$u = \frac{1}{y}, \tag{2.90}$$

to obtain

$$u' - ru = -\frac{r}{K(t)}. \tag{2.91}$$

The integrating factor is just $\mu(t) = \exp(-rt)$ and, hence,

$$u(t) = e^{-rt}\left(\frac{1}{y_0} - r\int_0^t \frac{e^{rs}}{K(s)}ds\right), \tag{2.92}$$

and after returning to substitution we obtain

$$y(t) = \frac{1}{e^{-rt}\left(\frac{1}{y_0} - r\int_0^t \frac{e^{rs}}{K(s)}ds\right)}. \tag{2.93}$$

Note that for constant K we obtain the logistic growth. □

2.5.2 Homogeneous equations

Another interesting class of first order equations are the so-called *homogenous* ones. Note, however, that the word "homogeneous" have different meaning in different situations. Here, a function $f = f(t, y)$ is homogeneous of degree n if $f(\lambda t, \lambda y) = \lambda^n f(t, y)$ for every $\lambda \in \mathbb{R}$. Now, a homogeneous differential equation is

$$y' = \frac{f(t, y)}{g(t, y)}, \tag{2.94}$$

where f and g are homogeneous of the same degree. This means that if we substitute

$$y = tu, \tag{2.95}$$

we obtain

$$u + tu' = \frac{f(t, tu)}{g(t, tu)} = \frac{f(1, u)}{g(1, u)}, \tag{2.96}$$

where we have used the homogeneity of f and g. Therefore, defining $h(u) = f(1, u)/g(1, u)$ we obtain a separable equation

$$tu' = h(u) - u, \tag{2.97}$$

which has an integral

$$\ln t + C = \int \frac{du}{h(u) - u}, \tag{2.98}$$

and the solution depends on the specific form of h.

The above class can also be characterized in a different way. From the homogeneity of f and g we have

$$\frac{f(\lambda t, \lambda y)}{g(\lambda t, \lambda y)} = \frac{f(t, y)}{g(t, y)}, \tag{2.99}$$

thus putting $\lambda = 1/t$ yields

$$\frac{f(t, y)}{g(t, y)} = \frac{f(1, \frac{y}{t})}{g(1, \frac{y}{t})} = h\left(\frac{y}{t}\right). \tag{2.100}$$

Therefore, a homogeneous equation is an ODE in which the right-hand side can be expressed as a function of the ratio y/t. This is the usual definition given in most textbooks.

Example. We solve

$$y' = \frac{t^2 + ty + y^2}{t^2}, \tag{2.101}$$

by noticing that after division by t^2 we obtain

$$y' = 1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2. \tag{2.102}$$

The substitution $y = tu$ leads to

$$u + tu' = 1 + u + u^2, \tag{2.103}$$

which is separable with

$$\ln t + C = \int \frac{du}{1+u^2} = \arctan u. \tag{2.104}$$

After going back to the original variable we obtain

$$y(t) = t \tan(\ln t + C), \tag{2.105}$$

and the integration constant can be determined from suitable conditions. □

2.5.3 Exact equations

Work in progress.

2.6 Real-world examples

Here we collect some additional real-world examples of first order ODEs. The list can, of course, be continued indefinitely.

Example. (*Hydrostatic pressure*) Assume that a medium filled with a certain fluid of density ρ is in the hydrostatic balance. Consider a small cylinder of cross section A and height Δz. Since we have a force balance, the difference in the pressure force on it top and bottom has to be balanced by gravity

$$A(p(z+\Delta z) - p(z)) = -g\rho A \Delta z, \tag{2.106}$$

where $A\Delta z$ is the volume of the cylinder. Dividing by Δz and passing to the limit we obtain the hydrostatic balance equation

$$\frac{dp}{dz} = -\rho g. \tag{2.107}$$

For example, when the density is constant, we obtain the well-known law $p = p_0 - \rho g z$.

However, in the atmosphere the density falls with height and cannot be assumed constant. Air can be assumed to be a ideal fluid, hence

$$p = \rho R T, \tag{2.108}$$

where R is a constant while T the temperature. Further, T also decreases with height and in troposphere (the lowest and most important for people layer of the atmosphere) this relation is almost linear

$$T(z) = T_0 - \Gamma z, \tag{2.109}$$

where Γ is called the lapse rate. Hence,

$$\rho(z) = \frac{1}{R} \frac{p(z)}{T_0 - \Gamma z}, \tag{2.110}$$

which can be plugged into the hydrostatic equation

$$\frac{dp}{dz} = -\frac{g}{R}\frac{p}{T_0 - \Gamma z}. \tag{2.111}$$

This is a separable equation with a solution

$$p(z) = p_0 \left(1 - \frac{\Gamma}{T_0} z\right)^{-\frac{g}{R\Gamma}}, \tag{2.112}$$

where p_0 is the atmospheric pressure at the surface. This accurate equation for the pressure distribution in troposphere is frequently used as a mean of calibrating altimeters. □

Example. (*Falling raindrop*) One of the mechanisms of rain is based on falling raindrops that fall through a mist and accumulate mass. Let the mass of the raindrop be m, density ρ, radius r, and downward velocity v. Then, from the momentum generated by the gravity

$$\frac{d}{dt}(m(t)v(t)) = m(t)g. \tag{2.113}$$

Moreover, assuming spherical droplets we have

$$m(t) = \frac{4}{3}\pi\rho r(t)^3. \tag{2.114}$$

Now, as the raindrop falls through a medium of water density ρ_0 it gains mass. A sensible approach would be to assume that the rate of mass collection is proportional to the ρ_0 (more dense medium gives more mass), cross section of the droplet (more area swept), and its velocity (faster accumulation)

$$\frac{dm}{dt} = \pi r(t)^2 v(t)\rho_0. \tag{2.115}$$

We neglect the air resistance, although in more realistic models it should be included.

Now, from (2.114) we can compute the radius and plug it in (2.115) to have

$$\frac{dm}{dt} = \alpha m^{\frac{2}{3}} v, \quad \alpha = \rho_0 \pi \left(\frac{3}{4}\frac{1}{\pi\rho}\right)^{\frac{2}{3}}. \tag{2.116}$$

Therefore the above with (2.113) constitute a closed system of two first order equations for $m = m(t)$ and $v = v(t)$. We can solve it by noticing that there is no independent variable involved in the equations and hence, we can use the chain rule to write

$$\frac{dv}{dt} = \frac{dv}{dm}\frac{dm}{dt}, \tag{2.117}$$

which, when plugged into (2.113), yields

$$v\frac{dv}{dm} + \frac{v^2}{m} = \frac{g}{\alpha}\frac{1}{m^{\frac{2}{3}}}, \tag{2.118}$$

where now our independent variable is the mass. This equation is very similar to Bernoulli ODE and we can simplify it by substituting $u = v^2$. This gives a linear equation

$$\frac{du}{dm} + \frac{2}{m}u = \frac{2g}{\alpha}\frac{1}{m^{\frac{2}{3}}}. \tag{2.119}$$

The integrating factor is $\mu(m) = m^2$, thus

$$\frac{d}{dm}\left(m^2 u\right) = \frac{2g}{\alpha}m^{\frac{4}{3}}. \tag{2.120}$$

Which along with condition $u(0) = 0$ gives

$$v = \beta m^{\frac{1}{6}}, \quad \beta = \left(\frac{6}{5}\frac{g}{\alpha}\right)^{\frac{1}{2}}, \tag{2.121}$$

that is, the velocity is proportional to $m^{1/6}$. We can now go back to (2.113) to obtain

$$\frac{dv}{dt} = g - \frac{1}{m}\frac{dm}{dt}v = g - \frac{\alpha m^{\frac{2}{3}}}{m}\beta^2 m^{\frac{2}{3}}. \tag{2.122}$$

Therefore, m cancels out and we are left with

$$\frac{dv}{dt} = g - \alpha\beta^2 = \frac{g}{7}, \tag{2.123}$$

and we have obtained an important result that the raindrop will fall with constant acceleration equal to $g/7$. The mass will thus increase as t^6. □

Example. (*Hourglass*) We will determine the optimal shape of a hourglass. Imagine an axisymmetric container with cross-sectional area at a height y equal to $A(y)$. The container is filled with water of height $h(t)$ that flows out through a small opening of area a at the bottom $y = 0$.

According to Torricelli Law (which follows from conservation of energy) it follows that the velocity of escaping water is equal to

$$v_a(h) = \sqrt{2gh}. \tag{2.124}$$

Next, if $V(h)$ is the volume of water at height h we have

$$\frac{d}{dt}V(h) = -av_w(h), \tag{2.125}$$

that is, in a small period of time Δt the volume of water flowing out of the container is equal to $av_w(h)\Delta t$. Now, since $V(h) = \int_0^h A(y)dy$ we have

$$A(h)\frac{dh}{dt} = -a\sqrt{2gh}. \tag{2.126}$$

The necessary property of a hourglass is the constant water level velocity, that is we require that $dh/dt = -C$=const. Then, we can measure time according to the height of water (or sand). This gives

$$A(h) = \frac{a}{C}\sqrt{2gh}, \tag{2.127}$$

but the cross-sectional area is equal to $A(h) = \pi f(h)^2$ where the graph of f denotes the shape of the container. For example, $f(h) = 1$ for a cylinder, $f(h) = h$ for a cone or $f(h) = \sqrt{1 - h^2}$ for sphere. Therefore,

$$f(h) = Dh^{\frac{1}{4}}, \tag{2.128}$$

where D is a new constant. Whence, we have shown that the shape of an hourglass should be proportional to $h^{\frac{1}{4}}$. □

2.7 Existence and uniqueness (optional)

A main objective of any theoretical study of differential equations is to ascertain whether a given problem has a unique solution. This can have very important consequences both in pure and applied mathematics. For example, when a model of a physical phenomenon gives us two solutions we should somehow be able to choose the one representing the actual situation or to rethink the model. Therefore, it is crucial to understand the conditions under which a general ODE, that might not be exactly solvable, possesses a unique solution.

We will probe that

$$y'(t) = f(t, y(t)), \quad y(0) = 0, \tag{2.129}$$

has a precisely one solution defined on the neighbourhood of $t = 0$. The zero initial condition can be prescribed without any loss of generality. Note that we have to prove our claim without the possibility of presenting an exact solution. This is the main difficulty of analysis of ODEs. The main and classical result is the following.

Theorem 2 (Picard-Lindelöf)**.** *Assume that f is continuous and has a continuous derivative $\partial f/\partial y$ on a rectangle $[-a, a] \times [-b, b]$. Then, there is a number $\epsilon > 0$, such that for $|t| \leq \epsilon \leq a$ there exists exactly one solution of (2.129)*[20].

Proof. First, we will prove that the solution exists. The proof technique is known as the Picard iteration and more generally, as the *Banach contraction theorem*. It has many applications not limited to the theory of differential equations. Note that upon integrating the ODE we can write it as

$$y(t) = \int_0^t f(s, y(s))ds. \tag{2.130}$$

Therefore, our function y satisfies an integral equation. The idea is to define a sequence of functions $y_n(t)$ that satisfy

$$y_{n+1}(t) := \int_0^t f(s, y_n(s))ds, \quad n \geq 0. \tag{2.131}$$

[20]In other words: there exists exactly one curve $y = y(t)$ defined for sufficiently small t that passes through $(0, 0)$.

We can also put $y_0(t) \equiv 0$. If we assume that such defined sequence is convergent (uniformly on $[-a, a]$) to some function $v(t)$, then

$$v(t) = \lim_{n\to\infty} y_{n+1}(t) = \lim_{n\to\infty} \int_0^t f(s, y_n(s))ds = \int_0^t f(s, \lim_{n\to\infty} y_n(s))ds = \int_0^t f(s, v(s))ds, \tag{2.132}$$

where we used the continuity of f when taking the limit under an integral sign. Therefore, $v(t)$ is a solution of (2.130), and hence of (2.129). We must, therefore, show that our sequence (2.131) is convergent. In this way we will construct the solution.

Since f is continuous over a closed and bounded set $[-a, a] \times [-b, b]$ it is bounded itself. Therefore, there exists a constant $M > 0$ such that

$$|f(t, y)| \leq M, \quad (t, y) \in [-a, a] \times [-b, b]. \tag{2.133}$$

Similarly, we infer that there is a constant $L > 0$ such that

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \leq L, \quad (t, y) \in [-a, a] \times [-b, b], \tag{2.134}$$

which implies Lipschitz continuity of f. To see this, let $u, v \in [-b, b]$ and $t \in [-a, a]$. Then, from the Lagrange mean value theorem there exists $\zeta = \zeta(t)$ for which we have

$$|f(t, u) - f(t, v)| = \left|\frac{\partial f}{\partial y}(t, \zeta(t))\right| |u - v| \leq L|u - v|. \tag{2.135}$$

We are now in position to prove the convergence of $y_n(t)$.

First, we have to ascertain that all terms of (2.131) are well-defined. It means that we have to have $y_n(t) \in [-b, b]$ for each $n \geq 0$ and $t \in [-a, a]$. We need that because $y_n(t)$ is the argument of f continuously defined on $[-a, a] \times [-b, b]$. If $y_n(t)$ was not inside $[-b, b]$, we would not be able to use the continuity assumption. We thus have

$$|y_{n+1}(t)| = \left|\int_0^t f(s, y_n(s))ds\right| \leq \int_0^{|t|} |f(s, y_n(s))|\, ds \leq M \int_0^{|t|} ds = M|t| \leq b, \tag{2.136}$$

where we used the boundedness of f (2.133). In order the above to be satisfied we should take $|t| \leq \epsilon$ for

$$\epsilon := \min\left\{a, \frac{b}{M}\right\}. \tag{2.137}$$

Now, observe that

$$y_{n+1}(t) = y_0(t) + (y_1(t) - y_0(t)) + (y_2(t) - y_1(t)) + ... + (y_{n+1}(t) - y_n(t)) = \sum_{i=0}^{n} (y_{i+1}(t) - y_i(t)), \tag{2.138}$$

since $y_0(t) \equiv 0$. We would like to show that the above series converges uniformly which will imply the continuity of the limit. We have,

$$|y_2(t) - y_1(t)| \leq \int_0^{|t|} |f(s, y_1(s)) - f(s, y_0(s))|\, ds \leq M|t|, \tag{2.139}$$

and further

$$\begin{aligned} |y_3(t) - y_2(t)| &\leq \int_0^{|t|} |f(s, y_2(s)) - f(s, y_1(s))| \, ds \\ &\leq L \int_0^{|t|} |y_2(s) - y_1(s)| \, ds \leq ML \int_0^{|t|} s ds = ML \frac{|t|^2}{2}, \end{aligned} \tag{2.140}$$

where we have used the previous inequality and (2.135). Inductively we show that

$$|y_{i+1}(t) - y_i(t)| \leq ML^{i-1} \frac{|t|^i}{i!}, \quad i \geq 0. \tag{2.141}$$

Hence,

$$\sum_{i=0}^{n} |y_{i+1}(t) - y_i(t)| \leq \frac{M}{L} \sum_{i=0}^{n} \frac{(L|t|)^i}{i!} \leq \frac{M}{L} \sum_{i=0}^{\infty} \frac{(L\epsilon)^i}{i!} = \frac{M}{L} e^{L\epsilon}. \tag{2.142}$$

Therefore, the series $\sum_{i=0}^{\infty} (y_{i+1}(t) - y_i(t))$ is uniformly convergent for $[-\epsilon, \epsilon]$ which implies that its sum is continuous (since all y_i are continuous). We can now define

$$y(t) := \sum_{i=0}^{\infty} (y_{i+1}(t) - y_i(t)), \tag{2.143}$$

which, thanks to (2.138), is the uniform limit of $y_n(t)$. From (2.132) we see that y is the solution of (2.130). As y is defined as a integral of a continuous function, it is differentiable and, hence, a solution of (2.129).

Having proved the existence we can proceed to uniqueness of the solutions to (2.130).To this end, assume that we have two solutions $u = u(t)$ i $v = v(t)$, satisfying (2.130). Let $t > 0$, we then have

$$|u(t) - v(t)| \leq \int_0^t |f(s, u(s)) - f(s, v(s))| \, ds \leq L \int_0^t |u(s) - v(s)| \, ds, \tag{2.144}$$

where we have used (2.135). Now, put

$$g(t) := \int_0^t |u(s) - v(s)| \, ds. \tag{2.145}$$

Immediately we see that $g(0) = 0$, $g(t) \geq 0$ and $g'(t) = |u(t) - v(t)|$. Thanks to (2.144) we can thus write

$$g'(t) - Lg(t) \leq 0. \tag{2.146}$$

The left-hand side is similar to a linear first order ODE and we will try to resolve it by integrating factor. After multiplication by e^{-Lt} we obtain

$$\left(e^{-Lt} g(t)\right)' \leq 0. \tag{2.147}$$

The above inequality can be integrated over $[0, t]$ to obtain

$$0 \geq \int_0^t \left(e^{-Ls} g(s)\right)' ds = e^{-Lt} g(t) - e^0 g(0) = e^{-Lt} g(t). \tag{2.148}$$

The expression on the right-hand side is non-negative. The only possibility for the above inequality to be true is $g(t) = 0$ for all $t \geq 0$. Whence, $0 = g'(t) = |u(t) - v(t)|$. Therefore, we must have $u(t) = v(t)$ for all $t \geq 0$, that is, there is only one solution to the equation (2.130). The proof of uniqueness for $t < 0$ is exactly the same [21]. □

From the proof of Picard-Lindelöf Theorem it follows that the solution of (2.129) exists at least *locally* (for $|t| \leq \epsilon$) but is unique *globally* (for $t \in \mathbb{R}$).

Example. (*Nonuniqueness*) The standard example of a nonunique solution of the ODE is

$$y' = y^{\frac{1}{2}}, \quad y(0) = 0. \tag{2.149}$$

Notice the right-hand side does not satisfy the assumption of Picard-Lindelöf Theorem since $y^{1/2}$ does not have a continuous derivative. One solution is of course $y \equiv 0$ and the other, by separation of variables,

$$y(t) = \frac{1}{4}t^2, \quad t \geq 0. \tag{2.150}$$

Therefore, the assumptions of the theorem are essential. □

Example. (*Successive approximations*) The method of Picard's iterations can be very useful in practice when we would like to obtain an approximate solution of our equation. Especially when an equation has a small parameter multiplying a term that makes the equation not solvable explicitly. This even works for algebraic equations. For example, suppose we have to study the following problem

$$y' = -y + \epsilon e^{-y}, \tag{2.151}$$

where ϵ is considered small. From our knowledge on autonomous equations we can show that there exists a stable stationary point y_c being a solution of the following algebraic equation

$$x = \epsilon e^{-x}. \tag{2.152}$$

Although we can solve this numerically for a given ϵ we would like to find some dependence of the solution to the parameter. We thus form the Picard's iteration sequence

$$x_{n+1} = \epsilon e^{-x_n}. \tag{2.153}$$

Since ϵ is small we can take $x_0 = 0$ and then compute

$$x_1 = \epsilon, \quad x_2 = \epsilon e^{-\epsilon}, \quad x_3 = \epsilon e^{-\epsilon e^{-\epsilon}} = \epsilon - \epsilon^2 + \frac{3}{2}\epsilon^3 + ..., \tag{2.154}$$

and so forth. For example, using the Taylor series for x_3 as an approximation we get a relative error of 0.2% for $\epsilon = 0.1$. The overall plot of the error is presented on Fig. 10. □

[21] The presented method of proving uniqueness is a fragment of the so-called *Grönwall's Lemma*.

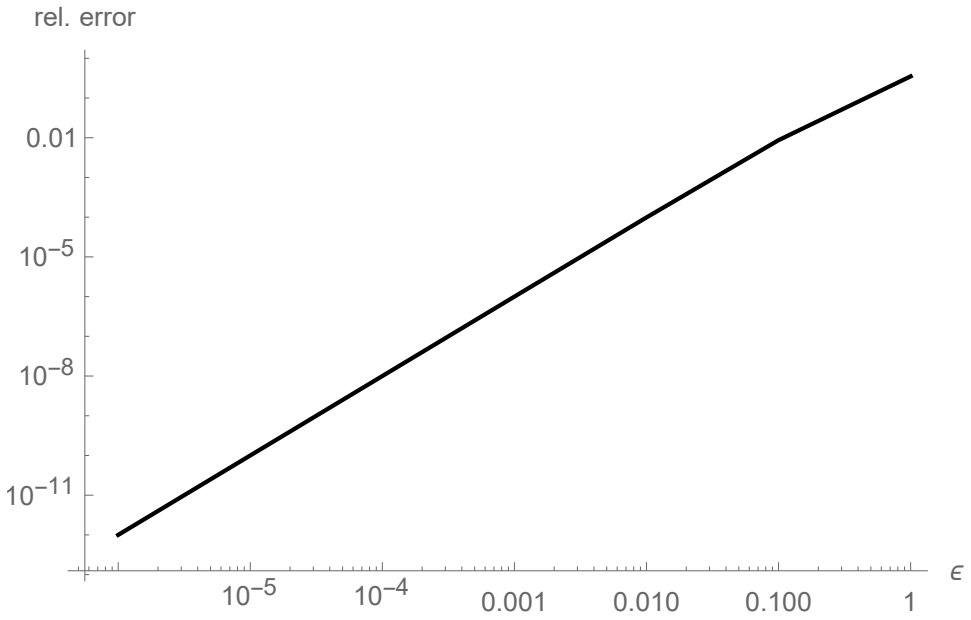

Figure 10: A relative error of approximating the solution of (2.153) with $\epsilon - \epsilon^2 + \frac{3}{2}\epsilon^3$.

3 Finite differences and Euler methods

3.1 Finite differences

Differential equations are constructed with derivatives and in numerical analysis we would like to find a way of approximating them numerically. Since derivatives are limits of difference quotients we have to find a way of representing these limits with a finite precision.

Let $y = y(x)$ be a function of variable. In what follows we will always assume sufficient smoothness of y unless otherwise stated. Then, of course

$$y'(x) = \lim_{h \to 0^+} \frac{y(x+h) - y(x)}{h}, \tag{3.1}$$

is the definition of the derivative. Since the above requires a limit passage we can truncate it to obtain a finite precision approximation in the form of *forward difference*

$$\delta_+ y(x) := \frac{y(x+h) - y(x)}{h}, \tag{3.2}$$

or *backward difference*

$$\delta_- y(x) := \frac{y(x) - y(x-h)}{h}, \tag{3.3}$$

where $h > 0$. We can also take the average of the two preceding operators and form the *centred difference*

$$\delta_0 y(x) := \frac{1}{2}\left(\delta_- y(t) + \delta_+ y(t)\right) = \frac{y(x+h) - y(x-h)}{2h}, \quad h > 0. \tag{3.4}$$

It should be clear that the centred approximation is usually better that either backward or forward difference.

Example. Let $y(x) = \sin x$. We would like to approximate $y'(x) = \cos x$. We have

$$\delta_+ y(x) = \frac{\sin(x+h) - \sin x}{h} = \cos\left(x + \frac{h}{2}\right) \frac{\sin\frac{h}{2}}{\frac{h}{2}} = \cos x + \frac{1}{2}\sin x\, h + O(h^2), \quad (3.5)$$

where we used the Taylor expansion when $h \to 0^+$. We can thus see that $\delta_+ y(x) - \cos x = (\sin x)\frac{h}{2} + O(h^2)$. Similarly, for the centred difference we have

$$\delta_0 y(x) = \frac{\sin(x+h) - \sin(x-h)}{2h} = \cos x - \frac{1}{6}\cos x\, h^2 + O(h^4) \quad \text{as} \quad h \to 0^+. \quad (3.6)$$

We can now see that if $h < 1$ then $h^2 \ll h$ and the error that we make when approximating the derivative with δ_0 should be much smaller than with the δ_+ (apart maybe from points of the form $x = n\pi$).

We see that various approximations of the derivatives are not equivalent and some of them are more accurate than the other. In many places below the error of some method will be proportional to some power of the step h, i.e.

$$E(h) \approx Ch^p, \quad p > 0, \quad (3.7)$$

where $C > 0$ is a constant independent on h. A useful way of visualizing this error during simulations will be to present it in a log-log scale because of the fact that

$$\ln E(h) \approx p \ln h + \ln C. \quad (3.8)$$

Therefore, $\ln E(h)$ is a linear function of $\ln h$ which slope is precisely equal to the *order* of the method p. We will use this tool many times.

The above example has indicated a useful method of obtaining estimates for the error of approximation - the Taylor expansion.

Definition 7. *The **truncation error** of a finite difference operator δ is*

$$\delta y(x) - y'(x). \quad (3.9)$$

When

$$|\delta y(x) - y'(x)| \leq Ch^p, \quad h > 0, \quad p > 0, \quad (3.10)$$

where $C > 0$ is independent of h, the number p is the order *of approximation.*

Of course, in a similar manner we can construct approximations of higher derivatives. Calculating explicitly

$$\delta_+ y(x) - y'(x) = \frac{y(x+h) - y(x)}{h} - y'(x) = \frac{1}{2}hy''(x) + \frac{1}{6}h^2 y'''(x) + O(h^3) \quad \text{as} \quad h \to 0^+, \quad (3.11)$$

and

$$\delta_0 y(x) - y(x) = \frac{1}{6}h^2 y'''(x) + O(h^4) \quad \text{as} \quad h \to 0^+. \tag{3.12}$$

Therefore, we see that δ_+ is of the first, while δ_0 of second order. We will see that centred approximations have larger order than the one-sided due to cancellations, that is the terms in the expansions of $y(x+h) - y(x-h)$ cancel out.

