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NUMERICAL METHOD FOR THE TIME-