One may ask whether we have been lucky in finding formulas for finite difference approximations. There are some systematic ways of deriving some useful operators. The simplest of them is the method of undetermined coefficients which utilises Taylor expansion. Suppose we want to find an approximation of $y^{(k)}(x^*)$ based on knowledge of y at points $\{x_i\}_1^n$ where $n \geq k+1$. This set of points is called the *stencil*. For every $i = 1, ..., n$ we have

$$y(x_i) = y(x^*) + y'(x^*)(x_i - x^*) + \frac{1}{2}y''(x^*)(x_i - x^*)^2 + ... + \frac{1}{k!}y^{(k)}(x^*)(x_i - x^*)^k + ... \tag{3.13}$$

We thus want a linear combination of the above to approximate a given derivative

$$\sum_{i=1}^{n} c_i y(x_i) = y^{(k)}(x^*) + O(h^p), \tag{3.14}$$

where $p > 0$ has to be as large as possible. Now, plugging (3.13) we arrive at

$$y^{(k)}(x^*) = \sum_{i=1}^{n} c_i \sum_{j=0}^{n-1} \frac{y^{(j)}(x^*)}{j!}(x_i - x^*)^j = \sum_{j=0}^{n-1} \frac{y^{(j)}(x^*)}{j!} \sum_{i=1}^{n} c_i (x_i - x^*)^j \tag{3.15}$$

Of course, the $j-$sum is terminated at $j = n$ since we would like to obtain a well-posed linear system. Now, the both sides of the above equation are equal if and only if

$$\frac{1}{j!}\sum_{i=1}^{n} c_i (x_i - x^*)^j = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases} \qquad j = 0, ..., n-1. \tag{3.16}$$

If all x_i are distinct, the above is just the Vandermonde's system. Unfortunately, for large n the resulting linear system is badly conditioned and hence, difficult to accurately solve numerically. There are however some useful bypasses over this problem.

Example. We will find a finite difference approximation to $y'(x)$ build on a stencil x, $x-h$, and $x-2h$. We write

$$\delta_2 y(x) = ay(x) + by(x-h) + cy(x-2h). \tag{3.17}$$

If we expand two right terms in the above and collect similar expressions we arrive at

$$\delta_2 y(x) = (a+b+c)y(x) - (b+2c)hy'(x) + \frac{1}{2}(b+4c)h^2 y''(x) - \frac{1}{6}(b+8c)h^3 y'''(x) + O(h^4) \quad \text{as} \quad h \to 0^+. \tag{3.18}$$

Now, since our approximation concerns y' we take

$$a + b + c = 0, \quad b + 2c = -\frac{1}{h}, \quad b + 4c = 0, \tag{3.19}$$

which has a solution

$$a = \frac{3}{2h}, \quad b = -\frac{2}{h}, \quad c = \frac{1}{2h}. \tag{3.20}$$

Therefore, our finite difference has the form

$$\delta_2 y(x) = \frac{1}{2h}\left(3y(x) - 4y(x-h) + y(x-2h)\right). \tag{3.21}$$

The error of the approximation can be readily calculated from the expansion yielding

$$\delta_2 y(x) - y'(x) = \frac{1}{12}h^2 y'''(\xi), \tag{3.22}$$

where ξ is some point. The approximation is thus second order accurate.

3.2 Euler methods

Now, we are able to proceed to design of some numerical methods for ODEs. We start from the simplest ones - Euler schemes. Suppose we would like to solve

$$y'(x) = f(x, y), \quad y(0) = y_0. \tag{3.23}$$

Let us introduce the grid $x_n = nh$ where $h > 0$ is the grid spacing. We can use any finite difference to approximate the derivative on the left-hand side. When we use $\delta_\pm$ we obtain

$$y'(x_n) = \delta_\pm y(x_n) + R_\pm = f(x_n, y(x_n)). \tag{3.24}$$

All numerical methods are based on truncating the remainder $R_\pm$. If y_n denotes the numerical approximation to $y(x_n)$ (notice that these are usually different quantities!), we obtain the *Euler forward* method

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}), \tag{3.25}$$

and *Euler backward* method

$$y_n = y_{n-1} + hf(x_n, y_n). \tag{3.26}$$

The difference is basically in the point at which the function f is evaluated. Note also that the forward method is *explicit*, i.e. the next step is calculated directly from the previous ones, while backward method is *implicit*, i.e. the next step additionally requires solving a nonlinear equation $z = y_{n-1} + f(x_n, z)$. This increased computational cost has its merits and advantages as we will see in the sequel.

Immediately, we ask a question how accurate are the methods (3.25) and (3.26)? Do they approximate the exact solution of (3.23) arbitrarily good when $h \to 0^+$? Naively thinking, we can expect that since $\delta_\pm$ is a first order operator, the Euler methods should also be first order accurate. This appears to be true however, is not that simple. When integrating a differential equation the error in each step is accumulated. We have to ascertain whether it not accumulates too much. This is the problem of *convergence*.

On the other hand, a numerical method can be convergent but to a different solution than the original ODE's. This is the problem of *consistency*. We can rigorously define the relevant terms.

Definition 8. *A* local truncation error *(LTE) is the remainder of the numerical scheme when* y_n *is replaced with the exact solution of the corresponding ODE, that is* $y(x_n)$*. If LTE vanishes as* $h \to 0^+$ *the method is said to be* ***consistent****.*

A consistent numerical method approximates the relevant differential equation. For forward Euler method we have

$$\begin{aligned} \text{LTE} &= \frac{y(x_n) - y(x_{n-1})}{h} - f(x_{n-1}, y(x_{n-1})) = y'(x_n) + \frac{1}{2}hy''(x_n) + O(h^3) - f(x_{n-1}, y(x_{n-1})) \\ &= \frac{1}{2}hy''(x_n) + O(h^3) \quad \text{as} \quad h \to 0^+, \end{aligned} \tag{3.27}$$

since $y'(x_n) = f(x_n, y(x_n))$. Therefore, Euler method is a consistent method. A consistent method however, may not be convergent to the exact solution since the error can accumulate to fast.

Example. Consider a numerical method

$$y_{n+1} = y_{n-1} - 2hy_n, \quad y_0 = 1, \quad y_1 = 1. \tag{3.28}$$

Computing the local truncation error we have

$$\text{LTE} = \frac{y(x_{n+1}) - y(x_{n-1})}{2h} + y(x_n) = \frac{1}{6}h^2y'''(\xi_n), \tag{3.29}$$

therefore the method is consistent with the following ODE

$$y' = -y, \quad y(0) = 1, \tag{3.30}$$

which has a solution $y(x) = e^{-x}$. Now, we recall that a linear recurrence can be solved by looking for power function solutions $y_n = Cr^n$ for some C and r. Plugging this ansatz we obtain from (3.28)

$$Cr^{n+1} + 2Chr^n - r^{n-1} = 0. \tag{3.31}$$

Cancelling yields

$$r^2 + 2hr - 1 = 0, \tag{3.32}$$

and hence

$$r_\pm = -h \pm \sqrt{1 + h^2}. \tag{3.33}$$

Hence $|r_-| > 1$ and the recurrence is divergence for a general initial condition.

The above method is called the *leap-frog* since it jumps two steps ahead. Notice that although the truncation error is of second order, the method is useless due to its lack of convergence. Now, we state what we will mean by a convergent numerical method.

Definition 9. *Fix* $x \in \mathbb{R}$*. A numerical method is* convergent *with order* $p > 0$ *if*

$$|y(x) - y_n| \leq Ch^p \quad \textit{as} \quad nh \to x \quad \textit{and} \quad h \to 0^+, \tag{3.34}$$

for some constant $C > 0$ *independent of* n *and* h*.*

Therefore, a convergent method yields an arbitrarily accurate approximation of $y(x)$ when the grid is refined with nh converging to x (for example take $h = x/n$ and $n \to \infty$). The convergence proofs are usually difficult to obtain especially for PDEs. In due course we will indicate the various issues appearing in them.

Theorem 3. *Let $y = y(x)$, $x \in [0, X]$ be a solution of (3.23) with $f(x, y)$ being continuosly twice differentiable with respect to y variable. Then, both of Euler methods (3.25)-(3.26) are first order convergent.*

Proof. We will proceed by induction and without any loss of generality we consider only the forward case. Assume that $y_0 = y(0)$ (this is a slight simplification since computing y_0 always contains a round-off error). Define the convergence error

$$e_n := y(x_n) - y_n. \tag{3.35}$$

Then, from (3.25) and Taylor expansion the first term can be bounded as follows

$$|e_1| = |y(x_1) - y_1| = |y(x_0) + y'(x_0)h + \frac{1}{2}y''(\xi_0)h^2 - y_0 - hf(x_0, y_0)| \leq \frac{1}{2}\max_{x\in[0,X]}|y''(x)|h^2 =: \tau, \tag{3.36}$$

since the initial conditions are the same and $y'(x_0) = f(x_0, y(x_0))$. Then, the inductive step can be carried over similarly

$$\begin{aligned} |e_n| &= |y(x_{n-1}) + hy'(x_{n-1}) + \frac{1}{2}h^2y''(\xi_{n-1}) - y_{n-1} - hf(x_{n-1}, y_{n-1})| \\ &\leq |e_{n-1}| + |f(x_{n-1}, y(x_{n-1})) - f(x_{n-1}, y_{n-1})|h + \tau, \end{aligned} \tag{3.37}$$

where we used the fact that $y'(x_{n-1}) = f(x_{n-1}, y_{n-1})$. Since f is Lipschitz with respect to the second variable we have $|f(x_{n-1}, y(x_{n-1})) - f(x_{n-1}, y_{n-1})| \leq L|e_{n-1}|$ for some $L > 0$. Therefore,

$$|e_n| \leq (1 + Lh)|e_{n-1}| + \tau. \tag{3.38}$$

Proceeding inductively, we obtain

$$|e_n| \leq (1+Lh)^2|e_{n-2}| + (1+Lh)\tau \leq \ldots \leq (1+Lh)^{n-1}|e_1| + \tau\sum_{i=0}^{n-2}(1+Lh)^i \leq \tau\sum_{i=0}^{n-1}(1+Lh)^i. \tag{3.39}$$

The last sum is geometric and hence

$$|e_n| \leq \tau\sum_{i=0}^{n-1}(1+Lh)^i = \frac{\tau}{Lh}\left((1-Lh)^n - 1\right) = \frac{1}{2L}\max_{x\in[0,X]}|y''(x)|\left(\left(1 - \frac{Lnh}{n}\right)^n - 1\right)h. \tag{3.40}$$

Now, since $(1 + 1/n)^n \leq e$ and $nh \leq X$ we have

$$|e_n| \leq \frac{1}{2L}\max_{x\in[0,X]}|y''(x)|\left(e^{LX} - 1\right)h = Ch. \tag{3.41}$$

This ends the proof. □

Euler methods are the simplest ones to solve ODEs and many times are not very useful due to their low accuracy. However, there are many important applications of using them in numerically solving PDEs since they are very easy to implement in time-advancement of a scheme. Moreover, sometimes it is not desirable to use high order methods due to unstable behaviour about which we will have much more to say.

4 Second order equations

Now we will deal with ordinary equations of the second order, which very often arise in all kinds of problems related to mechanics. One of the most important examples here is oscillations, and that is what we will focus on the most. First, let us define the general class of the equations studied.

Definition 10. *An initial value problem of the **second order** is a differential equation*

$$y'' = f(t, y, y'), \quad t \in (t_0, T), \tag{4.1}$$

along with initial conditions

$$y(0) = y_0, \quad y'(0) = y_0'. \tag{4.2}$$

Here, $y = y(t)$ is to be found while f is given.

First, notice that we have a pair of initial conditions: one for the function and one for the derivative. As we will see, this is necessary for having a well-posed problem. Intuitively, since we are dealing with an equation of a second order, when solving it we have to perform two integrations. These will produce two integration constant that has to be determined from two conditions. However, as we will see, this procedure is rarely the actual way of finding solutions to higher order equations.

Example. (*Pendulum*) One of the most important and archetypal second order equation is the one describing pendulum motion. The pendulum is assumed to be a weightless thread of length l with a bob of mass m attached at the end of it (see Fig. 11). By $\theta = \theta(t)$ denote the pendulum's angle subtended with the vertical. Then, balancing forces in normal coordinates[22] we have that

$$\underbrace{ml\frac{d^2\theta}{dt^2}}_{\text{mass} \times \text{angular acceleration} = \text{net force}} = \underbrace{-mg\sin\theta}_{\text{tangent coordinate of the gravity}}. \tag{4.3}$$

Therefore,

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0, \tag{4.4}$$

which is a second order nonlinear equation. If the maximal pendulum angle is small, usually taken to be $|\theta| \leq 6^\circ$, we can approximate $\sin\theta \approx \theta$ and obtain a *linear oscillator equation*

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0, \tag{4.5}$$

which plays a major role in modelling various oscillations. □

Example. (*Shock absorbers and dampers*) Shock absorbers are used in many kinds of mechanical devices to convert the kinetic energy into heat in order to prevent the system from excessive strain. For example, they are widely used in cars. We can devise a simple model for a damper with a spring to which a mass m is attached and a forcing

[22] The curvilinear basis spanned by tangent and normal vectors to the particle motion.

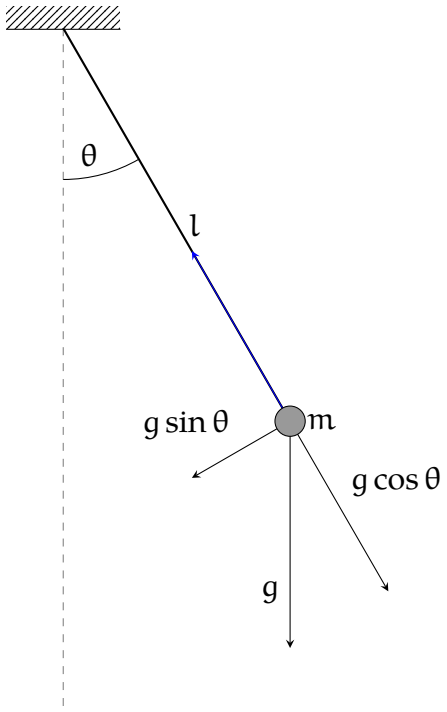

Figure 11: A schematic of a mathematical pendulum.

$F = F(t)$ introduces energy. The spring moves in a viscous fluid to dissipate energy. By $x = x(t)$ denote the spring deflection from equilibrium at time t.

From Hooke's Law we know that the elastic force acting on a spring is proportional to the deflection. That is,

$$F_e = -kx, \tag{4.6}$$

where k is the spring constant. Further, the dissipative force of the viscous medium can be assumed to be proportional to the velocity (this is Newton's law and the fluid is called Newtonian)

$$F_d = -\gamma \frac{dx}{dt}, \tag{4.7}$$

where γ is the damping constant. Balancing forces gives us

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t), \tag{4.8}$$

which is a linear oscillator equation with damping. □

Dealing with a general class of second-order equations is quite difficult and does not provide adequate physical conclusions. We will therefore examine a very special subclass of them, namely linear equations with constant coefficients.

Definition 11. ***A linear equation of the second order*** *is*

$$y'' + p(t)y' + q(t)y = f(t), \tag{4.9}$$

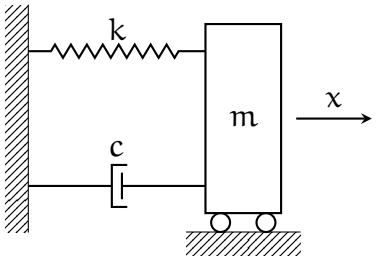

Figure 12: A schematic for the damper.

where p, q, *and* f *are given continuous functions. if* $f \equiv 0$ *then the equation is called* homogeneous *while in other case* - nonhomogeneous. *Where* p *and* q *are independent of* t, *then the equation has* constant coefficients.

Note that in general a second order linear equation can be written as

$$R(t)y'' + P(t)y' + Q(t)y = F(t), \tag{4.10}$$

however, if R is different than zero we can always divide by it and obtain (4.9). On the other hand, the points at which R vanishes are also interesting since at them the equation, formally, decreases one order. These are called *singular points* and are beyond the scope of our lecture.

4.1 Homogeneous equations with constant coefficients

The main object of our research will be equations with constant coefficients, which we will write as

$$ay'' + by' + cy = f(t), \tag{4.11}$$

where a, b, and c are constant. In general, they can be complex constants, but for us it is enough to consider the real coefficients. Let us start by finding a solution to the homogeneous equation

$$ay'' + by' + cy = 0. \tag{4.12}$$

Notice that in a specific case with $a = -c = 1$ and $b = 0$ we have $y'' = y$, which can be solved by both $y_1(t) = e^t$ and $y_2(t) = e^{-t}$. A moment of thought lets us to see that $y_3(t) = \sinh t$ and $y_4(t) = \cosh t$ are also solutions! This is a general case and is summarized in the following proposition.

Proposition 1. *Let* y_1 *and* y_2 *be solutions of (4.9) with* $f \equiv 0$. *Then, their linear combination is also a solution.*

Proof. Let y_1 and y_2 be solutions of the homogeneous equation. Now, it is enough to take advantage of the linearity of the derivative operator

$$\begin{aligned} (\alpha y_1 + \beta y_2)'' + p(t)(\alpha y_1 + \beta y_2)' + q(t)(\alpha y_1 + \beta y_2) = \\ \alpha\left(y_1'' + p(t)y_1' + q(t)y_1\right) + \beta\left(y_2'' + p(t)y_2' + q(t)y_2\right) = 0, \end{aligned} \tag{4.13}$$

which ends the proof. □

Therefore, we can see that if we find any two solutions to the homogeneous equation, let them be y_1 and y_2, the solution is also

$$y(t) = C_1 y_1(t) + C_2 y_2(t). \tag{4.14}$$

Later in the lecture we will see that provided that y_1 and y_2 are linearly independent, then every other solution of our equation could be written in the above form. We call this **general solution**. Then $C_{1,2}$ constants can be determined using the initial conditions[23]. The linearly independent solutions y_1 and y_2 are called **fundamental solutions**. The solution of the initial problem, that is with determined constants $C_{1,2}$, is called **special solution**.

Let us now return to the equation with constant coefficients (4.12) and assume that, by analogy with our simple special case, the solution is an exponential function. Let $y(t) = e^{rt}$, for some constant r. It is called an *ansatz*. Then after substitution

$$ar^2 + br + c = 0, \tag{4.15}$$

whence, r is a zero of the above equation called *characteristic polynomial*. We have to consider various cases.

- *Two real roots*. If $r_1 \neq r_2$ are these roots, then of course the fundamental solutions of the equation (4.12) will be $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$. The general solution is therefore

  $$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}, \tag{4.16}$$

  and we do not have any oscillations.

  Example. We will solve

  $$\begin{cases} y'' + 5y' + 6y = 0, \\ y(0) = 0,\ y'(0) = 3. \end{cases} \tag{4.17}$$

  The characteristic polynomial is $r^2 + 5r + 6 = 0$ with roots $r_1 = -2$ and $r_2 = -3$, therefore, the general solution is

  $$y(t) = C_1 e^{-2t} + C_2 e^{-3t}. \tag{4.18}$$

  In order to determine the constants we plug the initial conditions

  $$\begin{cases} 0 = y(0) = C_1 + C_2, \\ 3 = y'(0) = -2C_1 - 3C_2. \end{cases} \tag{4.19}$$

  After solving it we obtain $C_1 = 3$ and $C_2 = -3$. Finally, the special solution has the form

  $$y(t) = 3\left(e^{-2t} - e^{-3t}\right), \tag{4.20}$$

  and is presented on Fig. 13. There are no oscillations. □

[23] If they were not linearly independent, for example $y_2 = \alpha y_1$, then $y(t) = C_1 y_1(t) + C_2 \alpha y_1(t) = (C_1 + \alpha C_2) y_1(t) = D y_1(t)$ for some constant D. So, in general, we are not able to meet *two* initial conditions.

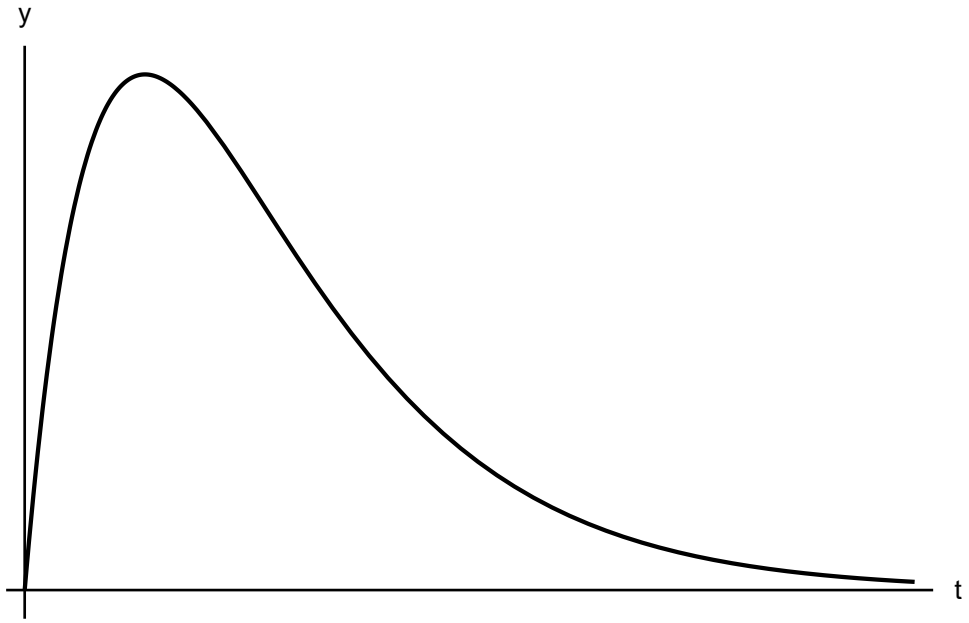

Figure 13: A graph of the solution $y(t) = 3\left(e^{-2t} - e^{-3t}\right)$.

- *Two complex roots*. The complex roots of a second order polynomial with real coefficients are always conjugate. Let $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$. Then we can immediately write the fundamental solutions as $y_1(t) = e^{\lambda t}e^{i\mu t}$ and $y_2(t) = e^{\lambda t}e^{-i\mu t}$. We could end there by now, but usually we are dealing with initial conditions that are real (e.g. position and speed of the pendulum). So we would like the solution to the differential equation to be real as well (although the complex form also gives a lot of benefits). Remember that a linear combination of solutions is also a solution, therefore they will be solutions

$$\begin{aligned} y_3(t) &= \frac{1}{2}\left(y_1(t) + y_2(t)\right) = e^{\lambda t}\frac{e^{i\mu t} + e^{-i\mu t}}{2} = e^{\lambda t}\cos\mu t, \\ y_4(t) &= \frac{1}{2i}\left(y_1(t) - y_2(t)\right) = e^{\lambda t}\frac{e^{i\mu t} - e^{-i\mu t}}{2} = e^{\lambda t}\sin\mu t, \end{aligned} \tag{4.21}$$

where we used Euler's formulas for the complex relationship of exponents with trigonometric functions. The general solution ultimately takes the form of

$$y(t) = e^{\lambda t}\left(C_1\cos\mu t + C_2\sin\mu t\right), \tag{4.22}$$

and thus, the angular frequency of the oscillations is μ while the time scale of the amplitude decay is λ^{-1}.

Example. A simple equation $y''+y = 0$ has the characteristic polynomial $r^2+1 = 0$ with two purely imaginary roots $r_{1,2} = \pm i$. Therefore, the solution is

$$y(t) = C_1\cos t + C_2\sin t, \tag{4.23}$$

which are clearly oscillatory with the same amplitude equal to $\sqrt{C_1^2 + C_2^2}$ (why?). They are presented on Fig. 14. □

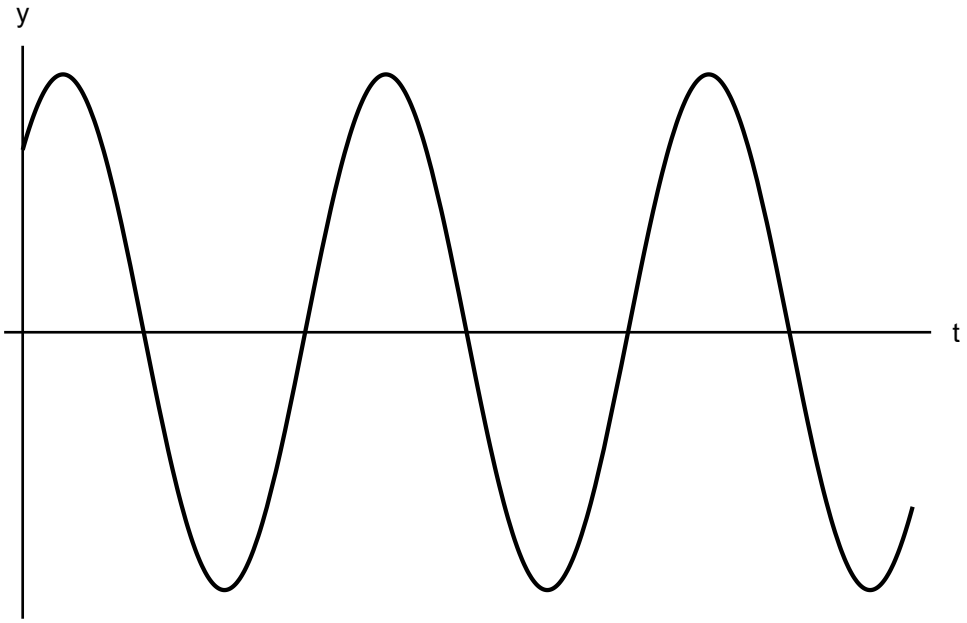

Figure 14: A graph of the solution $y(t) = y(t) = C_1 \cos t + C_2 \sin t$.

Example. The equation $y'' + y' + y = 0$ gives the characteristic polynomial $r^2 + r + 1 = 0$ with roots $r_{1,2} = -1/2 \pm i\sqrt{3}/2$. Therefore

$$y(t) = e^{-\frac{1}{2}t}\left(C_1 \cos\frac{\sqrt{3}}{2}t + C_2 \sin\frac{\sqrt{3}}{2}t\right), \tag{4.24}$$

which clearly shows oscillations of angular frequency $\sqrt{3}/2$ that are damped at a time scale 2 (see Fig. 15). □

- *One repeated root*. This root is represented by the formula $r_{1,2} = -\frac{b}{2a}$, which gives one fundamental solution $y_1(t) = e^{-\frac{b}{2a}t}$. How do we find the second one? Suppose it is very similar to the first, that is $y_2(t) = f(t)y_1(t)$ for an unknown function f. Then

$$\begin{aligned} y_2'(t) &= f'e^{-\frac{b}{2a}t} - \frac{b}{2a}fe^{-\frac{b}{2a}t}, \\ y_2''(t) &= f''e^{-\frac{b}{2a}t} - \frac{b}{a}f'e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}fe^{-\frac{b}{2a}t}. \end{aligned} \tag{4.25}$$

When we plug y_2 into (4.12) we obtain

$$af'' + (-b + b)f' + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)f = 0. \tag{4.26}$$

Since a characteristic polynomial has only one double root, its discriminant is equal to zero, that is, $b^2 = 4ac$. Thus, the factor of f in the above equation

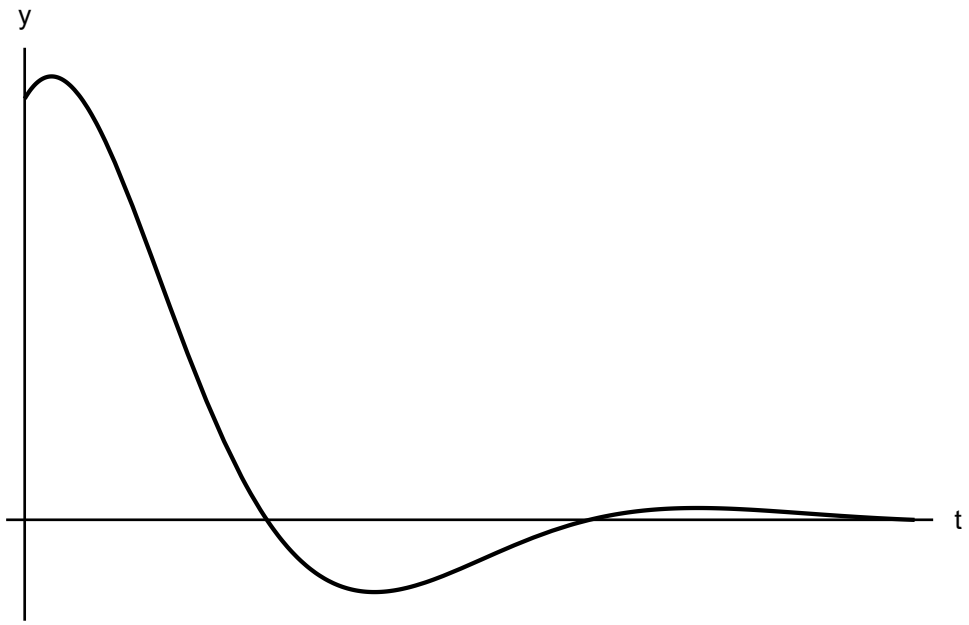

Figure 15: A graph of the solution $y(t) = e^{-\frac{1}{2}t}\left(C_1 \cos\frac{\sqrt{3}}{2}t + C_2 \sin\frac{\sqrt{3}}{2}t\right)$.

disappears, leaving $f'' = 0$, that is $f(t) = C_1 t + C_2$. Whence, we got a general solution

$$y(t) = C_1 e^{-\frac{b}{2a}t} + C_2 t e^{-\frac{b}{2a}t}, \tag{4.27}$$

where we slightly renamed the constants. So we can see that to get the second fundamental solution, it is enough to multiply the first by t.

Example. Let us solve

$$\begin{cases} y'' - y' + \frac{1}{4}y = 0, \\ y(0) = 2,\ y'(0) = \frac{1}{3}. \end{cases} \tag{4.28}$$

The characteristic polynomial is $r^2 - 1 + \frac{1}{4} = 0$ and has a double root $r_{1,2} = \frac{1}{2}$. Then,

$$y(t) = C_1 t e^{\frac{1}{2}t} + C_2 e^{\frac{1}{2}t}. \tag{4.29}$$

Substituting initial conditions gives

$$\begin{cases} 2 = y(0) = C_2, \\ \frac{1}{3} = y'(0) = C_1 + 1. \end{cases} \tag{4.30}$$

Therefore, the special solution has the form

$$y(t) = -\frac{2}{3} t e^{\frac{1}{2}t} + 2e^{\frac{1}{2}t}, \tag{4.31}$$

which is depicted on Fig. 16. □

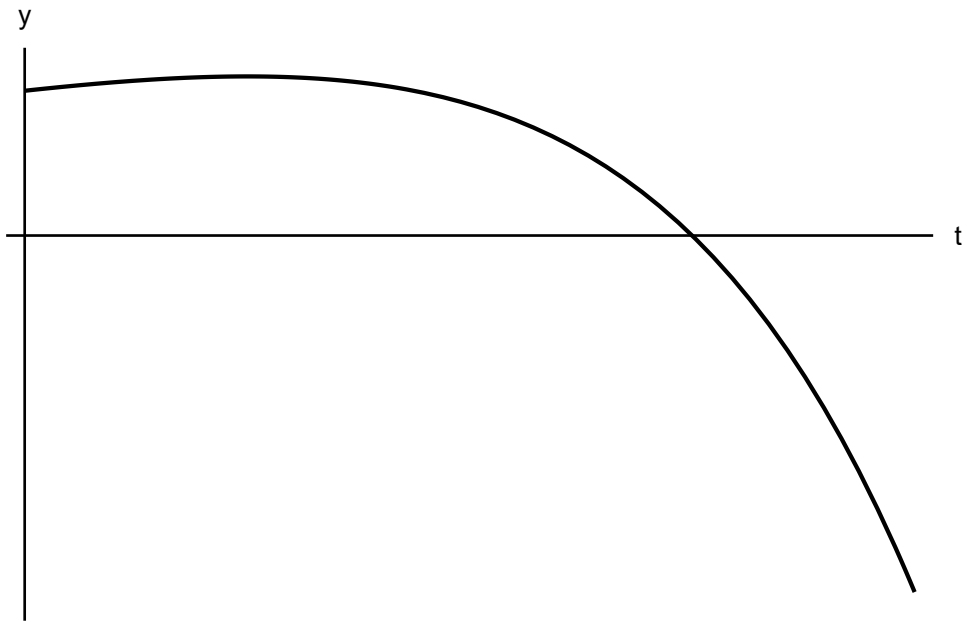

Figure 16: A graph of the solution $y(t) = -\frac{2}{3}te^{\frac{1}{2}t} + 2e^{\frac{1}{2}t}$.

4.2 Nonhomogeneous equations with constant coefficients

Having completely analysed homogeneous equations with constant coefficients we now look at the linear nonhomogeneous ones. Let us note immediately that if Y_1 and Y_2 are solutions of (4.9), then their difference satisfies the homogeneous equation

$$\begin{aligned}&(Y_1 - Y_2)'' + p(t)(Y_1 - y_2)' + q(t)(Y_1 - Y_2) = \\ &Y_1'' + p(t)Y_1' + q(t)Y_1 - (Y_1'' + p(t)Y_1' + q(t)Y_1) = f(t) - f(t) = 0.\end{aligned} \tag{4.32}$$

This means that the difference of solutions of the nonhomogeneous equation can be written as a linear combination of the fundamental solutions of the homogeneous equation $Y_1 - Y_2 = C_1y_1 + C_2y_2$. Thus, if φ is any solution of the nonhomogeneous equation (a special solution) then the general solution of (4.9) can be written as

$$y(t) = C_1y_1(t) + C_2y_2(t) + \varphi(t). \tag{4.33}$$

Therefore, in order to solve a nonhomogeneous equation we have to make the following steps.

1. Find the fundamental solutions of the corresponding homogeneous equation.
2. "Guess" any (special) solution φ of the inhomogeneous equation.
3. Write the solution as (4.33).

Example. Let us find a general solution of

$$y'' - 3y' + 4y = 3e^{2t}. \tag{4.34}$$

The characteristic polynomial of the homogeneous equation has the form $r^2-3r+4=0$, and its roots give us

$$y_1(t) = e^{\frac{3}{2}t}\cos\frac{\sqrt{7}}{2}t, \quad y_2(t)e^{\frac{3}{2}t}\cos\frac{\sqrt{7}}{2}t. \tag{4.35}$$

It is also easy to check that $\varphi(t) = -\frac{1}{2}e^{2t}$ is one of the solutions of our nonhomogeneous equation. Therefore, its general solution has the form

$$y(t) = e^{\frac{3}{2}t}\left(C_1\cos\frac{\sqrt{7}}{2}t + C_2\sin\frac{\sqrt{7}}{2}t\right) - \frac{1}{2}e^{2t}. \tag{4.36}$$

□

Of course, the word "guess" brings fear and trepidation to many students. Fortunately, there are systematic methods of finding special solutions of nonhomogeneous equations without any divine intervention.

4.2.1 Method of Undetermined Coefficients

It is also called the Prediction Method and is based on proposing the form φ and then checking that choice. However, it only works for some types of nonhomogeneities (that is, f). Let us assume that the f has the form of the product of the polynomial P_n (n - degree), the exponential and trigonometric functions, i.e.

$$f(t) = P_n(t)e^{\alpha t}\begin{cases}\sin\beta t,\\ \cos\beta t,\end{cases} \tag{4.37}$$

where α and β are constant. We can then show (with quite a long, tedious, but straightforward calculation) that the special solution is

$$\varphi(t) = t^s\left[\left(A_nt^n + A_{n-1}t^{n-1} + ... + A_0\right)e^{\alpha t}\cos\beta t + \left(B_nt^n + B_{n-1}t^{n-1} + ... + B_0\right)e^{\alpha t}\sin\beta t\right], \tag{4.38}$$

The unknown coefficients s, A_n and B_n can be found by substituting φ given by the above formula into the heterogeneous equation (4.9). There are a few rules to follow when guessing the φ form. First, if f contains a trigonometric function, φ should have them *both* (the point is that the derivative of the sine is a cosine and vice versa). Second, the number s is usually 0, but is positive in cases where a part of the nonhomogeneity f belongs to the set of fundamental solutions.

Example. We will solve

$$y'' - 3y' - 4y = -8e^t\cos 2t. \tag{4.39}$$

Here, $P_0(t) = -8$, $\alpha = 1$ and $\beta = 2$. We thus look for constants A and B such that

$$\varphi(t) = Ae^t\cos 2t + Be^t\sin 2t. \tag{4.40}$$

To substitute the above expression into the differential equation we need to compute the derivatives

$$\begin{aligned}\varphi'(t) &= (A+2B)e^t\cos 2t + (-2A+B)e^t\sin 2t,\\ \varphi''(t) &= (-3A+4B)e^t\cos 2t + (-4A-3B)e^t\sin 2t.\end{aligned} \tag{4.41}$$

For φ to be a solution, we must have $\varphi'' - 3\varphi' - 4\varphi = -8e^t \cos 2t$, which is equivalent to

$$(-10A - 2B)e^t \cos 2t + (2A - 10B)e^t \sin 2t = -8e^t \cos 2t. \tag{4.42}$$

Both sides of the above equation will be identically equal (for each t) if the coefficients of the respective functions are identical, i.e.

$$\begin{cases} -10A - 2B &= -8, \\ 2A - 10B &= 0. \end{cases} \tag{4.43}$$

Therefore, $A = \frac{10}{13}$ and $B = \frac{2}{13}$, that is

$$\varphi(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t. \tag{4.44}$$

Example. Now, consider

$$\begin{cases} y'' - 2y' + 1 = (t+2)e^t, \\ y(0) = 0,\ y'(0) = 1. \end{cases} \tag{4.45}$$

The general solution of the homogeneous equation is

$$y_h(t) = C_1 te^t + C_2 e^t. \tag{4.46}$$

We will see what the search for a solution to the inhomogeneous equation may look like. Suppose we are naive and make the ansatz

$$\varphi_1(t) = (At + B)e^t. \tag{4.47}$$

Then, after taking the derivatives and plugging into the equation we obtain

$$(t+2)e^t = \varphi_1'' - 2\varphi_1' + \varphi_1 = (At + B + 2A - 2At - 2A - 2B + At + B)e^t = 0, \tag{4.48}$$

which is a contradiction. We can see that φ_1 certainly cannot be a special solution. Let us go ahead and propose

$$\varphi_2(t) = (At^2 + Bt + C)e^t, \tag{4.49}$$

which gives

$$(t+2)e^t = \varphi_2'' - 2\varphi_2' + \varphi_2 = 2Ae^t. \tag{4.50}$$

Unfortunately, this is also not a good choice because we get a contradiction again: the left side of the above equality is variable and the right side is constant. Let it now take

$$\varphi_3(t) = (At^3 + Bt^2 + Ct + D)e^t, \tag{4.51}$$

and then

$$(t+2)e^t = \varphi_2'' - 2\varphi_2' + \varphi_2 = (2B + 6At)e^t, \tag{4.52}$$

which gives $2B + 3At = t + 2$, that is $A = 1/6$ and $B = 1$. The C and D constants are arbitrary, so we choose $C = D = 0$. Ultimately, $\varphi_3(t) = (t^3/6 + t^2)e^t$. The general solution of a heterogeneous equation has the form

$$y(t) = C_1 te^t + C_2 e^t + \left(\frac{1}{6}t^3 + t^2\right)e^t. \tag{4.53}$$

Only now can we take use the initial conditions and obtain

$$0 = y(0) = C_2, \quad 1 = y'(0) = C_1, \tag{4.54}$$

therefore, the solution of the nonhomogeneous problem is

$$y(t) = \left(\frac{1}{6}t^3 + t^2 + t\right) e^t. \tag{4.55}$$

Now consider why $\varphi_{1,2}$ was not the right choice. We can see that φ_1 is nothing but a linear combination of two fundamental solutions of the homogeneous equation. Therefore, substituting this function into the equation returns zero. On the other hand, if we write $\varphi_2(t) = At^2e^t + (Bt + C)e^t$, we can again see that the second term belongs to the set of fundamental solutions. This means that it satisfies the homogeneous equation. The A parameter alone is insufficient for φ_2 to satisfy the inhomogeneous equation because $t + 2$ belongs to the two-dimensional space of polynomials. Hence, we need another degree of freedom, and it is provided by φ_3. □

The Method of Undetermined Coefficient is self-correcting: if we assume too little, we get a contradiction; if too much, we will have to calculate more but the answer will be correct.

4.2.2 Method of Variation of Parameters

This method comes from Lagrange and allows us to write a solution to the equation (4.9) for any function f. We are not limited here to the functions for which the Method of Undetermined Coefficients works. The price to be paid for this generality is the more complicated form of the solution.

Let y_1 and y_2 be the fundamental solutions of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{4.56}$$

We know that then the general solution will be their linear combination $C_1y_1(t) + C_2y_2(t)$. Lagrange noticed that the special solution of the equation of the nonhomogeneous can be written in the form

$$\varphi(t) = C_1(t)y_1(t) + C_2(t)y_2(t). \tag{4.57}$$

This is where the name of the method comes from: we let C_1 and C_2 depend on time. This extra freedom turns out to be enough for φ to be the solution. When we calculate the derivative, we get

$$\varphi' = C_1'y_1 + C_1y_1' + C_2'y_2 + C_2y_2'. \tag{4.58}$$

Since we have some freedom in determining $C_{1,2}$ we can take

$$C_1'y_1 + C_2'y_2 = 0. \tag{4.59}$$

The reason is that after substituting φ in the equation (4.9) there will be no second derivatives of $C_{1,2}$. With the condition (4.59) the derivative expresses itself very simply

$$\varphi' = C_1y_1' + C_2y_2'. \tag{4.60}$$

Calculating the second derivative we have

$$\varphi'' = C_1'y_1' + C_1y_1'' + C_2'y_2' + C_2y_2'', \tag{4.61}$$

and after returning to the original ODE we obtain

$$C_1\left[y_1'' + p(t)y_1' + q(t)y_1\right] + C_2\left[y_1'' + p(t)y_1' + q(t)y_1\right] + C_1'y_1' + C_2'y_2' = f(t). \tag{4.62}$$

Since y_1 and y_2 are solutions of a homogeneous equation, the expressions in parentheses vanish. Combining this result with (4.59) we obtain a system of equations

$$\begin{cases} C_1'y_1 + C_2'y_2 &= 0, \\ C_1'y_1' + C_2'y_2' &= f(t) \end{cases} \tag{4.63}$$

It is a linear system of *algebraic* equations with unknowns C_1' and C_2'. Therefore, the solution is

$$C_1'(t) = -\frac{y_2(t)f(t)}{W(y_1,y_2)(t)}, \quad C_2'(t) = \frac{y_1(t)f(t)}{W(y_1,y_2)(t)}, \tag{4.64}$$

where the determinant

$$W(y_1,y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2, \tag{4.65}$$

is called the **Wronskian**. Integrating (4.64) we finally obtain

$$\varphi(t) = -y_1(t)\int \frac{y_2(t)f(t)}{W(y_1,y_2)(t)} + y_2(t)\int \frac{y_1(t)f(t)}{W(y_1,y_2)(t)}. \tag{4.66}$$

This formula is very important in many theoretical studies since it allows us to "invert" a differential equation to obtain an integral representation of the solution.

4.3 Wronskian and linear independence (optional)

We saw that the Wronskian appeared in the Method of Variation of Parameters. It is a determinant of fundamental importance in the theory of linear ordinary equations. As we will see in a moment, it also occurs in a completely different place - when finding a special solution.

Consider

$$\begin{cases} y'' + p(t)y' + q(t) = 0, \\ y(0) = y_0,\ y'(0) = y_0'. \end{cases} \tag{4.67}$$

Suppose we can find two solutions to the above homogeneous equation. Let them be y_1 and y_2. We will see what conditions they must meet in order for us to be able to satisfy both initial conditions. First of all, it must happen

$$\begin{cases} C_1y_1(0) + C_2y_2(0) = y_0, \\ C_1y_1'(0) + C_2y_2'(0) = y_0'. \end{cases} \tag{4.68}$$

It is a system of two linear algebraic equations that we can easily solve

$$C_1 = \frac{y_0y_2'(0) - y_0'y_2(0)}{y_1(0)y_2'(0) - y_1'(0)y_2(0)}, \quad C_2 = \frac{-y_0y_1'(0) + y_0'y_1(0)}{y_1(0)y_2'(0) - y_1'(0)y_2(0)}. \tag{4.69}$$

Note that the expression in the denominators is just Wronskian (4.65) taken at 0. Of course, for the above expressions to make sense, the denominator must always be different from zero. Therefore, we have proved the theorem.

Theorem 4. *Let y_1 and y_2 be solutions of the homogeneous equation (4.67). If their Wronskian does not vanish at $t = 0$, that is*

$$W(y_1, y_2)(0) \neq 0, \tag{4.70}$$

then (4.67) has a solution satisfying both initial conditions.

By direct calculation we can check that all the solutions we found for equations with constant coefficients have non-vanishing Wronskians, for example for $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ from $r_1 \, neq r_2$ we have

$$W(y_1, y_2)(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = (r_2 - r_1)\, e^{(r_1+r_2)t} \neq 0. \tag{4.71}$$

There are stronger results for solutions to second-order equations. They result from the theorem about the existence and uniqueness of solutions (Theorem 2). It turns out that while p and q in (4.67) are continuous on a certain interval, we are always able to find a unique solution to our problem. Moreover, there is always a set of fundamental solutions.

Wronskian is also important in the more algebraic aspects of differential equations. It is closely related to the linear independence of solutions to linear equations.

Theorem 5. *Let f and g be functions defined over a certain interval I. If $W(f, g)(t_0) \neq 0$ for some t_0 then f and g are linearly independent on I. Equivalently, if f and g are not linearly independent on I then $W(f, g)(t) = 0$ for each $t \in I$.*

Proof. Assume that

$$\alpha f(t) + \beta g(t) = 0. \tag{4.72}$$

For linear independence we have to show that the above equation implies $\alpha = \beta = 0$. Calculating the derivative we have

$$\alpha f'(t) + \beta g'(t) = 0. \tag{4.73}$$

When we substitute $t = t_0$ and write down the system of equations, we get

$$\begin{cases} \alpha f(t_0) + \beta g(t_0) = 0, \\ \alpha f'(t_0) + \beta g'(t_0) = 0. \end{cases} \tag{4.74}$$

The determinant of the above system is Wronskian $W(f, g)(t_0)$, which by assumption is different from zero. This means that $\alpha = \beta = 0$ is the solution. The functions f and g are therefore linearly independent. □

Note that despite the disappearance of Wronskian, the f and g functions can still be linearly independent. Examples include the functions $f(t) = |t|t^2$ and $g(t) = t^3$ on the $(-1, 1)$ interval.

We will now show a very important result that gives the exact formula for Wronskian. Often it can be used to find the second fundamental solution as long as only the first is known.

Theorem 6 (Abel). *Let y_1 and y_2 be solutions of the homogeneous equation*

$$y'' + p(t)y' + q(t)y = 0, \tag{4.75}$$

for p and q continuous over some interval I. Then, the Wronskian can be written as

$$W(y_1, y_2)(t) = C \exp\left(-\int p(t)dt\right), \tag{4.76}$$

where C is a constant that depends on $y_{1,2}$. Thus, W is either identically zero for all $t \in I$ ($C = 0$), or is not zero for $t \in I$ ($C \neq 0$).

Proof. From the assumption we have

$$\begin{cases} y_1''(t) + p(t)y_1'(t) + q(t)y_1 = 0, \\ y_2''(t) + p(t)y_2'(t) + q(t)y_2 = 0. \end{cases} \tag{4.77}$$

Multiplying the first equation by y_2 and the second by y_1 and subtracting both sides, we get

$$(y_1y_2'' - y_1''y_2) + p(t)\,(y_1y_2' - y_1'y_2) = 0. \tag{4.78}$$

If we observe that $W' = y_1y_2'' - y_1''y_2$ then we arrive at

$$W' + p(t)W = 0. \tag{4.79}$$

It is a very well-known separable first order equation with a solution

$$W(t) = C \exp\left(-\int p(t)dt\right). \tag{4.80}$$

The integration constant C can be determined from the knowledge of $y_{1,2}$. □

Therefore, if we know y_1 then using Abel's Theorem we can write

$$y_1y_2' - y_1'y_2 = C \exp\left(-\int p(t)dt\right), \tag{4.81}$$

which is a linear first order equation for y_2.

Example. We know that $y_1(t) = t^{-1}$ is a solution of

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0, \quad t > 0. \tag{4.82}$$

We also have $\exp\left(-\int p(t)dt\right) = t^{-3/2}$. We will now find the second solution by writing

$$y_2' + t^{-1}y_2 = Ct^{-\frac{1}{2}}. \tag{4.83}$$

We can apply the integrating factor right away, but it's easier to multiply the above equation by t to get

$$ty_2' + y_2 = Ct^{\frac{1}{2}}, \tag{4.84}$$

what is

$$(ty_2)' = Ct^{\frac{1}{2}}. \tag{4.85}$$

That gives

$$y_2(t) = Dt^{\frac{1}{2}} + Et^{-1}. \tag{4.86}$$

Since Et^{-1} is linearly dependent on y_1, we only need to take $y_2(t) = t^{\frac{1}{2}}$.

Let us now summarize our considerations on solutions to second-order linear differential equations (identical conclusions carry over to higher orders). Let y_1 and y_2 be solutions of the homogeneous equation (4.67) on the interval I. Then the following statements are equivalent.

1. The y_1 and y_2 functions constitute fundamental system of solutions in the I range.
2. Functions y_1 and y_2 are linearly independent.
3. $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$.
4. $W(y_1, y_2)(t) \neq 0$ for all $t \in I$.

Many, if not all, considerations of linear second order equations can be easily generalized to higher order equations. Linearity is the fundamental concept and associations with algebra sculpt the general structure of solution spaces.

4.4 Forced oscillation and resonance

We will now consider a very important example of a forced linear oscillator (4.8) which, for convenience, can be written in a standard form

$$x'' + 2\beta x' + \omega_0^2 x = f(t), \tag{4.87}$$

where $\beta = \gamma/(2m)$, $\omega_0 = \sqrt{k/m}$, and $f(t) = F(t)/m$. This is a simple ODE that we already know how to solve. Here, we will consider a periodically forced case, that is

$$f(t) = f_0 \cos \omega t, \tag{4.88}$$

with the amplitude f_0 and forcing frequency ω. The characteristic polynomial being $r^2 + 2\beta r + \omega_0^2$ with roots

$$r_{1,2} = -b \pm \sqrt{b^2 - \omega_0^2}, \tag{4.89}$$

therefore we have three cases:

1. *undamped* when $\beta = 0$,
2. *weakly damped* when $0 < \beta < \omega_0$,
3. *critically damped* when $\beta = \omega_0$,
4. *strongly damped* when $\beta > \omega_0$.

We will only consider weakly damped case for then the nontrivial behaviour occurs. The solution of the forced equation can be written as

$$x(t) = e^{-bt}\left(C_1 \cos\left(\sqrt{\omega^2 - b^2}t\right) + C_2 \sin\left(\sqrt{\omega^2 - b^2}t\right)\right) + \varphi(t), \tag{4.90}$$

where φ is the special solution of the nonhomogeneous equation. It can be written as[24]

$$\varphi(t) = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \cos\left(\omega t - \arctan\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right). \tag{4.91}$$

Now, since the solution of the homogeneous equation (4.90) has an exponentially vanishing amplitude at a time scale β^{-1}, it is called the *transient* solution. On the other hand, the solution of the forced equation (4.91) is a persisting oscillation with angular frequency ω with a given amplitude and phase with the former being the most important.

Notice that the amplitude of forced oscillations

$$A(\omega) = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}, \tag{4.92}$$

depicted on Fig. 17, is unbounded for $\beta = 0$ (undamped oscillations). For our case, it is continuous with a clearly visible peak at ω_0 equal to

$$A_{max} = \frac{f_0}{2\beta\omega_0}. \tag{4.93}$$

Therefore, for even very small forcing f_0 the amplitude of oscillations can become very large when the damping is light. This phenomenon is called **resonance**. This can be very dangerous when constructing building or machines - the external oscillations can resonate with internal ones and lead to a collapse. On the other hand, in electronic devices or seismographs resonance is harnessed to detect very weak signals. Resonance is being also utilized, for example, in acoustics (tuning musical instruments, amplifying sound) and lasers (by creating coherent light in the cavity).

4.5 Boundary value problems

Boundary value problems appear very often in the analysis of partial differential equations. They are related to solving a given problem in a certain region on the boundary of which it is necessary to impose some conditions for the unknown solution. The problem of heat propagation in the rod can be used as an example. The situation is different if one end of the bar is cooled to a certain temperature and other whether it is insulated. Boundary value problems are not only important for partial equations, they are also a fundamental object in the application of ordinary equations.

Example. (*Hanging cable*) An important problem in engineering and construction is to determine the shape of a hanging cable or chain. Since this is the solution of lowest

[24] This requires a little bit of work!

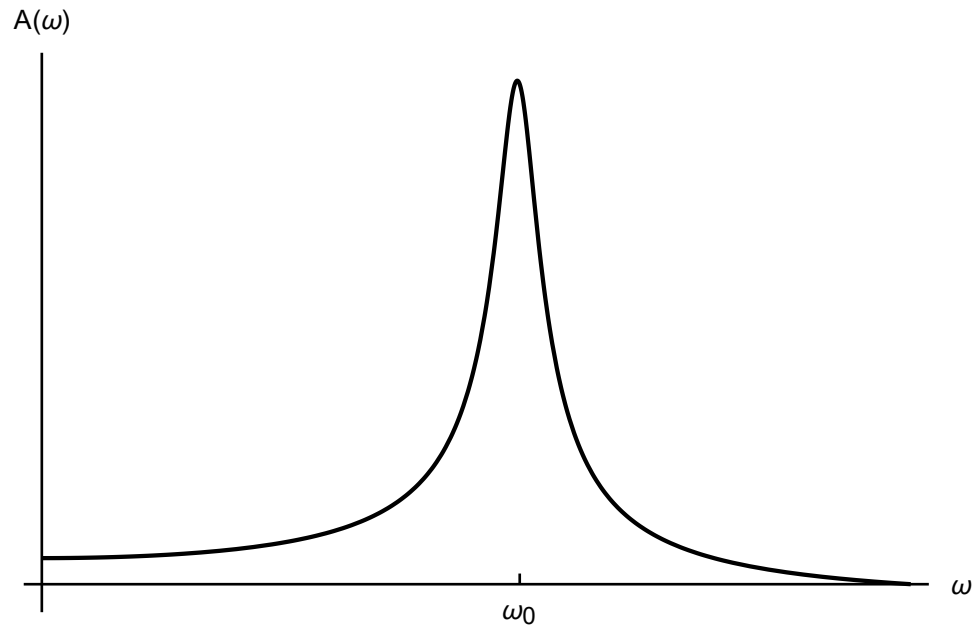

Figure 17: Amplitude of oscillations for $0 < \beta < \omega_0$ versus ω.

energy it is being used in designing arches of highest sustainable load or calculating the break-points of electrical cables (wires) that can be stressed by heavy snow or frost. Moreover, the solution of the problem is used in constructing suspension bridges or anchoring heavy marine objects.

We would like to find the ODE governing the shape of the steady-state of hanging cable of linear density ρ (see Fig. 18). Let the graph of the function $y = y(x)$ denote it and we fix its ends at $x = \pm L$

$$y(-L) = H_1, \quad y(L) = H_2. \tag{4.94}$$

In order to balance forces, that is cable tension **T** with gravity, we have to consider only a small portion of it, say $(x, x + \Delta x)$ and then pass to the limit with $\Delta x \to 0$. This is needed since all the tension changes from point to point. Looking at the bottom of Fig. 18 we see that the horizontal balance of forces is just

$$T(x + \Delta x) \cos \varphi(x + \Delta x) - T(x) \cos \varphi(x) = 0, \tag{4.95}$$

were φ denotes the angle subtended between the tangent to the cable and the horizontal. Dividing by Δx and taking the limit yields

$$(T(x) \cos \varphi(x))' = 0, \tag{4.96}$$

where prime denotes the derivative with respect to x. This means that

$$T(x) \cos \varphi(x) = T_0, \tag{4.97}$$

where the constant T_0 is the maximal tension of the cable (since $\cos \varphi(x) = 1$ then).

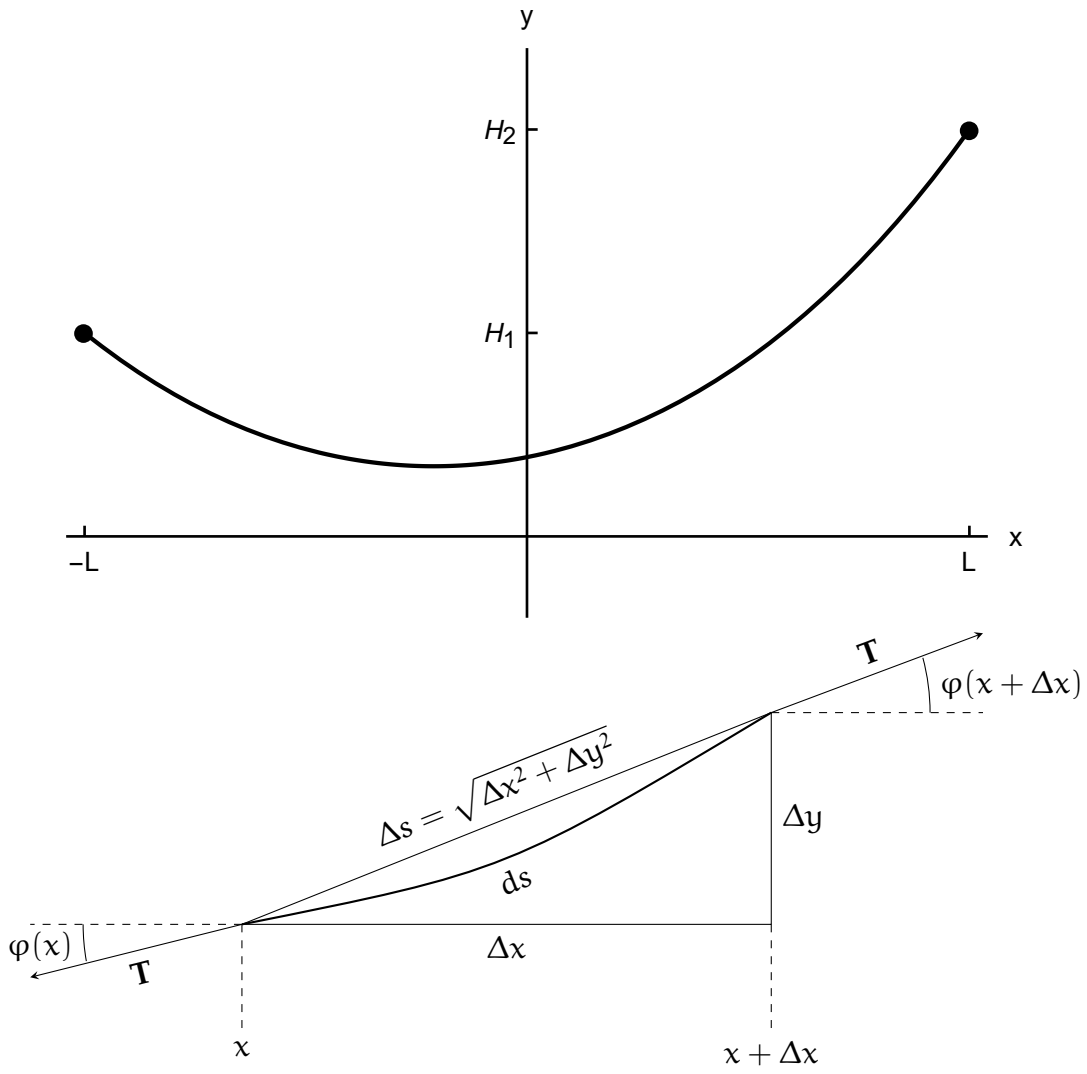

Figure 18: A schematic of a hanging cable or chain (top) and its small portion (bottom).

The balance in the vertical direction involves the gravity. The mass of our small increment is

$$m = g\int_x^{x+\Delta x} \rho(s)\sqrt{1+(y'(s))^2}ds = \rho(\xi)g\sqrt{1+(y'(\xi))^2}\Delta x, \tag{4.98}$$

where we have used the mean-value theorem for integrals with $x < \xi < x + \Delta x$. Therefore, the vertical balance gives

$$T(x+\Delta x)\sin\varphi(x+\Delta x) - T(x)\sin\varphi(x) = \rho(\xi)g\sqrt{1+(y'(\xi))^2}\Delta x, \tag{4.99}$$

or after the limit

$$(T(x)\sin\varphi(x))' = \rho(x)g\sqrt{1+(y'(x))^2}. \tag{4.100}$$

Now, because of (4.97) and the fact that $\tan\varphi(x) = y'(x)$ we can get rid of the angle and arrive at the final equation

$$T_0 y'' = \rho(x)g\sqrt{1+(y')^2}, \tag{4.101}$$

which is a nonlinear second order equation. Before we solve it, notice that when the cable deflection is small, that is $|y'(x)|$ is small, then the square root in the ODE can be accurately approximated by one. Then, the simplified equation becomes $T_0 y'' = \rho(x)g$ which can immediately can be solved by double integration.

For the full nonlinear case we notice that the given ODE is just a first order equation in disguise. That is, when we plug $u = y'$ we obtain

$$u' = a\sqrt{1+u^2}, \quad a = \frac{\rho g}{T_0}, \tag{4.102}$$

where we assumed that the density is constant (however, it can also be integrated for general distributions of mass). This is a separable ODE with an integral

$$u(x) = \sinh(ax + C), \tag{4.103}$$

with integration constant C. Further, going back to the substitution leads us to

$$y(x) = \frac{1}{a}\cosh(ax + C) + D. \tag{4.104}$$

And after plugging the boundary conditions (4.94) we have

$$y(x) = H - \frac{\cosh(aL) - \cosh(ax)}{a}. \tag{4.105}$$

Therefore, the graph of the hanging cable or a chain is the hyperbolic cosine (and not a parabola!). Due to that, this curve is called *catenary* from the Latin *catena*. It also has an important property that it forms a surface of revolution, called *catenoid*, with prescribed boundary conditions and with the smallest surface area.

Other interesting situations can also be modelled. For example, when our cable is a part of the suspension bridge that is loaded with a road of mass density w, we would obtain

$$T_0 y'' = \rho(x)g\sqrt{1+(y')^2} + wgx, \tag{4.106}$$

in many situations the weight of the road is much larger than that of the cable, and then ρ can be neglected and the result is a parabola. Another generalization is the equation modelling the shape of a swinging jumping rope. In this case the tension balances the centrifugal force (when the gravity is negligible), and hence

$$T_0 y'' = -\rho(x)\omega^2 y\sqrt{1+(y')^2}, \tag{4.107}$$

where ω is the angular velocity. The factor y comes from the fact that it is equal to the radius of revolution. The boundary conditions can, for example, be

$$y'(0) = 0, \quad y(L) = 0. \tag{4.108}$$

The resulting curve is called *troposkein* (from Greek "turning rope") and is being used in designing wind turbines. □

We will now proceed to formal definitions.

Definition 12. *A **two-point boundary value problem** is a second order ordinary differential equation defined on an interval along with conditions prescribed at its endpoints called **boundary conditions**.*

Definition 13. *The* Sturm-Liouville *boundary value problem is a differential equation*

$$-(p(x)y')' + q(x)y = \lambda r(x)y, \quad x \in (0,1), \tag{4.109}$$

along with homogeneous boundary conditions

$$a_1 y(0) + a_2 y'(0) = 0, \quad b_1 y(1) + b_2 y'(1) = 0. \tag{4.110}$$

*Any nonzero solution y of the above problem is called the **eigenfunction** with corresponding constant λ called an **eigenvalue**. Here, p, q, and r are given continuous functions.*

The Sturm-Liouville problem is very general and includes almost all boundary problems found in applications of differential equations. Note that we can consider the $(0,1)$ interval without losing generality, since the independent variable can always be scaled by a linear function. Note that if we define a linear operator

$$Ly := -(p(x)y')' + q(x)y, \tag{4.111}$$

this Sturm-Liouville problem can be written as

$$Ly = \lambda r(x)y, \tag{4.112}$$

which immediately reminds us of the eigenvalue problem for matrices known from algebra. Hence, the similar nomenclature.

Example. One of the simplest Sturm-Liouville problems is the case for $p(x) = 1$, $q(x) = 0$ and $r(x) = 1$, i.e.

$$y'' = -\lambda y. \tag{4.113}$$

For the boundary conditions let us take

$$y(0) = y(1) = 0. \tag{4.114}$$

We have to find all of the solutions and in order to do that we have to consider three cases dependeing on the sign of λ.

- $\lambda = 0$ gives us $y'' = 0$, therefore

$$y(x) = ax + b. \tag{4.115}$$

The condition at $x = 0$ gives us $b = 0$ and the other $0 = y(1) = a$. This means that $y \equiv 0$ is the solution to our problem for $\lambda = 0$. It is a trivial solution that is uninteresting from the point of view of applications.

- $\lambda < 0$ implies

$$y(x) = C_1 e^{\sqrt{\lambda}x} + C_2 C_1 e^{-\sqrt{\lambda}x}. \tag{4.116}$$

Now, $0 = y(0) = C_1 + C_2$, hence $C_2 = -C_1$. Therefore,

$$y(x) = C_1 \left(e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x} \right) \tag{4.117}$$

Using the second condition gives us

$$0 = y(1) = C_1 \left(e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}} \right) = 2C_2 \sinh \sqrt{-\lambda}. \tag{4.118}$$

Now we have to have $C_2 = 0$ which means the trivial solution $y \equiv 0$ or $\sinh \sqrt{\lambda} = 0$. We know that the hyperbolic sine vanishes only for $\lambda = 0$, which contradicts the assumption. So in this case, the trivial solution is the only one we can count on.

- $\lambda > 0$ forces

$$y(x) = C_1 \cos \sqrt{\lambda}x + C_2 \sin \sqrt{\lambda}x. \tag{4.119}$$

The condition at zero gives us

$$0 = y(0) = C_1, \tag{4.120}$$

therefore $y(x) = C_2 \sin \sqrt{\lambda}x$. The second one implies

$$0 = y(1) = C_2 \sin \sqrt{\lambda}. \tag{4.121}$$

If we take $C_2 = 0$ we obtain $y \equiv 0$, which is a trivial solution. This forces $\sin \sqrt{\lambda} = 0$ what is satisfied only for

$$\lambda_k = k^2 \pi^2, \quad k \in \mathbb{Z}. \tag{4.122}$$

We can see that our problem can have solutions only for particular values of λ. Moreover, the eigenfunctions are then

$$y_k(x) = A_k \sin(k\pi x), \quad k \in \mathbb{Z}. \tag{4.123}$$

The A_k constants cannot be determined without an additional condition. We will see below that the existence of a very natural and elegant way to deal with this ambiguity.

The above example teaches us many things. First, the homogeneous Sturm-Liouville problem always has an uninteresting trivial solution. Second, non-trivial solutions exist only for certain eigenvalues. Therefore, solving boundary problems consists in finding eigenfunctions and their corresponding eigenvalues simultaneously.

Many properties of the above simple example can be carried over into more general boundary value problems. One of the most important of these is the Lagrange Identity.

Theorem 7 (Lagrange)**.** *Let operator* L *be defined according to (4.111). Then, for* $u, v \in C^2(0,1)$ *we have*

$$\int_0^1 (vLu - uLv)\,dx = 0. \tag{4.124}$$

Proof. Immediately we have

$$\int_0^1 vLu\,dx = \int_0^1 (-(p(x)u')'v + quv)\,dx. \tag{4.125}$$

If we integrate the first term of the integrand twice by parts, we get

$$\int_0^1 vLu\,dx = -p(x)\left[u'(x)v(x) - u(x)v'(x)\right]_0^1 + \int_0^1 uLv\,dx. \tag{4.126}$$

Using the boundary conditions (4.110) we can calculate (for $a_2 \neq 0$ and $b_2 \neq 0$)

$$\begin{aligned} -p(x)\left[u'(x)v(x) - u(x)v'(x)\right]_0^1 &= -p(1)\left[u'(1)v(1) - u(1)v'(1)\right] + p(0)\left[u'(0)v(0) - u(0)v'(0)\right] \\ &= -p(1)\left[-\frac{b_1}{b_2}u(1)v(1) + \frac{b_1}{b_2}u(1)v(1)\right] + p(0)\left[-\frac{b_1}{b_2}u(0)v(0) + \frac{b_1}{b_2}u(0)v(0)\right] = 0, \end{aligned} \tag{4.127}$$

which ends the proof (case when $a_2 = 0$ or $b_2 = 0$ is analogous). □

If we define the scalar product by

$$\langle u, v\rangle = \int_0^1 u(x)\bar{v}(x)dx, \tag{4.128}$$

then, the Lagrange Identity can be written as

$$\langle Lu, v\rangle = \langle u, Lv\rangle, \tag{4.129}$$

which means that L is symmetric with respect to that product. We now show some results about the Sturm-Liouville problems. They all assume that (4.109) - (4.110) has solutions. The proof of this claim is very complex and requires additional machinery. It can be found in more advanced textbooks.

Proposition 2. *All the eigenvalues of the problem (4.109)-(4.110) are real.*

Proof. Let λ be the eigenvalue corresponding to φ. Then

$$\langle L\varphi, \varphi\rangle = \langle \varphi, L\varphi\rangle = \overline{\langle L\varphi, \varphi\rangle}. \tag{4.130}$$

Since φ is a solution we have $L\varphi = \lambda r(x)\varphi$, that is

$$\lambda\langle r\varphi, \varphi\rangle = \overline{\lambda}\,\overline{\langle r\varphi, \varphi\rangle} = \overline{\lambda}\,\overline{\langle \varphi, r\varphi\rangle} = \overline{\lambda}\,\langle r\varphi, \varphi\rangle, \tag{4.131}$$

because r is a real-valued function. Therefore

$$\left(\lambda - \overline{\lambda}\right)\int_0^1 r(x)\,|\varphi(x)|^2\,dx = 0. \tag{4.132}$$

We therefore must have $\lambda = \overline{\lambda}$. □

Further important result concerns orthogonality.

Proposition 3. *If φ_1 and φ_2 are eigenfunctions of (4.109)-(4.110) with corresponding eigenvalues λ_1 and λ_2, with $\lambda_1 \neq \lambda_2$, then*

$$\int_0^1 r(x)\varphi_1(x)\varphi_2(x)dx = 0. \tag{4.133}$$

Proof. Similarly as above we have

$$\lambda_1\langle r\varphi_1, \varphi_2\rangle = \langle L\varphi_1, \varphi_2\rangle = \langle \varphi_1, L\varphi_2\rangle = \lambda_2\langle \varphi_1, r\varphi_2\rangle. \tag{4.134}$$

Using the result that eigenfunctions are real we further obtain

$$(\lambda_1 - \lambda_2)\,\langle \varphi_1, r\varphi_2\rangle = 0, \tag{4.135}$$

which forces $\langle \varphi_1, r\varphi_2\rangle = 0$. □

Orthogonality allows us to properly norm the set of functions that are solutions to the boundary problem. This allows for the determination of the constants appearing when solving the differential equation (see above example).

Definition 14. *The set $\{\varphi_n\}_n$ is called* orthonormal *with respect to r, if*

$$\langle \varphi_n, r\varphi_m\rangle = \int_0^1 \varphi_n(x)\varphi_m(x)r(x)dx = \delta_{nm}, \tag{4.136}$$

where δ_{nm} is the Kronecker delta.

To sum up, the Sturm-Liouville problem has countably many solutions that are eigenfunctions φ_n with corresponding eigenvalues λ_n. Eigenfunctions are orthogonal with respect to the weight of r and can be made orthonormal by appropriate normalization. It turns out that the much stronger result about completeness of the set of eigenfunctions is true. This means that if $f \in C^2(0,1)$ (this assumption can be seriously weakened) and $\{\varphi_n\}_n$ is the orthonormal set of eigenfunctions of the Sturm-Liouville problem, we have

$$f(x) = \sum_{n=1}^{\infty} \langle f, r\varphi_n\rangle\varphi_n(x). \tag{4.137}$$

This immediately reminds us of similar results for unitary finite-dimensional spaces learned in algebra or Fourier series in calculus. This is indeed the origin of these matters.

5 Laplace transform

There is a remarkable method of transforming systems of linear equations with constant coefficients that greatly utilize finding their solution. This transformation takes differential into algebraic equations which, then, can be solved with known Gaussian elimination or other means. The idea of the method goes back to Laplace. Today, it has numerous applications not only in mathematics but also in physics and engineering (especially electrical).

5.1 Solving linear constant coefficients equations

Here, we will see how the Laplace transform can be utilized in solving linear equations with constant coefficients. First, we define what it is exactly.

Definition 15. ***Laplace transform*** *of a function* $f = f(t)$ *is*

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^\infty f(t)e^{-st}dt, \tag{5.1}$$

provided the integral converges. The function f *is usually called the* ***original*** *of the transform* F.

Depending on the context, sometimes it is more useful to write the Laplace transform as an operator $\mathcal{L}$ or the image F. The exponential inside the integral contains the new independent variable s. Before we proceed to studying some properties of Laplace transform, we have to ascertain that it is always well-defined for a given class of functions. Usually, the following result is satisfactory.

Theorem 8. *Let a piecewise continuous function* f *is of* exponential order*, that is there are constants* M, a, *and* T *such that for* $t > T$ *we have* $|f(t)| \leq M\exp(at)$. *Then, the Laplace transform (5.1) exists for* $s > a$.

Proof. Left as an exercise - comparison criterion for improper integrals. □

Therefore, in what follows we will always assume that we are dealing with functions satisfying the above theorem. We start with some simple examples.

Example.

1. If
$$f(t) = H(t) := \begin{cases} 1, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{5.2}$$
then
$$\mathcal{L}\{H\}(s) = \int_0^\infty e^{-st}dt = \frac{1}{s}, \tag{5.3}$$
where $s > 0$. The function H is called the **Heaviside function**.

2. The Laplace transform of $f(t) = e^{at}$ is

$$\mathcal{L}\left\{e^{at}\right\}(s) = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = \frac{1}{s-a}, \tag{5.4}$$

for $s > a$. We see that multiplying by an exponential results in translating the transform.

3. The Laplace transform of $f(t) = \sin(at)$ is

$$\mathcal{L}\{\sin(at)\}(s) = \int_0^\infty \sin(at)e^{-st}dt = \frac{a}{s^2+a^2}, \tag{5.5}$$

for $s > 0$. Similarly,

$$\mathcal{L}\{\cos(at)\}(s) = \frac{s}{s^2+a^2}. \tag{5.6}$$

As we mentioned above, the Laplace transform is an operator acting on a space of functions of exponential order. It is easy to see that it in fact linear. Further, since the exponential is an eigenfunction of the derivative, we see the reason why Laplace transform can change differentiation into multiplication and, hence, switch differential into algebraic equations.

Proposition 4. *Let f be of exponential order. Then, we have the following.*

1. *Linearity*

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}, \quad \alpha, \beta \in \mathbb{R}. \tag{5.7}$$

2. *Transform of a derivative*

$$\mathcal{L}\left\{f^{(n)}(t)\right\}(s) = s^n\mathcal{L}\{f(t)\} - \sum_{i=0}^{n-1} s^{n-1-i}f^{(i)}(0), \tag{5.8}$$

in particular

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0). \tag{5.9}$$

Proof. The linearity follows from liniearity of an integral. Therefore, we focus on the second property. Note that,

$$\begin{aligned}\mathcal{L}\{f'(t)\}(s) &= \int_0^\infty f'(t)e^{-st}dt \overset{\text{parts}}{=} \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}dt \\ &= -f(0) + s\mathcal{L}\{f(t)\}(s).\end{aligned} \tag{5.10}$$

And the rest follows by induction. □

These two properties are essential in solving linear differential equations with constant coefficients. It applied not only to ODEs but to PDEs as well.

Example. We will solve $y'' - y' - 2y = 0$ with conditions $y(0) = 1$, $y'(0) = 0$. The application of the transform and using linearity and the transform of a derivative (5.8) gives

$$0 = \mathcal{L}\{y'' - y' - 2y\} = s^2Y(s) - sy(0) - y'(0) - sY(s) + y(0) - 2Y(s). \tag{5.11}$$

Solving for the unknown Y yields

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{1}{3}\frac{1}{s-2} + \frac{2}{3}\frac{1}{s+1}, \tag{5.12}$$

which is the Laplace transform of our solution. We have found in (5.4) that these transforms correspond to an exponential originals, and hence

$$y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}, \tag{5.13}$$

which is the solution to our initial value problem. □

Notice that in one step, the method of Laplace transform finds a solution of an ODE with applied initial conditions. Probably the most difficult part in this process is to find out which original corresponds to a given Laplace transform. In many cases we can look them up in tables after simplifying the form of the transform. The latter can frequently be accomplished by expansion into partial fractions. In general we can do the following steps.

1. We start with equation of the form[25]

$$ay'' + by' + cy = f(t), \tag{5.14}$$

   with initial conditions

$$y(0) = y_0, \quad y'(0) = y_0'. \tag{5.15}$$

2. Apply Laplace transform, use linearity, and change differentiation into multiplication by s (formula (5.8))

$$a(s^2Y(s) - sy(0) - y'(0)) + b(sY(s) - y(0)) + cY(s) = F(s). \tag{5.16}$$

3. Solve for Y,

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}. \tag{5.17}$$

4. Expand into partial fractions noting that in the denominator we have the characteristic polynomial of the ODE.

5. Look up in the tables what are the corresponding originals to the given transforms.

[25] Everything here can be conducted for higher order equations and even systems.

All of the above steps can be illustrated in the following example.

Example. We solve a nonhomogeneous problem $y'' + y = \sin 2t$ with $y(0) = 2$ and $y'(0) = 1$. We have

$$s^2Y - sy(0) - y'(0) + Y = \frac{2}{s^2+4} \to Y(s) = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}. \tag{5.18}$$

After expansion into partial fractions we obtain

$$Y(s) = 2\frac{s}{s^2+1} + \frac{5}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4}. \tag{5.19}$$

Each term can be identified (5.5)-(5.6) to yield

$$y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin(2t). \tag{5.20}$$

Note that in this last step we also used linearity. □

The simplicity of the Laplace transform method is evident. It can even be incorporated into symbolic computation environments to implement efficient solvers for systems of constant coefficient linear equations. The success of the above method is based on a seemingly hard to notice property of the Laplace transform - the uniqueness. Why do we know that a given transform corresponds to a unique original? If this were not the case, the method would be useless in applications. Fortunately, Laplace transform is unique and we will give the proof of it in Section 5.4.

5.2 Step functions and Dirac delta

Another very useful property of Laplace transform is equal difficulty with dealing with piecewise continuous nonhomogeneities in our ODEs. A solution of such a problem would be very exhausting when using normal methods for then, we would have to solve additional algebraic equations in order to ascertain smooth junctions of the partial solutions in the jump points. Laplace transform does it automatically. This is useful in dealing with problems modelling switching.

To start, let us redefine the Heaviside function

$$H_c(t) := H(t-c) = \begin{cases} 1, & t \geq c, \\ 0, & t < c. \end{cases} \tag{5.21}$$

In that way we can model switching

$$f(t)H_c(t) = \begin{cases} f(t), & t \geq c, \\ 0, & t < c. \end{cases} \tag{5.22}$$

The above is called a *causal* signal since it has its own well-defined origin at $t = c$ (see Fig. 19). The transform of such a causal function can be computed from the definition

$$\mathcal{L}\{f(t)\,H_c(t)\} = \int_c^\infty f(t)e^{-st}dt = \int_0^\infty e^{-s(x+c)}f(x+c)dx = e^{-sc}\mathcal{L}\{f(t+c)\}. \tag{5.23}$$

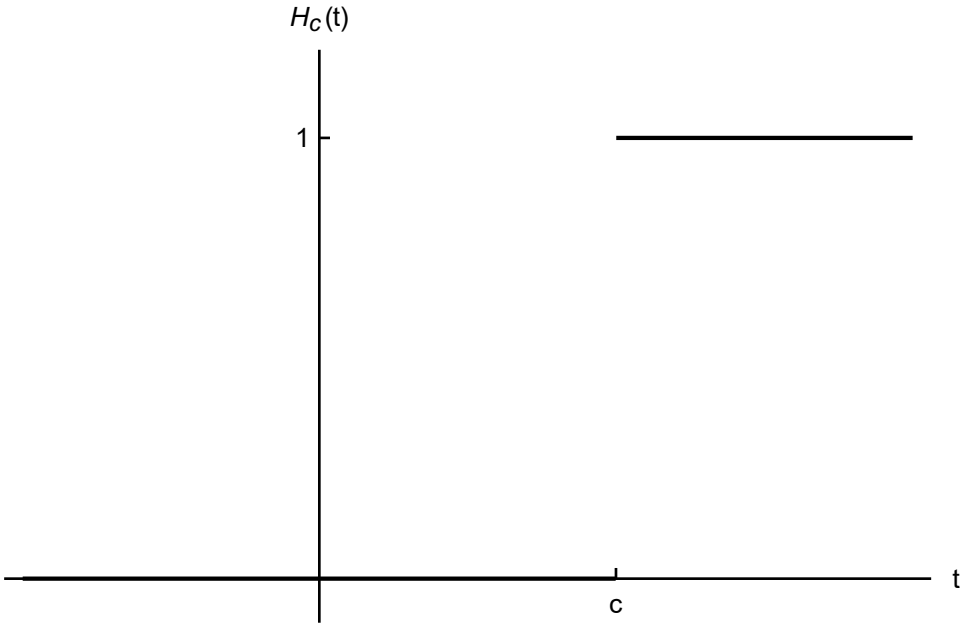

Figure 19: A Heaviside function.

Which, after substitution $f(t) \mapsto f(t-c)$, can be written as

$$\mathcal{L}\{f(t-c)H_c(t)\} = e^{-cs}\mathcal{L}\{f(t)\}, \tag{5.24}$$

therefore, a multiplication of the Laplace transform by e^{-cs} results in a original that is switched on at $t = c$. This result has important applications in solving problems with piecewise continuous force.

Example. Let us solve $2y'' + y' + y = g(t)$ with zero initial conditions and (see Fig. 20)

$$g(t) = \begin{cases} 1, & 2 \le t \le 15, \\ 0, & 0 \le t < 2 \text{ and } t > 15. \end{cases} \tag{5.25}$$

We can write it as $g(t) = H_2(t) - H_{15}(t)$. This is an important part since it allows us to easily compute the

$$(2s^2 + s + 1)Y(s) = \frac{e^{-2s} - e^{-15s}}{s}, \tag{5.26}$$

which is

$$Y(s) = e^{-2s}Z(s) - e^{-15s}Z(s), \tag{5.27}$$

where $Z(s) = (s(2s^2 + s + 1))^{-1}$ that after inverting gives

$$z(t) = 1 - \frac{1}{\sqrt{7}}e^{-t/4}\left(\sqrt{7}\cos\frac{\sqrt{7}}{4}t + \sin\frac{\sqrt{7}}{4}t\right). \tag{5.28}$$

Finally, using (5.24)

$$y(t) = H_2(t)z(t-2) - H_{15}(t)z(t-15), \tag{5.29}$$

which is presented on Fig. 5.25. Notice that the function oscillates freely after switching on the force at $t = 2$ and then, suddenly falls at turning it off at $t = 15$. The remaining vibrations decay with a time scale 4. □

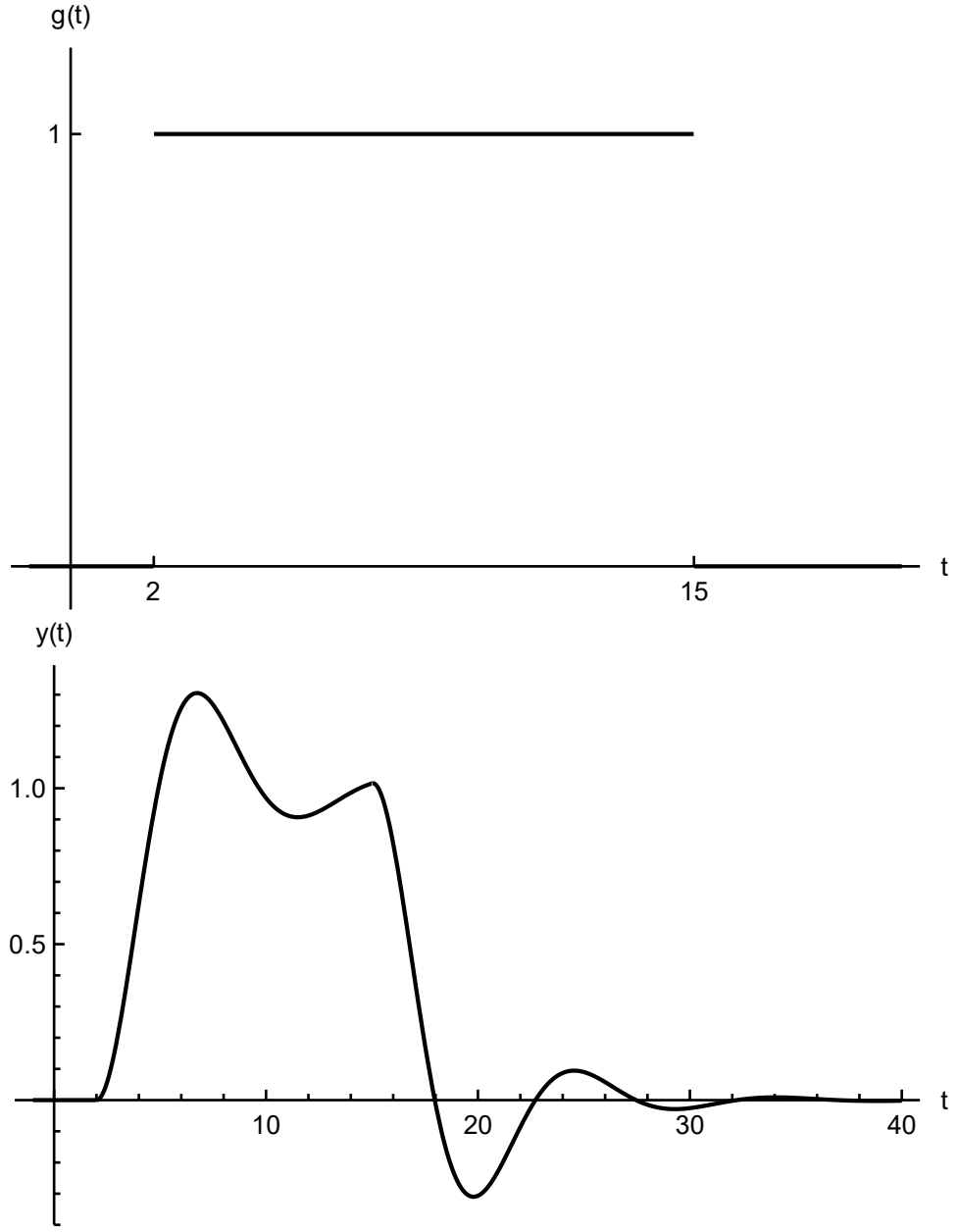

Figure 20: The causal signal as in (5.25) (top) and the solution (5.29) (bottom).

The piecewise step functions can be very useful in modelling sudden switching behaviour in electrical circuits. However, doing that we are quickly being led to a daunting task of calculating the derivative of such a function. For example, suppose we have an RLC circuit with a current source. If we suddenly turn the current on, that is $I = I(t)$ is a step function, the voltage drop on the inductor is LdI/dt. This derivative does not exists in the classical sense. For then, a step function is piecewise constant

and it has a derivative equal to zero except at one point. However, it makes a profound effect on our circuit. Other examples can be found in mechanics.

Example. (*Unit impulse*) Imagine a situation when a point mass is momentarily and very briefly struck by another one. We can think about bouncing billiard balls or a piano hammer striking the string. The latter can be modelled by a linear oscillator with forcing

$$my'' + \gamma y' + ky = f_{\Delta t}(t), \tag{5.30}$$

where $f_{\Delta t}$ models the strike with duration Δt. Since the exchange of the momentum P is completed in a very short interval of time Δt we have

$$f(t) = \begin{cases} \dfrac{P}{\Delta t}, & 0 < t < \Delta t, \\ 0, & t \geq \Delta t. \end{cases} \tag{5.31}$$

Then, the total momentum at time t is

$$p_{\Delta t}(t) = \int_0^t f(s)ds = \begin{cases} 0, & t \leq 0, \\ \dfrac{Pt}{\Delta t}, & 0 < t \leq \Delta t, \\ P, & t > \Delta t. \end{cases} \tag{5.32}$$

In the limit $\Delta t \to 0^+$ we have

$$p(t) = \int_0^t f(s)ds = \begin{cases} 0, & t \leq 0, \\ P, & t > 0, \end{cases} \tag{5.33}$$

which is the Heaviside function $PH_0(t)$. However, in that limit the force $f_{\Delta t}$ becomes infinite at $t = 0$ and zero for $t \neq 0$. Therefore, the short impulse of momentum can be modelled by an idealized spike of force. □

To train our intuition about the possible generalized derivative of a step function consider its smooth approximations. On Fig. 21 we see several smooth approximations of the Heaviside function. They are becoming steeper with each iteration. Steeper the function, the larger the derivative. However, if H_n denotes the approximation we have

$$\int_{-\infty}^{\infty} H_n'(x)dx = H_n(\infty) - H_n(-\infty) = 1 - 0 = 1, \tag{5.34}$$

and hence, the integral of the derivative is always constant. The area under the spike is always the same regardless its size. This is because the higher the impulse, the more concentrated it is near zero. In the limit H_n' would be an infinitely high spike centred at $x = 0$. This clearly cannot be a function.

Nevertheless, the concept of such an impulse function dates back even to Fourier. Then, Heaviside and other engineers used it as a useful tool in solving problems in electricity. Later, Dirac gave it a working definition, called the *Dirac delta*, and revolutionized formulation of quantum mechanics. All these practitioners were right

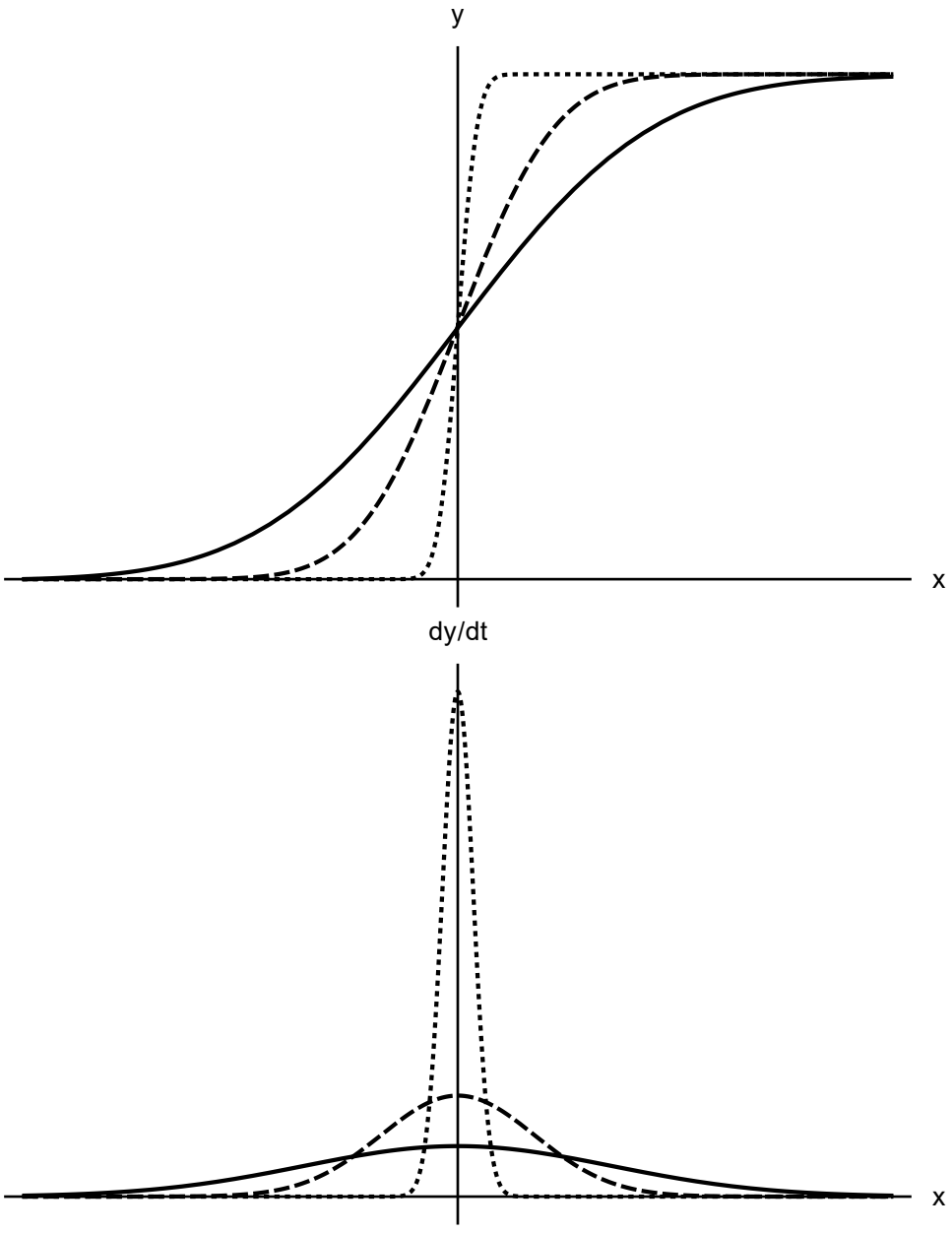

Figure 21: Exemplary approximations to the Heaviside functions (top) and their derivatives (bottom).

- there exists something bizarre as a derivative of a unit step. But it was only until Laurent Schwartz to give a rigorous mathematical definition of the derivative of it. It appears that there is a larger and more general than functions class of mappings - the distributions (also called generalized functions even though they are not functions in general). They are continuous functionals over a space of the so-called test functions. The continuity is regarded in the sense of a precisely given topology and the space of test functions is also clearly defined (for example they have to be infinitely smooth). Technical details of distribution theory is, unfortunately, beyond the scope of our elementary lecture on ODEs and we have to abstain from giving a rigorous definition here. However, we will present a working, non-rigorous way of summarizing our observations.

Definition 16. ***Dirac delta*** *is an object δ (a "generalized function") satisfying*

1. *(Concentration near zero)*
$$\delta(t) = 0 \quad \textit{for} \quad t \neq 0. \tag{5.35}$$

2. *(Normalization)*
$$\int_{\mathbb{R}} \delta(t)dt = 1. \tag{5.36}$$

If δ were a function the first property would be in contradiction with the second. For then, a function that is zero almost everywhere has a zero integral. Moreover, the second property can be generalized to the so-called *filtering* or *sifting* property

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(t-t_0)f(t) &= \int_{-\infty}^{\infty} \delta(t-t_0)f(t_0)dt + \int_{-\infty}^{\infty} \delta(t-t_0)(f(t)-f(t_0))dt \\ &= f(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)dt = f(t_0),\end{aligned} \tag{5.37}$$

since $\delta(t-t_0)(f(t)-f(t_0)) = 0$. We will use this properties to find the Laplace transform of Dirac delta. From the definition (5.1) we have

$$\mathcal{L}\{\delta(t)\}(s) = \int_0^{\infty} \delta(t)e^{-st}dt = e^{-st}|_{t=0} = 1, \tag{5.38}$$

and the Laplace transform is constant. Now, we are able to solve differential equations with impulsive forcing.

Example. Let us go back to (5.30) and for simplicity assume that $m = k = 1 = P$ and $\gamma = 0$. We have

$$y'' + y = \delta(t), \tag{5.39}$$

which models an idealized oscillator forced by an unit impulse at $t = 0$. Assume that we have zero initial conditions. Taking the Laplace transform when $t > 0$ leads to

$$Y(s) = \frac{1}{s^2+1}, \tag{5.40}$$

which yields

$$y(t) = \begin{cases} \sin t, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{5.41}$$

which is continuous with discontinuous derivative. Taking second derivative produces the Dirac delta. □

5.3 The convolution (optional)

There is a fundamental mathematical operation known as *convolution* that is closely associated with Laplace transform.

Definition 17. *The (one-sided)* ***convolution*** *of two functions* f *and* g *is defined by*

$$f * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau, \tag{5.42}$$

provided the integral exists.

The convolution integral above arises in many applied situations where the past values of some quantity affect the present state. We can meet convolution in biology, hydrology, physics, and engineering. It can be thought as a generalized linear superposition of a functions values at different time.

$$\sum_{j=1}^{n} f(t_j)g_j \approx \int_0^t f(\tau)g(t-\tau)d\tau, \tag{5.43}$$

where g_j are some weights.

There are times when we are dealing with products of transforms. It would be an error to think that the original of such is the product of originals, that is in general

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}. \tag{5.44}$$

However, there is a remarkable result joining products of Laplace transform with convolutions. First, we consider an important example.

Example. (*Transfer function*) In many fields of engineering, such as electronics, one considers complicated systems that process signals (e.g. filters). Usually, the modelling can be done with a black-box that takes an input signal f and changes it into output signal y. If the system is linear this operation can be written as a system of linear ODEs with constant coefficients. For simplicity, with a second order equation we would have

$$ay'' + by' + cy = f(t). \tag{5.45}$$

Taking the Laplace transform we would obtain

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}. \tag{5.46}$$

The first term is associated only with the system itself, while the second is the external input. By linearity we can focus only on the latter and deduce that in order to find the system response we should be able to invert the product $H(s)F(s)$, where

$$H(s) = \frac{1}{as^2 + bs + c}, \tag{5.47}$$

is called the **transfer function** since it depends only on the characteristics of the system. The inversion, i.e. $h(t) = \mathcal{L}^{-1}\{H(s)\}$ is then called **impulse response**. □

The main result is the following.

Theorem 9 (Convolution theorem)**.** *Let* f *and* g *be piecewise functions of exponential order. Then, if* $H(s) = F(s)G(s)$ *with* $h(t) = \mathcal{L}^{-1}\{H(s)\}$ *we have*

$$h(t) = f * g(t). \tag{5.48}$$

Proof. From the definition of the Laplace transform (5.1) we have

$$F(s)G(s) = \left(\int_0^\infty f(t)e^{-st}dt\right)\left(\int_0^\infty g(\tau)e^{-s\tau}dt\right), \tag{5.49}$$

which can be written as an iterated integral

$$F(s)G(s) = \int_0^\infty g(\tau)d\tau \int_0^\infty f(t)e^{-s(t+\tau)}dt. \tag{5.50}$$

Now, a change of the variable $\xi = t + \tau$ yields

$$F(s)G(s) = \int_0^\infty g(\tau)d\tau \int_t^\infty f(\xi - \tau)e^{-s\xi}d\xi. \tag{5.51}$$

When we use Fubini Theorem and interchange the order of integration we obtain

$$F(s)G(s) = \int_0^\infty e^{-s\xi}d\xi \int_0^t f(\xi - \tau)g(\tau)d\tau, \tag{5.52}$$

or

$$F(s)G(s) = \int_0^\infty \left(\int_0^t f(\xi - \tau)g(\tau)d\tau\right)e^{-s\xi}d\xi, \tag{5.53}$$

which is what we had to prove. □

Therefore, the product of Laplace transforms corresponds to a convolution of the originals. We will end this section with a very important historical example that can neatly be solved with the use of convolution theorem.

Example. (*The tautochrone*) A classical problem in mechanics is to determine a shape of the curve over which a particle will slide without friction and under the gravity to the bottom in the same time regardless of the initial position. It arose when constructing pendulum clock that were suppose to work on ships. There were many approaches

to find the the tautochrone (from Green - "same time"): Huygens, Leibnitz, Jacob Bernoulli, and Abel. We will follow that of Abel.

Suppose the particle starts at height y_0 and slides over a curve $y = y(x)$. The conservation of mechanical energy gives

$$\frac{1}{2}m\left(\frac{dl}{dt}\right)^2 = mg(y_0 - y), \tag{5.54}$$

where ds/dt is the tangential velocity. Therefore,

$$\frac{dl}{dt} = -\sqrt{2g(y_0 - y)}, \tag{5.55}$$

where the minus sign denotes the velocity pointing downwards. Upon integration it follows that the time of fall equals

$$T = -\int \frac{dl}{\sqrt{2g(y_0 - y)}} = \int_0^{y_0} \frac{1}{\sqrt{2g(y_0 - y)}} \frac{dl}{dy} dy, \tag{5.56}$$

where in the last equality we have substituted $l = l(y)$, used the chain rule, and interchanged the endpoints of integration. The above is called the Abel integral equation since it requires us to find dl/dt provided we know T.

For a tautochrone we have $T =$const. and we can take the Laplace transform of the equation. The left-hand side is easy

$$\mathcal{L}\{T\} = \frac{T}{s}, \tag{5.57}$$

while on the right-hand side of the integral equation we notice the convolution. Therefore, thanks to Theorem 9 we know that the Laplace transform will be a product of two transforms, hence

$$\frac{T}{s} = \frac{1}{2g}\mathcal{L}\left\{\frac{1}{\sqrt{y}}\right\}\mathcal{L}\left\{\frac{dl}{dy}\right\}. \tag{5.58}$$

What remains it to compute the Laplace transform of $1/\sqrt{y}$ which is

$$\mathcal{L}\left\{\frac{1}{\sqrt{y}}\right\} = \int_0^\infty y^{-\frac{1}{2}} e^{-sy} dy = \frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2} dx = \sqrt{\frac{\pi}{s}}, \tag{5.59}$$

where we substituted $x = sy$ and used the Poisson's integral. Therefore,

$$\mathcal{L}\left\{\frac{dl}{dy}\right\} = \sqrt{\frac{2g}{\pi}}\sqrt{s}\frac{T}{s} = \sqrt{\frac{2g}{\pi}}\frac{T}{\sqrt{s}}. \tag{5.60}$$

Whence, using (5.59) again we arrive at

$$\frac{dl}{dy} = \frac{T\sqrt{2g}}{\pi}\frac{1}{\sqrt{y}}. \tag{5.61}$$

Finally, using the well-known formula for a differential of arc-length we have

$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \frac{T\sqrt{2g}}{\pi}\frac{1}{\sqrt{y}}. \tag{5.62}$$

The solution of the above is called the *cycloid* and is a curve that is traced by a bicycle valve when riding with constant speed. It can be obtained in a parametric form. Therefore, the tautochrone is an arc of a cycloid. □

5.4 Uniqueness of the Laplace transform (optional)

We end this section with a proof that the Laplace transform is uniquely associated with a given function. That is, the Laplace transform operator is one to one. Before we proceed to the main result, we need a fundamental theorem from calculus.

Theorem 10 (Weierstrass)**.** *If* $f : [a, b] \to \mathbb{R}$ *is continuous, then for each* $\epsilon > 0$ *there exists a polynomial* $P = P_\epsilon(t)$ *such that*

$$\max_{t\in[a,b]} |f(t) - P_\epsilon(t)| < \epsilon. \tag{5.63}$$

This is the famous Weierstrass approximation theorem stating that a continuous function on a closed and bounded interval can be uniformly approximated by polynomials. Next, we prove the following lemma.

Lemma 1. *Let* $f = f(t)$ *be a continuous function defined on* $[0, 1]$*. If for each* $n \in \mathbb{N}$ *we have*

$$\int_0^1 f(t)t^n dt = 0 \tag{5.64}$$

then it follows that $f \equiv 0$*.*

Proof. Fix $\epsilon > 0$ and from Weierstrass theorem let us take a polynomial P_ϵ. Since f is a continuous function on a bounded interval we have $|f(t)| \leq M$ for some $M > 0$. Then, from the assumption and linearity of the integral we have $\int_0^1 f(t)P_\epsilon(t)dt = 0$. We can also write

$$\int_0^1 f(t)^2 dt = \int_0^1 f(t)(f(t)-P_\epsilon(t))dt + \int_0^1 f(t)P_\epsilon(t)dt \leq M \max_{t\in[a,b]} |f(t)-P_\epsilon(t)| < M\epsilon. \tag{5.65}$$

Since ϵ is arbitrary we have $\int_0^1 f(t)^2 dt = 0$, which forces $f \equiv 0$. □

Finally, we state the main result.

Theorem 11. *Let* f, g *be continuous functions on* $[0, \infty)$ *of exponential order. Then* $\mathcal{L}\{f\} = \mathcal{L}\{g\}$ *forces* $f = g$*.*

Proof. From the linearity of the Laplace transform it suffices to show that $\mathcal{L}\{f\} = 0$ implies $f \equiv 0$. Assume that

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt = 0, \quad s > a, \tag{5.66}$$

for some $a \in \mathbb{R}$. Fix an arbitrary constant $s_0 > a$. After substitution $x = e^{-t}$ we obtain

$$\mathcal{L}\{f\}(s) = \int_0^1 f(-\ln x)x^{s-1}dx = 0. \tag{5.67}$$

Let now $s = s_0 + n + 1$ for any fixed $n \in \mathbb{N}$. Then,

$$\int_0^1 [f(-\ln x)x^{s_0}]\, x^n dx = 0. \tag{5.68}$$

Since $f(-\ln x)x^{s_0}$ is continuous (f is of exponential order) from the above lemma we have $f \equiv 0$. □

The above theorem can also be proved under the assumption that f and g are merely piecewise continuous. However, the proof is a little bit more technical and we omit it.

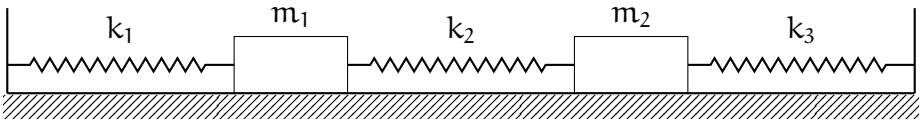

Figure 22: A cartoon of the three springs problem.

6 Systems of equations

In modelling we are frequently in need of describing interactions between several different species or objects. Not only each of them has its own dynamics, all of them can interact with each other producing interesting effects. Gravitational attraction between masses or coexistence of various animal populations are prominent examples.

Example. (*The* n *body problem*) One of the most important and profound tasks in mathematical physics is to study the gravitational attraction of n point masses. The main problem is to determine the future motion of a set of particles provided we know their position at some time. Given n masses $\{m_i\}_{i=1}^n$ we can write the Newton equations of motion in $\mathbb{R}^3$

$$m_i \frac{d^2\mathbf{x}_i}{dt^2} = G \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_i m_j}{\|\mathbf{x}_j - \mathbf{x}_i\|^2} \frac{\mathbf{x}_j - \mathbf{x}_i}{\|\mathbf{x}_j - \mathbf{x}_i\|}, \quad 0 \leq i \leq n. \tag{6.1}$$

Here, $\mathbf{x}_i \in \mathbb{R}^3$ denotes the position of the i-th particle. The above system of second order ordinary equations with variable coefficients is a basis of *celestial mechanics* and attracted attention of almost every great mathematician from the times of Newton. The two body problem can completely be solved (Kepler's problem). Moreover, there is also a lot of knowledge of three body problem. For larger number of masses, the n body problem poses a great difficulty both for analytical and numerical solutions. □

Example. (*Three springs*) Two masses are interconnected by three springs (see Fig. 22). Assuming no friction and Hooke's Law we will find the equations of motion. Let x_i denote the distance from the equilibrium position of the i-th spring with constant k_i. This means that the spring elongations are, respectively: x_1, $x_2 - x_1$, and $-x_2$. From Hooke's Law we thus have

$$\begin{cases} m_1 \dfrac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1), \\ m_2 \dfrac{d^2x_1}{dt^2} = -k_2(x_2 - x_1) - k_3x_2, \end{cases} \tag{6.2}$$

which is a system of two ordinary linear equations of the second order. □

Example. (*Electrical circuit*) Systems of linear ODEs arise very frequently in analysis large electrical circuits. They are a fundamental tool for each electrical engineer and along Laplace transform they constitute the basis of circuit theory.

A simple example of a parallel RLC circuit is presented on Fig. 23. As usual, we use Kirchhoff's Law to obtain a balance of voltage and currents in the loop. Its states that

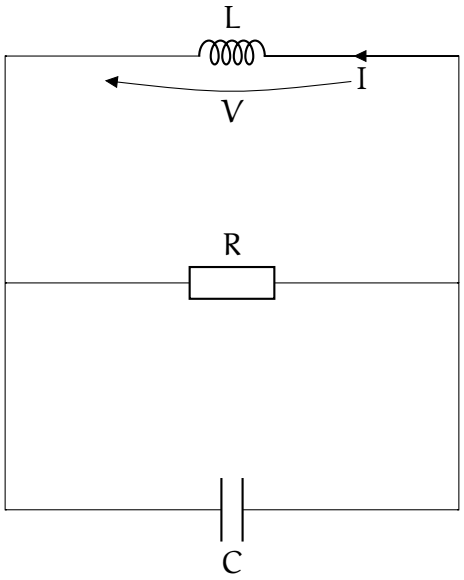

Figure 23: A schematic of an electrical circuit.

the sum of currents in each subloop is equal to zero. By I and V we denote the current and voltage drop across the inductor, and similarly I_R, I_C. From elementary physics concerning electrical circuits we know the constitutive relations for each element

$$\begin{cases} I_R &= \dfrac{V}{R}, \\ I_C &= C\dfrac{dV}{dt}, \\ \dfrac{dI}{dt} &= \dfrac{V}{L}, \end{cases} \tag{6.3}$$

where R is the resistance, C capacity, and L inductance. Therefore, from Kirchhoff's Law we have

$$I_C = -I - I_R \quad \to \quad C\frac{dV}{dt} = -I - \frac{V}{R}. \tag{6.4}$$

From which we can find the change in the voltage

$$\frac{dV}{dt} = -\frac{I}{C} - \frac{V}{RC}. \tag{6.5}$$

The equation for the change of the current follows from the constitutive relation for the inductor

$$\frac{dI}{dt} = \frac{V}{L}. \tag{6.6}$$

Whence, two above equations are a closed system of linear first order equations with constant coefficients. Application of Kirchhoff's Laws can be generalized for multiple loops and more complex electrical networks. □

Example. (*Population interactions*) Let $x = x(t)$ and $y = y(t)$ denote numbers of two populations living on the same area. In general, their change can be modelled by a

system

$$\begin{cases} \dfrac{dx}{dt} = xF(x,y), \\ \dfrac{dy}{dt} = yG(x,t), \end{cases} \tag{6.7}$$

where F and G model interactions. For example, if $F = -G = 1$ we would have two independent populations from which x grows and y dies exponentially. The interesting cases arise when there is a nontrivial interaction. Probably the simplest model of two populations from which one preys on the other is due to Lotka and Volterra

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy, \\ \dfrac{dy}{dt} = \gamma y + \delta xy, \end{cases} \tag{6.8}$$

where α, β, γ, and δ are positive constants. The product xy models interaction since it is positive only when both populations are present. The interaction is positive for predators y, and negative for prey x. The growth, on the other hand, is positive for prey, and negative for predators. This makes sense since without the food (i.e. $x = 0$), predators should die out. But when there is plenty of prey, the amount of predators increases proportionally to the product xy - more hunters can acquire more food. This further can be generalized to become a more realistic description of interacting species. For example, we can assume logistic growth of prey and saturation of hunting abilities for predators (there should be a limit of the food gathering abilities with respect to the population number). Within the framework of our model we can also describe competing or symbiotic species. Possibilities are infinite and they constitute the basics of mathematical ecology. □

Example. (*A n order equation is equivalent to a system of n first order equations*) Let us consider an equation

$$y^{(n)}(t) = F(t, y, y', ..., y^{(n)}). \tag{6.9}$$

Then, after substitution

$$x_1 = y, \quad x_2 = y', \quad x_3 = y'', \quad \cdots \quad x_n = y^{(n-1)}, \tag{6.10}$$

we obtain

$$\begin{cases} x_1' = y' = x_2, \\ x_2' = y'' = x_3, \\ \cdots \\ x_n' = y^{(n)} = F(t, x_1, x_2, ..., x_n), \end{cases} \tag{6.11}$$

which is a system of first order, possibly nonlinear, equations. This technique of transforming a single equation into a system is very useful not only in theoretical considerations. □

6.1 Definitions

As usual, we will start our investigations from analysis systems of linear ODEs. They will greatly aid us in understanding the more difficulty nonlinear ones. We start with formal definitions.

Definition 18. *A an initial value problem for a* ***system of ordinary differential equations*** *of order* n *is*

$$\begin{cases} x_1' = F_1(t, x_1, \cdots, x_n), \\ x_2' = F_2(t, x_1, \cdots, x_n), \\ \cdots \\ x_n' = F_n(t, x_1, \cdots, x_n). \end{cases} \quad \textit{with initial conditions} \quad \begin{cases} x_1(0) = x_1^0, \\ x_2(0) = x_2^0, \\ \cdots \\ x_n(0) = x_n^0. \end{cases} \tag{6.12}$$

Functions F_i *along with initial conditions are given.*

The above is the most general system of n differential equations. Under the assumption of continuity of F_j and $\partial F_i/\partial x_j$ it can be shown that the above problem has a unique local solution (see Sec. 2.7). The linear systems arise when all F_j are linear with respect to the unknowns y_i. In what follows with a bold face font we will denote vectors and use matrix notation to simplify visual appearance.

Definition 19. *A* ***linear system*** *has the form*

$$\boldsymbol{x}' = P(t)\boldsymbol{x} + \boldsymbol{f}, \tag{6.13}$$

where $\boldsymbol{x} = (x_1, x_2, ..., x_n)^T$ *and*

$$P(t) = \begin{pmatrix} p_{11}(t) & p_{21}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1}(t) & p_{2n}(t) & \cdots & p_{nn}(t) \end{pmatrix}, \quad \boldsymbol{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \cdots \\ f_n(t) \end{pmatrix}. \tag{6.14}$$

When $\boldsymbol{f} \equiv 0$ *the system is* ***homogeneous*** *and otherwise it is* ***nonhomogeneous****. If* $P(t) =$ *const. then the system has* ***constant coefficients****.*

A linear system, similarly to a n-th order equation, has n linearly independent solutions

$$\mathbf{x}^{(k)} = (x_1^{(k)}, \cdots, x_n^{(k)}), \tag{6.15}$$

where $1 \le k \le n$. The Wronskian is then

$$W(x_1, ..., x_n) = \det \begin{pmatrix} x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(n)} \\ \vdots & \vdots & \cdots & \vdots \\ x_n^{(1)} & x_n^{(2)} & \cdots & x_n^{(n)} \end{pmatrix}. \tag{6.16}$$

Whence, the general solution can always be written as

$$\mathbf{x} = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}, \tag{6.17}$$

which is very similar to the case of a single equation of second order.

In what follows we will limit ourselves to studying systems of two equations. This is due to the fact that they are the most basic block of the theory and are found in a multitude of different applied situations. One of the most useful methods of studying, especially nonlinear, system is the geometrical method that we have already met with first order equations. Here, it is the very place when this ingenious idea of Poincaré proves to be indispensable.

Definition 20. *A **phase space** of a system (6.12) is the collection of all trajectories of its solutions corresponding to all initial conditions. That is, if $\boldsymbol{x}$ is a solution of (6.12) with a given initial condition, then the **trajectory***

$$\{(x_1(t), x_2(t), ..., x_n(t)) \in \mathbb{R}^n : t \in \mathbb{R}\} \tag{6.18}$$

*is a element of the phase space[26]. For a two-dimensional case the phase space becomes the **phase plane**. We will also denote $x = x_1$ and $y = x_2$.*

The nomenclature is taken from thermodynamics where the phase space consists of all trajectories of all possible states of a system. An example is presented on Fig. 24 where arrows indicate the direction of the solution when t increases. This method has a virtue of being completely geometrical where many quantitative features of the solution are evident. For example, the figure shows circles in the centre which correspond to periodic solutions. This could be very difficult to prove otherwise. Note also that using phase plane analysis does not require solving the system (which can very rarely be accomplished).

In two dimensions the situation is a little bit easier. Even without solving the nonlinear system

$$\begin{cases} x' = F(x, y), \\ y' = G(x, y), \end{cases} \tag{6.19}$$

we can write an ODE describing its trajectories

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{G(x, y)}{F(x, y)}. \tag{6.20}$$

Solving it for different initial conditions yields the complete phase plane.

6.2 Linear systems with constant coefficients

Here, we will focus on a complete analysis of homogeneous linear systems with constant coefficients which can be written as

$$\mathbf{x}' = A\mathbf{x}, \tag{6.21}$$

where A is a 2×2 constant matrix. The nonhomogeneous systems can be dealt with essentially the same techniques as single second order equations (see Problems). Similarly to them, we look for exponential solutions

$$\mathbf{x} = \xi e^{\lambda t}, \tag{6.22}$$

[26] It is a parameterically given curve in the $x_1 - x_2 - ... - x_n$ plane.

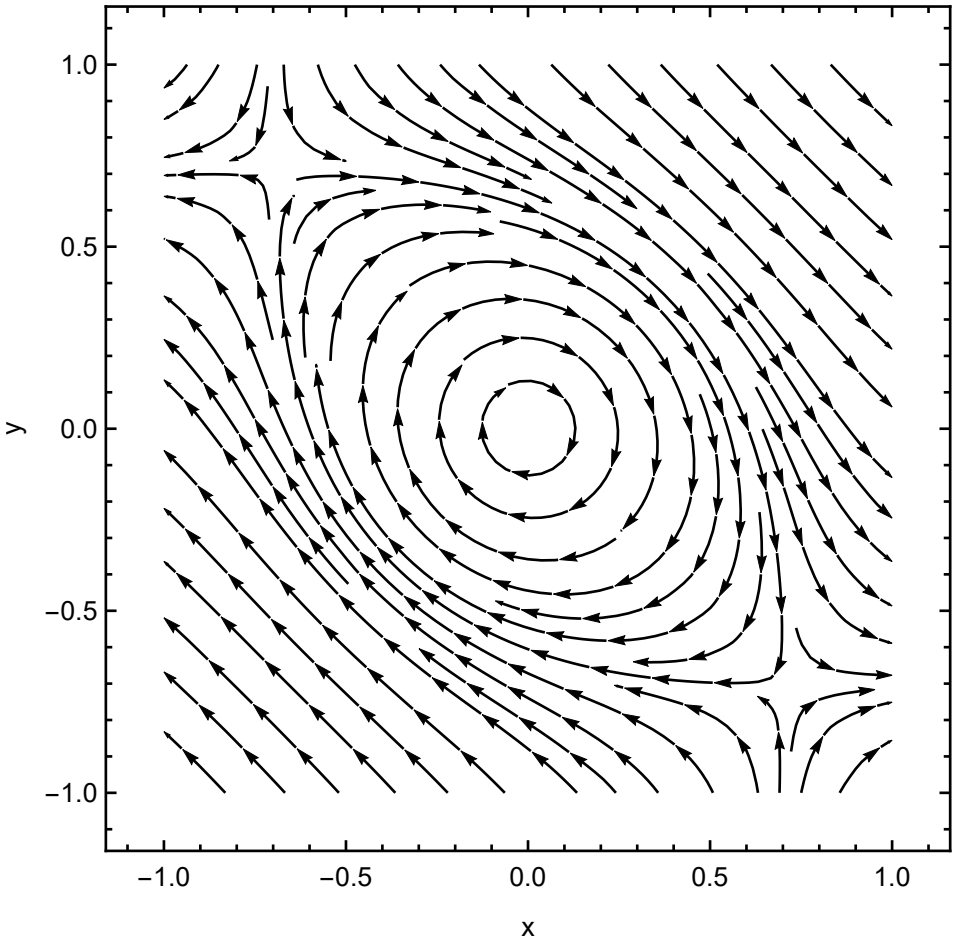

Figure 24: An exemplary phase plane.

where ξ and λ are to be found. When we plug the above ansatz into (6.21) we obtain

$$A\xi = \lambda\xi, \tag{6.23}$$

which is an *eigenvalue* problem for the matrix A. Therefore, ξ is an eigenvector with corresponding eigenvalue λ. Whence, we are left in considering various cases.

- *Two real and distinct eigenvalues*. In that case the general solution is

$$\mathbf{x}(t) = C_1\xi^{(1)}e^{\lambda_1 t} + C_2\xi^{(2)}e^{\lambda_2 t}, \tag{6.24}$$

where $\xi_{1,2}$ are eigenvectors of A with corresponding eigenvalues $\lambda_{1,2}$.

Example. For a system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\mathbf{x} = A\mathbf{x}, \tag{6.25}$$

we have

$$(A - \lambda I) = 0 \iff 0 = \det\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda - 3. \tag{6.26}$$

Therefore, $\lambda_1 = 3$ and $\lambda_2 = -1$. The first eigenvector is calculated as follows

$$0 = (A - 3I)\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \to \xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{6.27}$$

Similarly, the second one is

$$\xi^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{6.28}$$

The general solution is then

$$\mathbf{x}(t) = C_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} + C_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t}. \tag{6.29}$$

The phase plane can we drawn remembering that we have to include all parametrically given curves, that is, for each constant $C_{1,2}$ we draw

$$x(t) = C_1e^{3t} + C_2e^{-t}, \quad y(t) = 2C_1e^{3t} - 2C_2e^{-t}. \tag{6.30}$$

This can be done either with the help of elementary calculus techniques or with the aid of a computer. Moreover, note that we can distinguish two special curves: one for $C_1 = 0$ and the other for $C_2 = 0$

$$\begin{cases} x(t) = C_2e^{-t}, \\ y(t) = -2C_2e^{-t}, \end{cases} \qquad \begin{cases} x(t) = C_1e^{3t}, \\ y(t) = 2C_1e^{3t}, \end{cases} \tag{6.31}$$

or

$$y = -2x, \quad \text{or} \quad y = 2x. \tag{6.32}$$

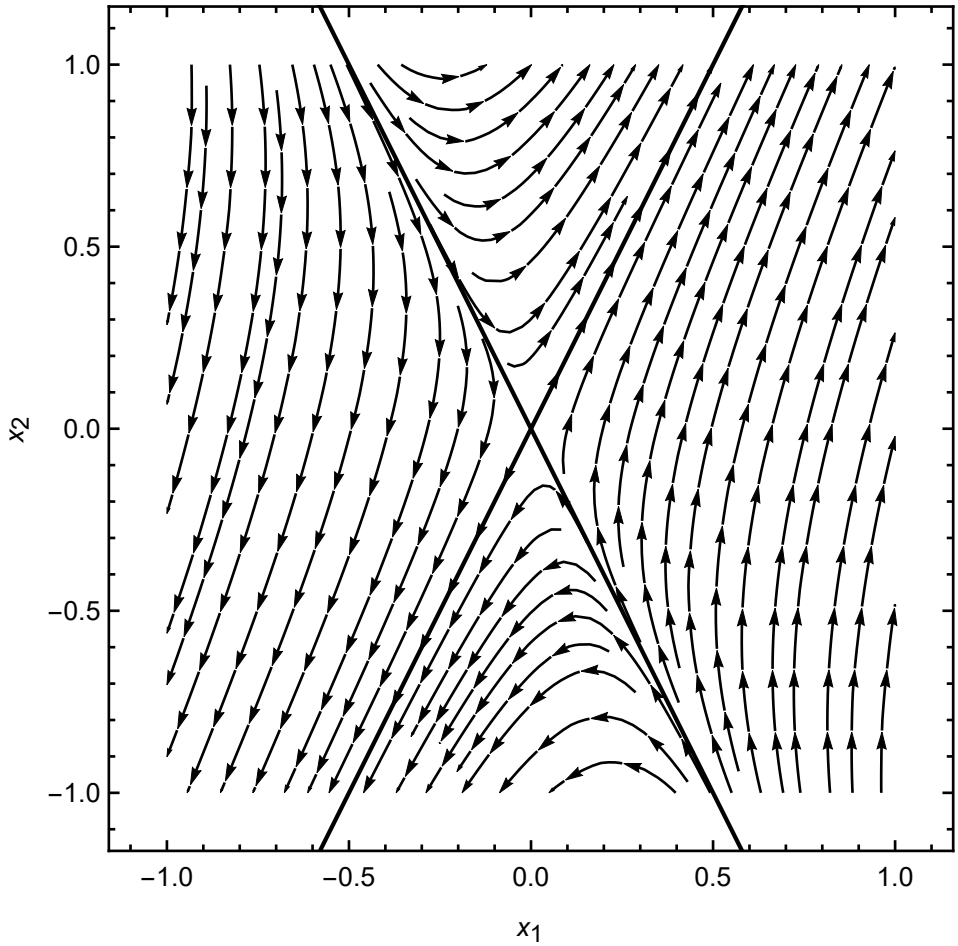

Figure 25: A phase portrait of the solution (6.30). The separatrices are drawn in bold.

Note that these lines are generated by eigenvectors and are called *separatrices* or *invariant manifolds*. The full depiction of the phase plane is presented on Fig. 25. Note that trajectories approach the origin but then are suddenly repelled to infinity. □

Two distinct real eigenvalues
In general, for the case of two distinct real eigenvalues we can have three cases. Depending on the sign of eigenvalues $\lambda_{1,2}$. The typical phase portraits are depicted on Fig. 26.

1. $\lambda_1, \lambda_2 < 0$. The solutions will approach the origin when $t \to \infty$. The origin is then called the **stable node**.
2. $\lambda_1\lambda_2 < 0$. The solutions approach and then are repelled from the origin as $t \to \pm\infty$. The origin is then called the **saddle point**.
3. $\lambda_1, \lambda_2 > 0$. The solutions will approach the origin when $t \to -\infty$. The origin is then called the **unstable node**.

- *Two conjugate complex eigenvalues*. We have then $\lambda_{1,2} = \mu \pm i\omega$ and $\boldsymbol{\xi}^{(1,2)} = \mathbf{a} \pm i\,\mathbf{b}$. Similarly as in the case of second order equations, it leads to complex solutions

$$\begin{aligned}\mathbf{x}^{(1,2)}(t) &= \boldsymbol{\xi}^{(1,2)} e^{(\mu\pm i\omega)t} = e^{\mu t}\,(\mathbf{a} \pm i\mathbf{b})\,(\cos\omega t \pm i \sin\omega t) \\ &= e^{\mu t}\,[\mathbf{a}\cos\omega t - \mathbf{b}\sin\omega t \pm i\,(\mathbf{a}\sin\omega t + \mathbf{b}\cos\omega t)]\,.\end{aligned} \tag{6.33}$$

In order to obtain real functions we may take a linear combination of the above. For example, we can extract real and imaginary parts to obtain oscillatory solutions

$$\begin{aligned}\mathbf{x}^{(3)} &= \frac{1}{2}\left(\mathbf{x}^{(1)} + \mathbf{x}^{(2)}\right) = e^{\mu t}\,(\mathbf{a}\cos\omega t - \mathbf{b}\sin\omega t)\,, \\ \mathbf{x}^{(4)} &= \frac{1}{2i}\left(\mathbf{x}^{(1)} - \mathbf{x}^{(2)}\right) = e^{\mu t}\,(\mathbf{a}\sin\omega t + \mathbf{b}\cos\omega t)\,.\end{aligned} \tag{6.34}$$

The general solution is therefore

$$\mathbf{x}(t) = C_1\mathbf{x}^{(3)} + C_2\mathbf{x}^{(4)}. \tag{6.35}$$

Note that the real part of the eigenvalues tells us about the stability (decaying amplitude) while the imaginary part corresponds to the frequency of oscillations.

Example. For a system

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix}\mathbf{x}, \tag{6.36}$$

the eigenvalues are $\lambda_{1,2} = -\frac{1}{2} \pm i$, and eigenvectors

$$\boldsymbol{\xi}^{(1,2)} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.37}$$

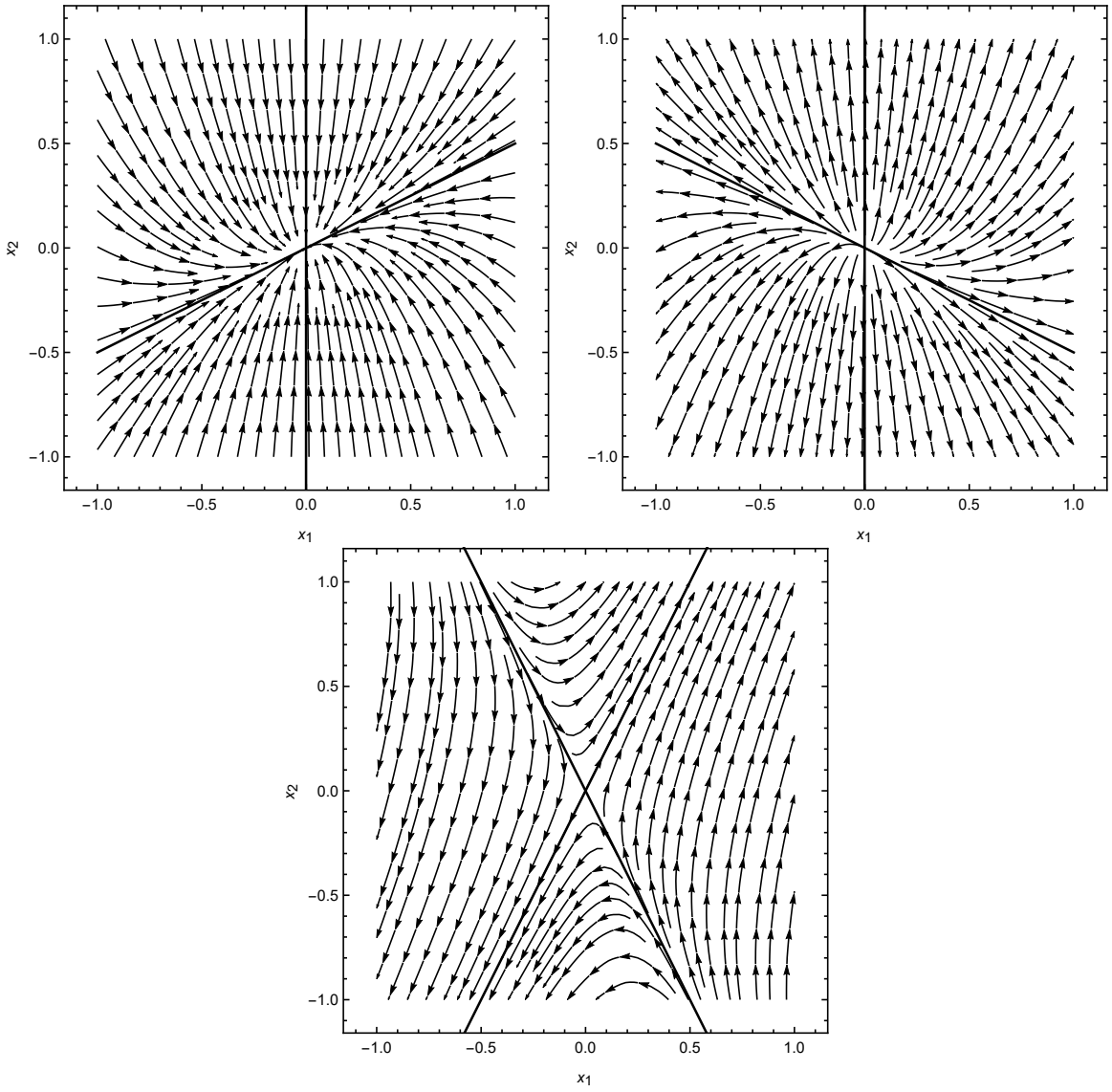

Figure 26: Different types of phase portraits for two distinct real eigenvalues. Top: **stable node** (λ_1, $\lambda_2 < 0$), and **unstable node** (λ_1, $\lambda_2 > 0$). Bottom: **saddle point** ($\lambda_1 < 0 < \lambda_2$).

Fundamental solutions are then

$$\begin{aligned}\mathbf{x}^{(1)}(t) &= e^{-\frac{1}{2}t}\left(\begin{pmatrix}1\\0\end{pmatrix}\cos t - \begin{pmatrix}0\\1\end{pmatrix}\sin t\right),\\ \mathbf{x}^{(2)}(t) &= e^{-\frac{1}{2}t}\left(\begin{pmatrix}1\\0\end{pmatrix}\sin t + \begin{pmatrix}0\\1\end{pmatrix}\cos t\right).\end{aligned} \tag{6.38}$$

The trajectory corresponding to $\mathbf{x}^{(1)}(t)$ has coordinates

$$\begin{cases} x_1(t) = e^{-\frac{1}{2}t}\cos t,\\ x_2(t) = -e^{-\frac{1}{2}t}\sin t.\end{cases} \tag{6.39}$$

We thus have $x_1^2 + x_2^2 = e^{-t}$ which is an equation for a spiral - the radius decreases exponentially with time. The direction of rotation can be found from the system itself since the derivative $\mathbf{x}'$ is the tangent vector. For example, if we take $(x_1, x_2) = (1, 0)$ we obtain

$$\mathbf{x}'(1,0) = \begin{pmatrix}-\frac{1}{2} & 1\\ -1 & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\-1\end{pmatrix}, \tag{6.40}$$

therefore, the tangent vector to a trajectory at $(1, 0)$ points into down-left direction. The phase portrait is presented on Fig. 27. For the complex eigenvalues case there always will be some oscillations. □

Two complex eigenvalues
In general, for the case of two distinct real eigenvalues we can have three cases. Depending on the sign of $\mu = \operatorname{Re}\lambda_{1,2}$. The typical phase portraits are depicted on Fig. 28.

1. $\mu < 0$. The solutions will approach the origin when $t \to \infty$. The origin is then called the **stable spiral (focus)**.
2. $\mu = 0$. The solutions will not approach nor be repelled from the origin when $t \to \pm\infty$. The origin is then called the **centre**.
3. $\mu > 0$. The solutions will approach the origin when $t \to -\infty$. The origin is then called the **unstable spiral (focus)**.

- *Repeated real eigenvalues and one eigenvector*. Immediately we have one fundamental solution

$$\mathbf{x}^{(1)}(t) = \xi e^{\lambda t}. \tag{6.41}$$

The other is sought in the form

$$\mathbf{x}^{(2)}(t) = \xi t e^{\lambda t} + \eta e^{\lambda t}, \tag{6.42}$$

where we added t (similarly as in the case of second order equations) and the term with η. The latter is crucial. We have

$$\underline{te^{\lambda t}A\xi} + e^{\lambda t}A\eta = A\mathbf{x}^{(2)} = \frac{d\mathbf{x}^{(2)}}{dt} = \xi e^{\lambda t} + \underline{\lambda\xi t e^{\lambda t}} + \lambda\eta e^{\lambda t}. \tag{6.43}$$

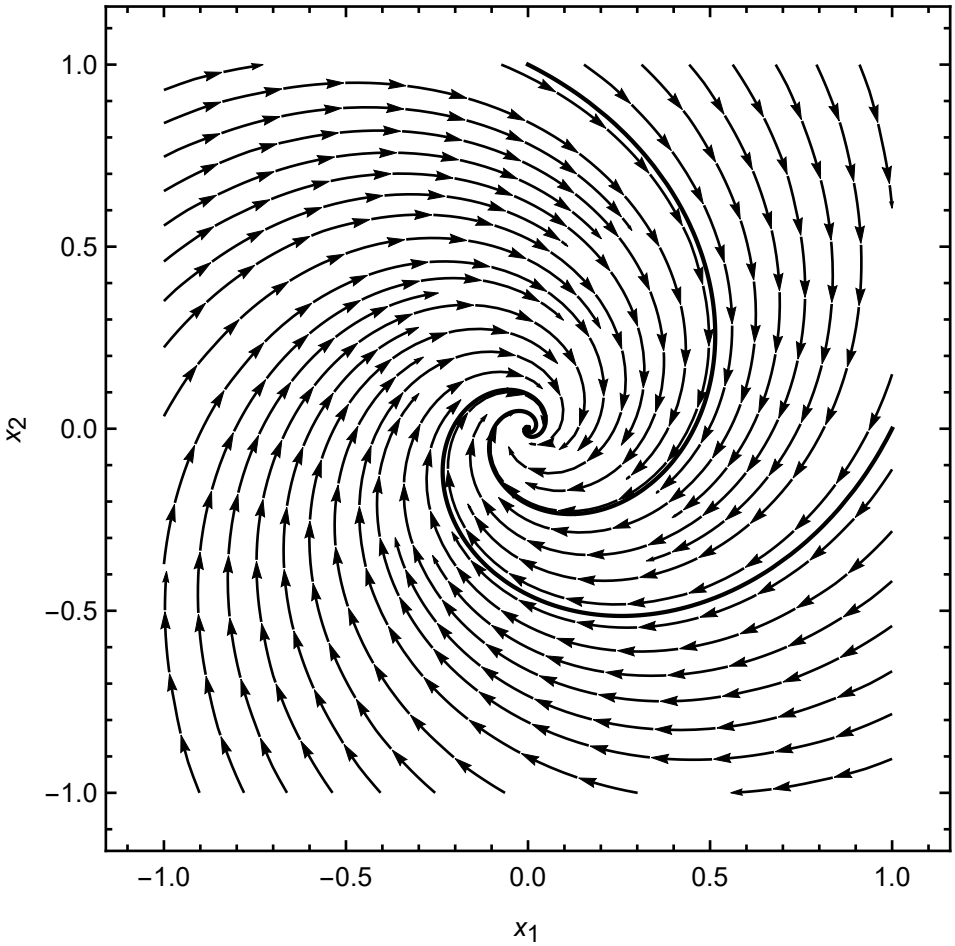

Figure 27: The phase portrait for the example with two complex eigenvalues.

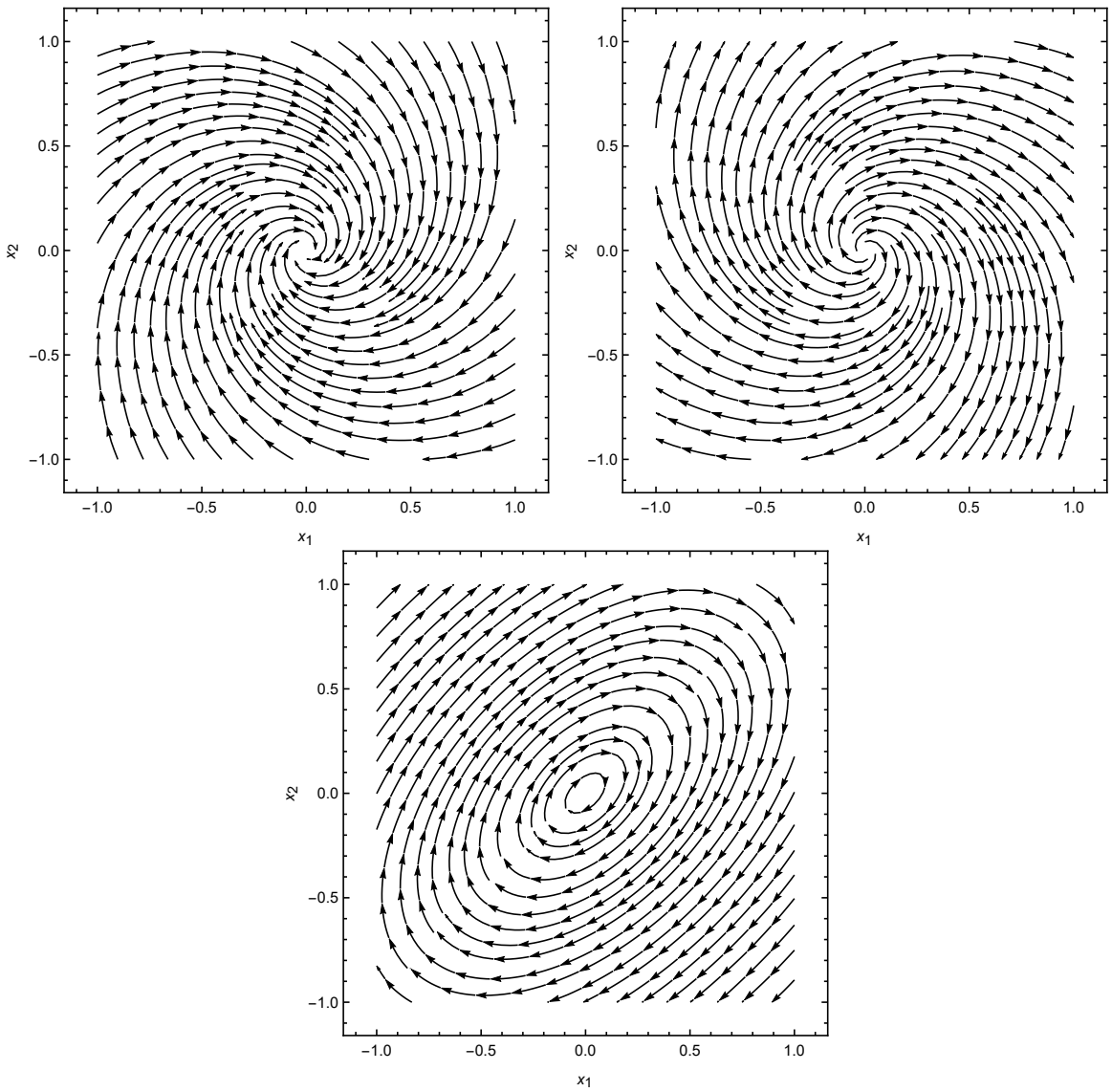

Figure 28: Phase portraits corresponding to commplex eigenvalues. On top: **stable spiral (focus)** ($\mu < 0$) and **unstable spiral (focus)** ($\mu > 0$). Bottom: **centre** ($\mu = 0$).

Since ξ is the eigenvector of A we finally obtain

$$(A - \lambda I)\eta = \xi, \tag{6.44}$$

therefore η is the **generalized eigenvector** corresponding to ξ. The phase portrait is called the **improper node**.

Example. The system

$$\mathbf{x}' = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix} \mathbf{x}, \tag{6.45}$$

has only one eigenvalue $\lambda = -2$ with eigenvector

$$\xi = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \tag{6.46}$$

The generalized eigenvector satisfies $(A + 2I)\eta = \xi$, that is

$$\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}. \tag{6.47}$$

The solution is

$$\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \tag{6.48}$$

for an arbitrary constant $k \in \mathbb{R}$. We neglect the multiple of ξ and finally

$$\mathbf{x}(t) = \left((C_1 + C_2 t) \begin{bmatrix} 3 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{-2t}. \tag{6.49}$$

The phase portrait is presented on Fig. 29. We can see that it resembles a node, however, is more "flattened" due to one characteristic direction. □

- *Repeated real eigenvalues and two independent eigenvector* In this case we immediately have

$$\mathbf{x} = \left(C_1 \xi^{(1)} + C_2 \xi^{(2)} \right) e^{\lambda t}. \tag{6.50}$$

The phase portrait can be obtained explicitly by extracting the independent variable

$$\begin{cases} x_1 = \left(C_1 \xi_1^{(1)} + C_2 \xi_1^{(2)} \right) e^{\lambda t} \\ x_2 = \left(C_1 \xi_2^{(1)} + C_2 \xi_2^{(2)} \right) e^{\lambda t} \end{cases} \rightarrow x_2 = \frac{C_1 \xi_1^{(1)} + C_2 \xi_1^{(2)}}{C_1 \xi_2^{(1)} + C_2 \xi_2^{(2)}} x_1. \tag{6.51}$$

It is a set of rays comming out of the origin. It is called the **star** (see Fig. 30).

We can thus see that in order to solve a linear system we have to find eigenvalues and eigenvectors of the matrix A. Sometimes only the determination of the type of the phase portrait is needed without finding explicit solutions. This can be done very quickly with the help of the following simple observation.

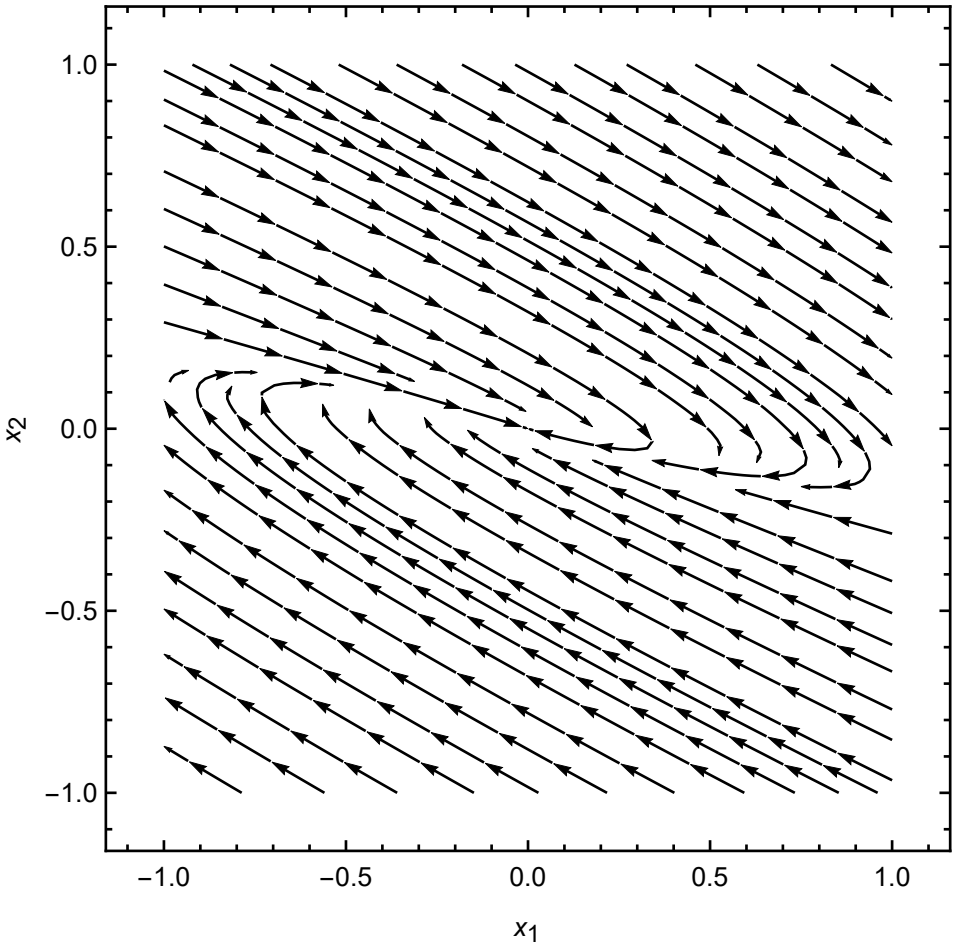

Figure 29: A phase portrait for (6.49).

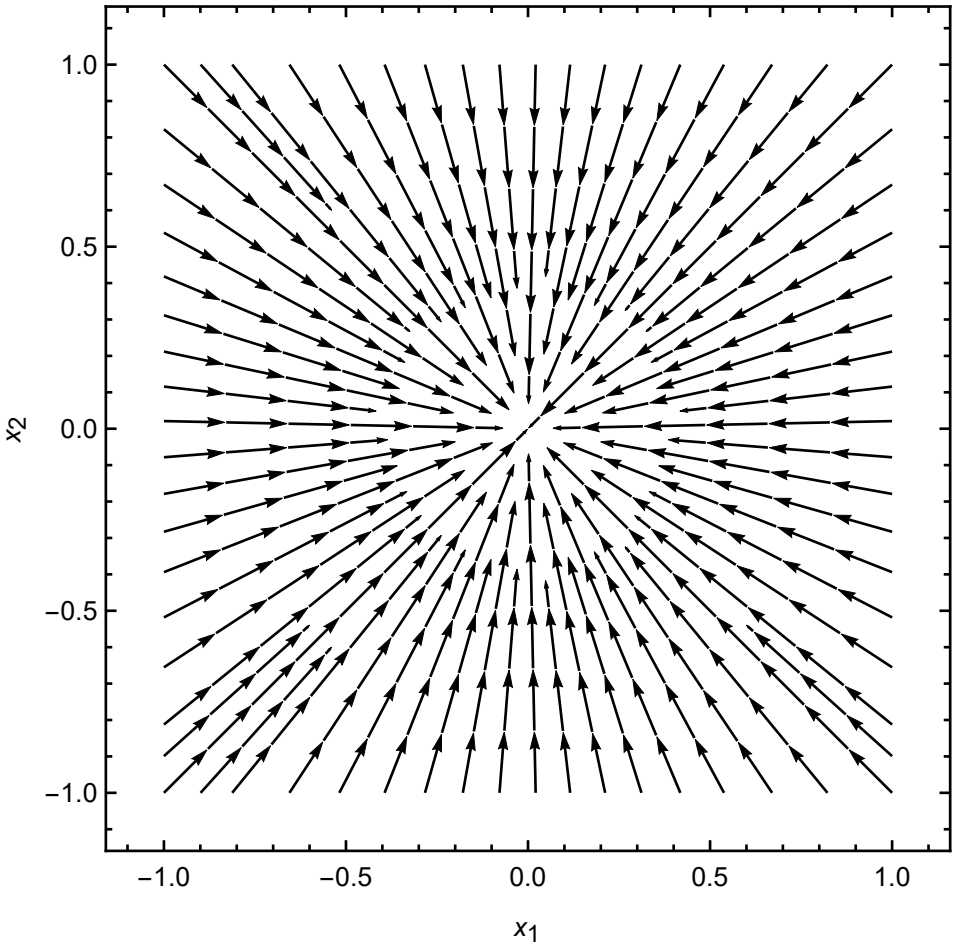

Figure 30: A stable star phase portrait.

Let the matrix of the system has a form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{6.52}$$

Then, the characteristic polynomial has the form

$$\lambda^2 - \tau\lambda + \Delta = 0, \tag{6.53}$$

where $\tau = a + d$ is the trace of A while $\Delta = ad - bc$ is its determinant. The eigenvalues are then

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right). \tag{6.54}$$

From Viéte's formulas we have

$$\tau = \lambda_1 + \lambda_2, \quad \Delta = \lambda_1\lambda_2. \tag{6.55}$$

from which it is straightforward to classify the signs of eigenvalues. Therefore, in order to determine the type of the phase portrait we have only to compute τ and Δ, observe their sign, and compute $\tau^2 - 4\delta$. This will allow us to find a point (τ, Δ) on the $\tau - \Delta$ plane corresponding to a particular phase portrait. For example, when $\Delta < 0$ we always have a saddle since then eigenvalues have opposite signs. Similarly, when $\Delta > 0$ with $\tau^2 - 4\Delta < 0$ the phase portrait is an unstable spiral. The overall classification is depicted on Fig. 31.

6.3 Systems of two nonlinear autonomous equations

The most interesting applications of systems of differential equations can be found in nonlinear ones. The general situation is, of course, very difficult since even for single equations we cannot obtain all information of their solutions. However, in what follows we will learn how the phase plane analysis is useful in finding information about quantitative behaviour of their solutions.

6.3.1 Critical points and stability

The crucial notion is the stationary or critical point which we have already met in the case of single equations. In what follows we will analyse only *autonomous* systems, that is, these where there is no explicit dependence on t (compare Definition 5). They are also called **dynamical systems**.

Definition 21. *A **critical (stationary) point** of the system*

$$\begin{cases} x' = F(x, y), \\ y' = G(x, y), \end{cases} \tag{6.56}$$

is a point $\boldsymbol{x}_c = (x_c, y_c)$, for which $F(\boldsymbol{x}_c) = G(\boldsymbol{x}_c) = 0$.

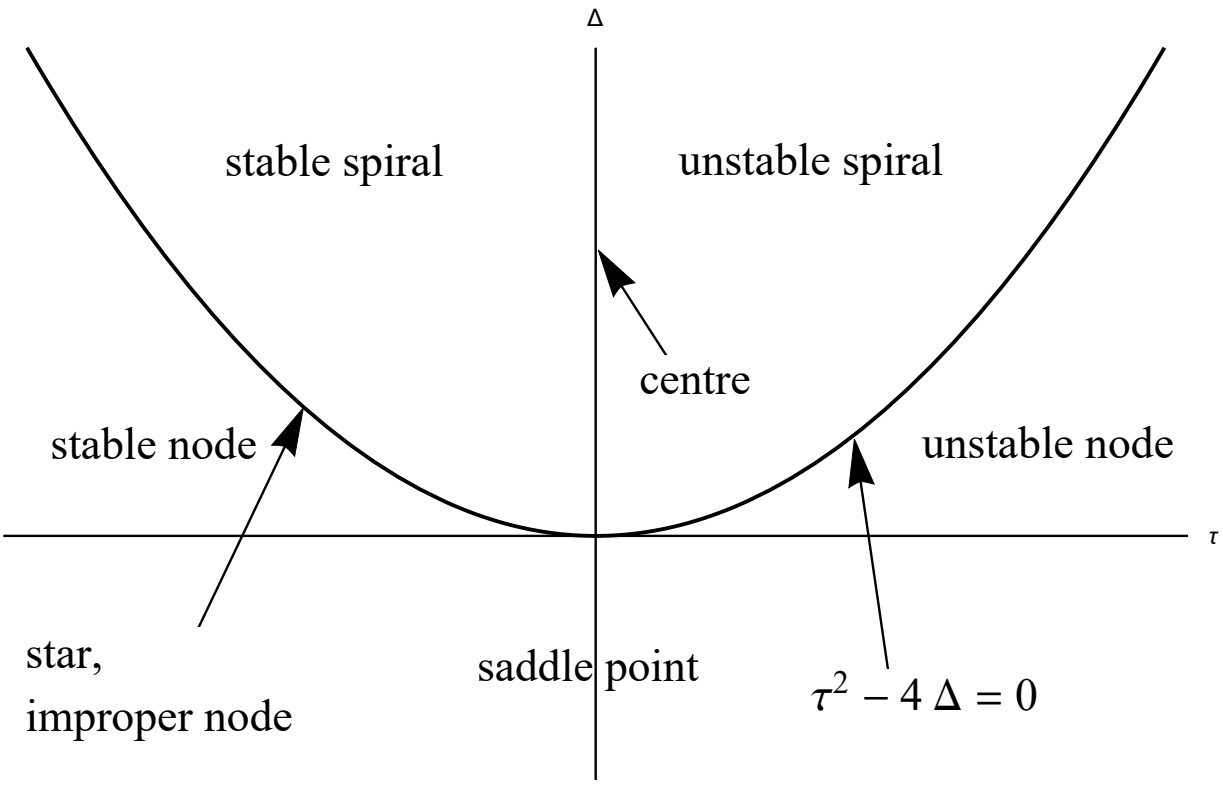

Figure 31: Classification of the phase portraits according to the trace τ and determinant Δ of the matrix A.

Critical points are fundamental since at these, the system does not evolve - derivatives vanish. Nondenegerate, i.e. $\det A \neq 0$, linear systems have a unique critical point at the origin. We expect that a given nonlinear system possesses several stationary points that attract or repel nearby trajectories. In models of real phenomena we rarely can find out about initial conditions (unless we are conducting an experiment in laboratory). What matters the most is the long-time behaviour of the system and what happens with the trajectories.

Definition 22. *Let $\boldsymbol{\phi}$ be a solution of the autonomous system (6.56) with a critical point $\boldsymbol{x}_c$. Then $\boldsymbol{x}_c$ is*

- ***stable*** *if for $t > 0$ we have*

$$\forall_{\epsilon>0}\, \exists_{\delta>0}\, \|\boldsymbol{\phi}(0) - \boldsymbol{x}_c\| < \delta \implies \|\boldsymbol{\phi}(t) - \boldsymbol{x}_c\| < \epsilon. \tag{6.57}$$

- ***asymptotically stable*** *if it is stable and*

$$\exists_{0<\delta_0<\delta}\, \|\boldsymbol{\phi}(0) - \boldsymbol{x}_c\| < \delta_0 \implies \lim_{t\to\infty} \boldsymbol{\phi}(t) = \boldsymbol{x}_c. \tag{6.58}$$

- ***unstable*** *if it is not stable.*

The definitions of stability can be best understood graphically. On Fig. 32 we see two situations that depict differences between stability and asymptotic stability. The

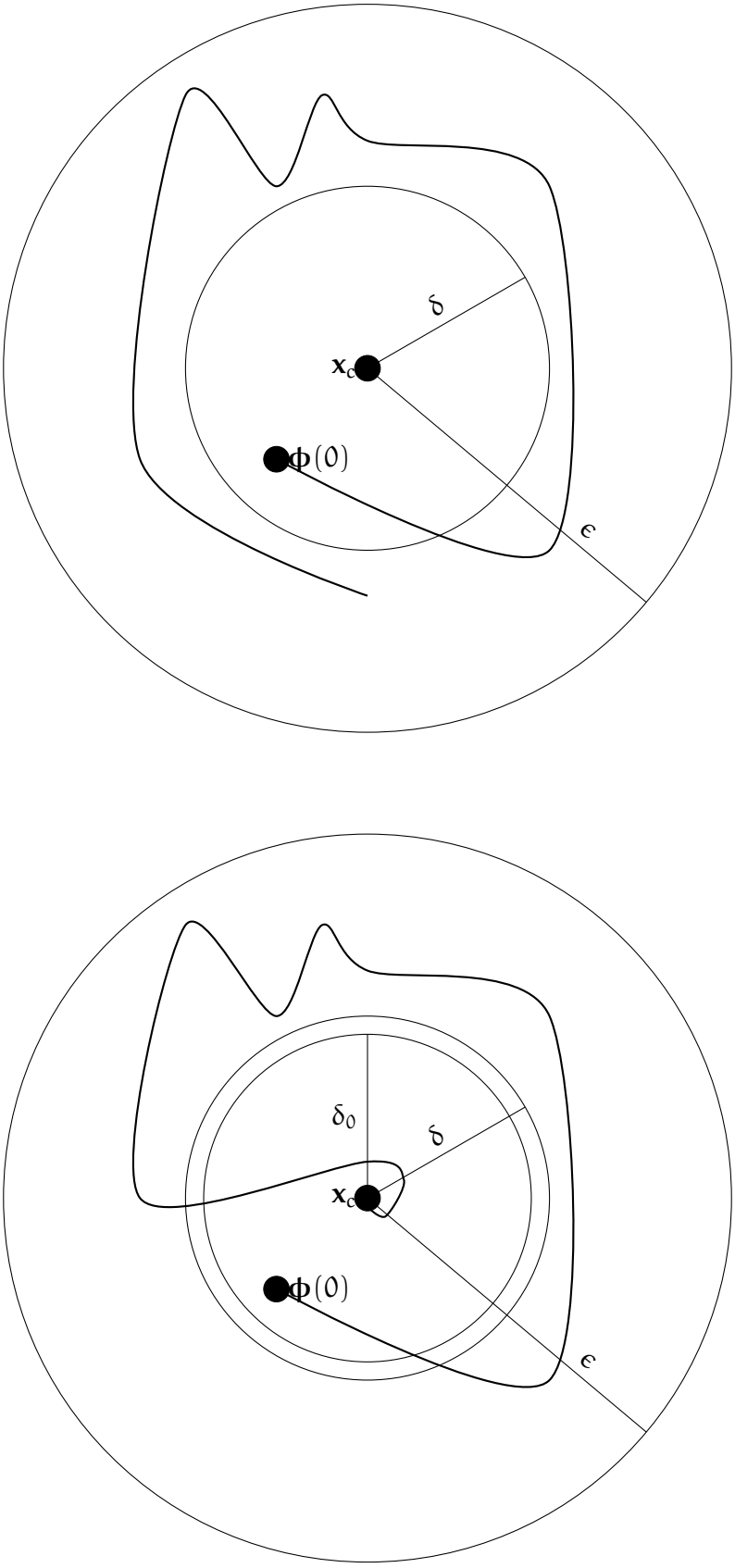

Figure 32: Top: stable critical point - trajectories starting from δ-ball always stay in ϵ-ball. Bottom: asymptotic stability - trajectories starting from δ_0-oball do not leave ϵ-ball and asymptotically approach $\mathbf{x}_c$.

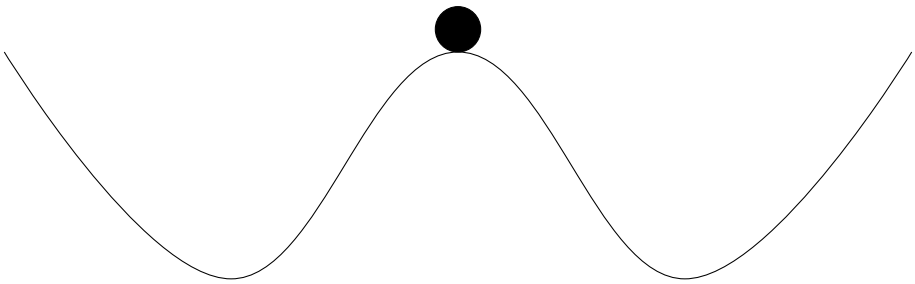

Figure 33: An intuitive interpretation of stable and unstable critical point.

most important part is the fact that a point is stable when nearby trajectories are always confined in an arbitrary neighbourhood. In asymptotic stability the trajectories has also approach the critical point for large times.

An intuitive definition of the stability can also be given in physical terms. On Fig. 33 we can see a ball that has been placed on the top of a mountain. It is a stationary point since it is at rest. However, an arbitrarily small perturbation will force it to fall in either way oscillating around the bottom of the valley. If the friction is present the bottom will be asymptotically stable and, otherwise, it will be a stable critical point of the system. Note that sufficiently large perturbations can force the ball outside our scenery. Hence, our notion of stability is only *local*.

Example. (*Pendulum*) The pendulum equation has the form

$$\frac{d^2\theta}{dt^2} + 2\beta\frac{d\theta}{dt} + \omega_0^2 \sin\theta = 0. \tag{6.59}$$

Let $x = \theta$ and $y = \theta'$. Then,

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -\omega_0^2 \sin x - 2\beta y. \end{cases} \tag{6.60}$$

Critical points satisfy $y = 0$ and $2y = \omega_0^2 \sin x$. That means

$$x = \pm n\pi, \quad y = 0, \quad n \in \mathbb{N}. \tag{6.61}$$

These are the points in which the pendulum is either on the top (n - odd) or at the bottom (n - even). Intuitively we know that the former are unstable while the latter stable. However, to show that rigorously is a completely different matter... □

Checking stability for linear systems is easy - it straightforwardly follows from the definition and availability of the exact solution. Note that we have already indicated the stability of various phase portraits in their name.

Theorem 12. *Let $\boldsymbol{x}' = A\boldsymbol{x}$ be a system of two linear equations with constant coefficients. Let the matrix A be nonsingular with eigenvalues $\lambda_{1,2}$. Then, there exists a unique critical point $\boldsymbol{x} = 0$, which is*

- *asymptotically stable when Re* $\{\lambda_{1,2}\} < 0$ *(stable nodes and spirals),*
- *stable when Re* $\{\lambda_{1,2}\} = 0$ *(centre),*
- *unstable when Re* $\{\lambda_1\} > 0$ *or* $\Re\{\lambda_2\} > 0$ *(unstable nodes and spirals).*

6.3.2 Linearisation

Linear systems are very important in finding the behaviour of nonlinear equations near their stationary points. Assume is the critical point of (6.56). When we use Taylor series to expand F and G we obtain

$$\begin{aligned} F(x,y) &= F(x_c,y_c) + \frac{\partial F}{\partial x}(x_c,y_c)(x-x_c) + \frac{\partial F}{\partial y}(x_c,y_c)(y-y_c) + R_1, \\ G(x,y) &= G(x_c,y_c) + \frac{\partial G}{\partial x}(x_c,y_c)(x-x_c) + \frac{\partial G}{\partial y}(x_c,y_c)(y-y_c) + R_2, \end{aligned} \tag{6.62}$$

where $\mathbf{R} = (R_1, R_2)$ is the remainder. Now, since $F(x_c, y_c) = G(x_c, y_c) = 0$ we can write

$$\begin{cases} (x-x_0)' = \dfrac{\partial F}{\partial x}(x_c,y_c)(x-x_c) + \dfrac{\partial F}{\partial y}(x_c,y_c)(y-y_c) + R_1, \\ (y-y_0)' = \dfrac{\partial G}{\partial x}(x_c,y_c)(x-x_c) + \dfrac{\partial G}{\partial y}(x_c,y_c)(y-y_c) + R_2. \end{cases} \tag{6.63}$$

Now, if we denote $\mathbf{u} = \mathbf{x} - \mathbf{x}_c$ then we obtain

$$\mathbf{u}' = A\mathbf{u} + \mathbf{R}, \tag{6.64}$$

where

$$A = \begin{pmatrix} \dfrac{\partial F}{\partial x}(x_c,y_c) & \dfrac{\partial F}{\partial y}(x_c,y_c) \\ \dfrac{\partial G}{\partial x}(x_c,y_c) & \dfrac{\partial G}{\partial y}(x_c,y_c) \end{pmatrix} \tag{6.65}$$

is the Jacobi matrix of (F, G). Since $\|\mathbf{R}\| = O(\|\mathbf{u}\|^2)$ when $\|\mathbf{u}\| \to 0$ (that is $\mathbf{x} \to \mathbf{x}_c$) we expect that neglecting the remainder will not alter the behaviour of the system near the critical point. This is indeed the case and the full statement is one of the most important results in theory of dynamical systems.

Theorem 13 (Hartman-Grobman)**.** *Let the dynamical system* $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$ *has a linearisation* $\boldsymbol{u}' = A\boldsymbol{u}$ *near the critical point* $\boldsymbol{x}_c$*. Assume that all of the eignevlaues of Jacobi matrix* A *have nonzero real part. Then, there exists a neighbourhood of* $\boldsymbol{x}_c$ *in which the solutions of the nonlinear equations are topologically equivalent to the solutions of corresponding linearisation.*

In plain words, the Hartman-Grobman Theorem says that near the critical points, the phase portraits of a nonlinear system and its linearisation are the same up to a continuous mapping with a continuous inverse. The main assumption is the nonvanishing real part of all of the eigenvalues. When this is not the case, a more interesting situations can happen. We can now summarize all we have found concerning the

analysis of dynamical systems.

Nonlinear systems
In order to analyse the behaviour of solutions of a dynamical system we usually proceed as follows.

1. Find all critical points of the system by solving $F(x_c, y_c) = G(x_c, y_c) = 0$.
2. Calculate the Jacobi matrix (6.65).
3. Analyse the type and stability of each critical point by studying the Jacobi matrix.
4. Sketch the phase portrait.

Example. (*Competition*) Consider two populations that compete for common resources (ex. sheep and rabbits eating grass). If the growth of both populations is logistic we have

$$\begin{cases} \dfrac{dx}{dt} = r_1 x \left(1 - \dfrac{x}{K_1}\right) - a_1 xy, \\ \dfrac{dy}{dt} = r_2 x \left(1 - \dfrac{x}{K_2}\right) - a_2 xy. \end{cases} \tag{6.66}$$

Note that in each case the interaction term is negative since one specie steals food from the other. Although it is possible to analyse the general case we will focus on a specific example.

$$\begin{cases} \dfrac{dx}{dt} = x(1 - x - y) = F(x, y), \\ \dfrac{dy}{dt} = y\left(\dfrac{3}{4} - y - \dfrac{1}{2}x\right) = G(x, y), \end{cases} \qquad A(x, y) = \begin{bmatrix} 1 - 2x - y & -x \\ -\dfrac{1}{2}y & \dfrac{3}{4} - 2y - \dfrac{1}{2}x \end{bmatrix}. \tag{6.67}$$

The critical points are found from the system

$$\begin{cases} x(1 - x - y) = 0, \\ y\left(\dfrac{3}{4} - y - \dfrac{1}{2}x\right) = 0, \end{cases} \tag{6.68}$$

therefore we have four solutions: $(0, 0)$ (both populations die), $(0, 3/4)$ (only the first population dies), $(1, 0)$ (only the second population dies), and $(1/2, 1/2)$ (an equilibrium).

- For the point $(0, 0)$ we have

$$A(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{bmatrix}, \tag{6.69}$$

  which has eigenvalues $\lambda_1 = 1$, $\lambda_2 = \frac{3}{4}$ and vectors $\xi^{(1)} = (1, 0)$ and $\xi^{(2)} = (0, 1)$. It is an *unstable node*.

- For the point $(1, 0)$ we have

$$A(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & \frac{1}{4} \end{bmatrix}, \tag{6.70}$$

whence, $\lambda_1 = -1$, $\lambda_2 = \frac{1}{4}$, and $\xi^{(1)} = (1, 0)$ with $\xi^{(2)} = (4, -5)$. It is a *saddle point*.

- For the point $(0, 3/4)$ we have

$$A(0,3/4) = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{3}{4} \end{bmatrix}, \tag{6.71}$$

that is, $\lambda_1 = \frac{1}{4}$, $\lambda_2 = -\frac{3}{4}$ and $\xi^{(1)} = (8, -3)$ with $\xi^{(2)} = (0, 1)$. It is a *saddle point*.

- For the point $(1/2, 1/2)$ we have

$$A(1/2,1/2) = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix}, \tag{6.72}$$

that is, $\lambda_1 = \frac{-2+\sqrt{2}}{4} < 0$, $\lambda_2 = -\frac{2+\sqrt{2}}{4} < 0$ and $\xi^{(1)} = (\sqrt{2}, -1)$ with $\xi^{(2)} = (\sqrt{2}, 1)$. It is a *stable node*.

Therefore, we see that only the point with both populations living in equilibrium is stable which is a good news. The phase portrait is depicted on Fig. 34. All trajectories, regardless their initial conditions, will approach this stable point. □

Example. (*Lotka-Volterra predator-prey model*) We go back to one of our initial examples concerning predator-prey model

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy, \\ \dfrac{dy}{dt} = -\gamma y + \delta xy. \end{cases} \tag{6.73}$$

Critical points are $(0, 0)$ and $(\gamma/\delta, \alpha/\beta)$. The Jacobi matrix can be written as

$$A(x,y) = \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & -\gamma + \delta x \end{bmatrix}. \tag{6.74}$$

We see that

$$A(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix} \tag{6.75}$$

and

$$A(\gamma/\delta, \alpha/\beta) = \begin{pmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{pmatrix}. \tag{6.76}$$

Therefore, the point $(0, 0)$ is a saddle. Similarly we can show that the eigenvalues of A at the other critical point $(\gamma/\delta, \alpha/\beta)$ are $\pm i\sqrt{\alpha\gamma}$. They are *purely imaginary*. We cannot, then, use Hartman-Grobman theorem to make any conclusions. However, we expect that the system oscillates near that point.

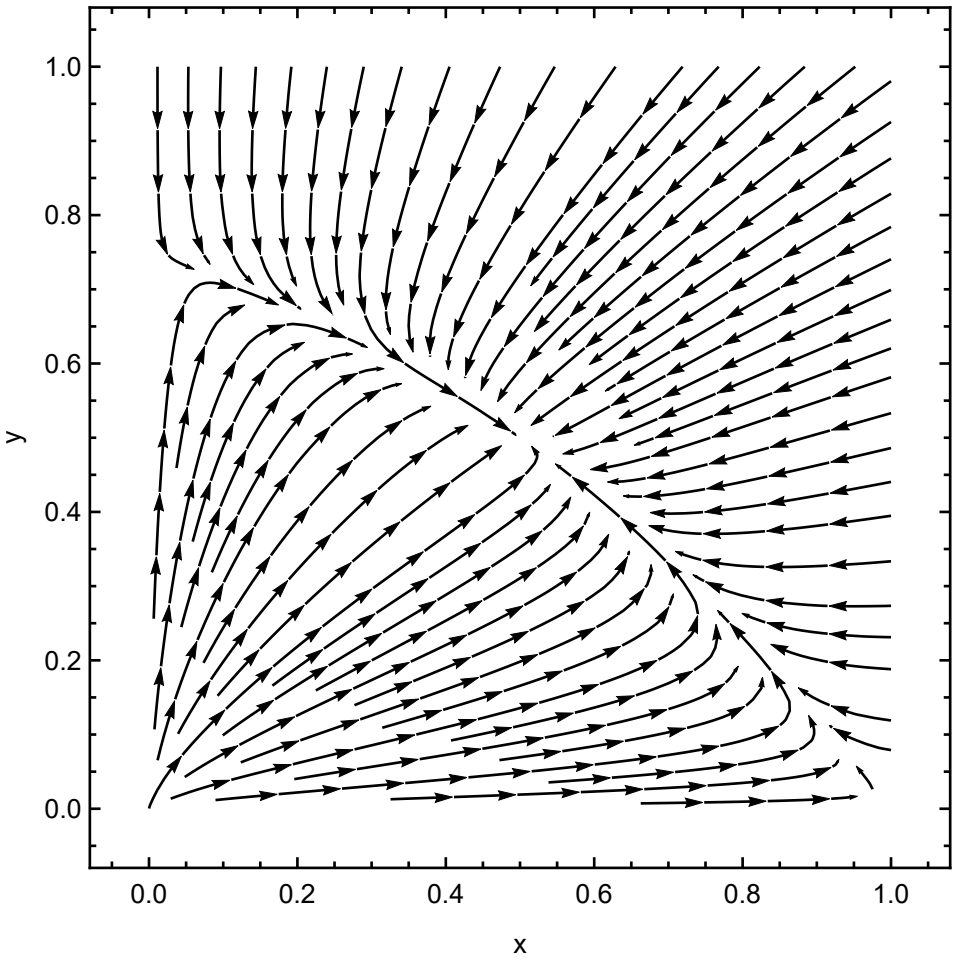

Figure 34: The phase portrait for competitive species model.

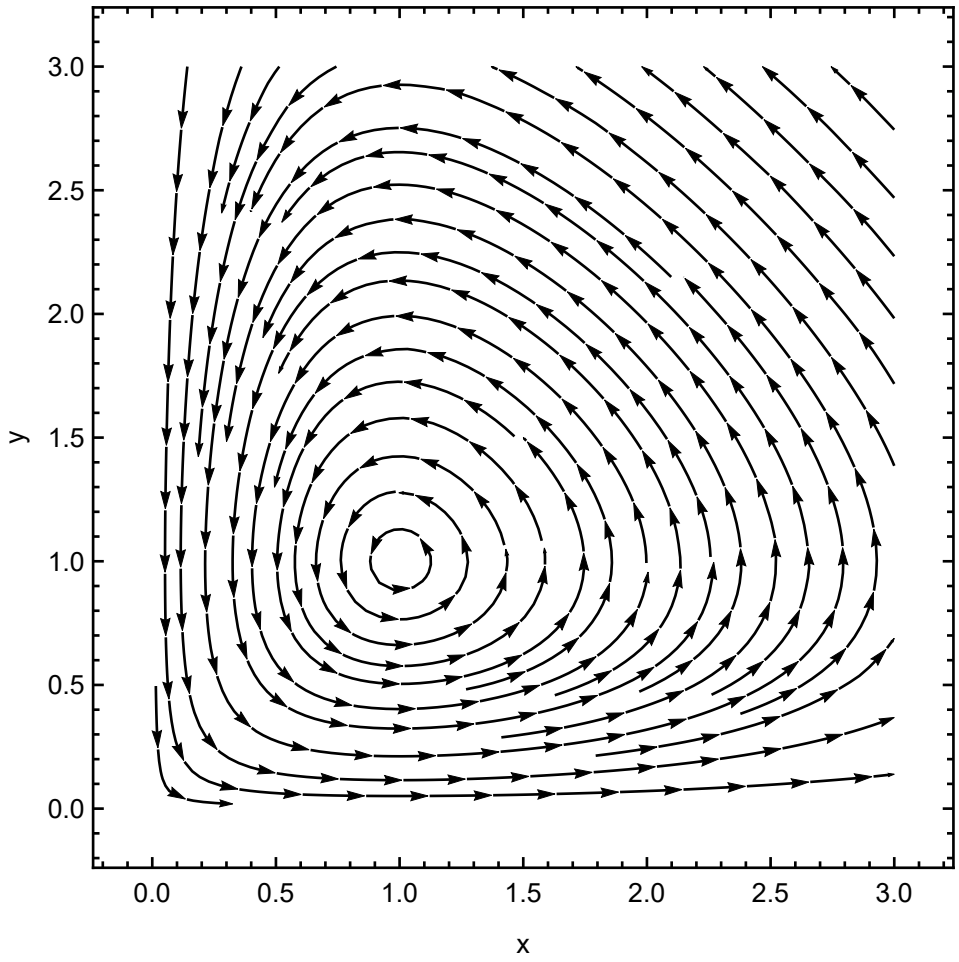

Figure 35: The phase portrait for the Lotka-Volterra system.

To prove this claim we write the ODE for trajectories (6.20)

$$\frac{dy}{dx} = \frac{-\gamma y + \delta xy}{\alpha x - \beta xy} \to \int \frac{\alpha - \beta y}{y} dy = \int \frac{-\gamma + \delta x}{x} dx. \tag{6.77}$$

After integration we obtain $C = \alpha \ln y - \beta y + \gamma \ln x - \delta x = f(x, y)$ which is an conserved quantity, that is constant on each trajectory. Therefore, each trajectory is a level curve of a function $f = f(x, y)$. Moreover, it is easy to show that f has a global maximum at $(\gamma/\delta, \alpha/\beta)$. Therefore, each level curve is closed yielding a periodic solution of the system. The phase portrait is presented on Fig. 35.

We can immediately draw an important conclusion. Since solutions are oscillatory, the populations will periodically change their numbers. This can be observed in nature since when predation is high it can quickly decrease the number of prey. As a consequence, predators die out due to a lack of sufficient amount of food. Then, the chance of survival of remaining prey increases leading to proliferation of this population. The cycle repeats from this point because predators can acquire more food and thus rebuild their numbers. A real-world dta concerning Lynx (predator) and Hare (prey) is depicted on Fig. 36. Oscillations are evident. □

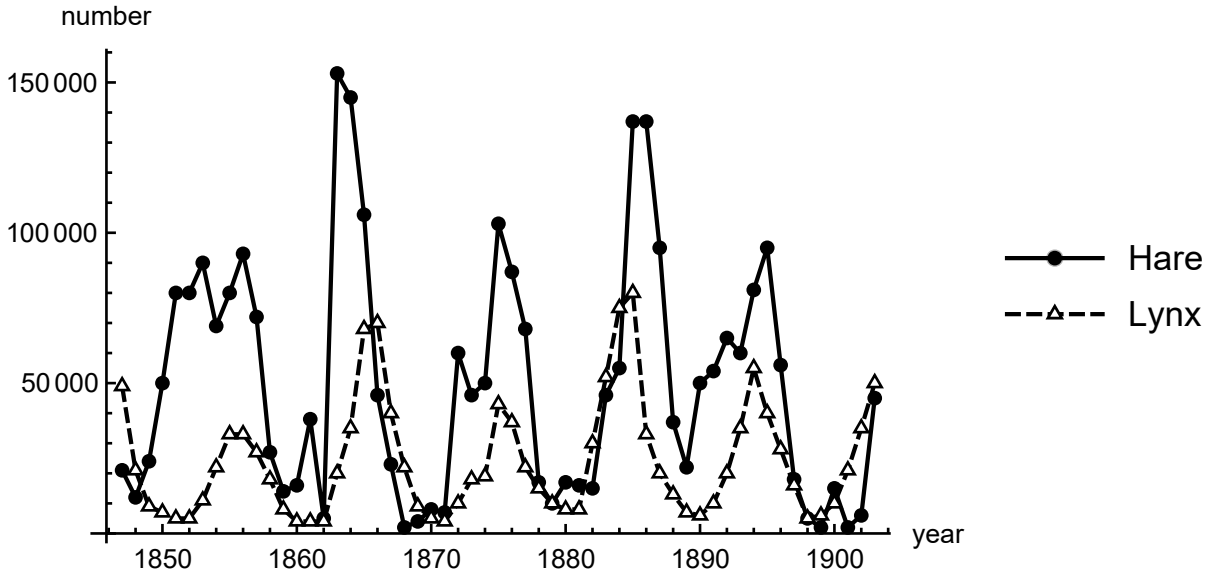

Figure 36: A real data concerning populations of Lynx and Hare from Hudson Bay Company.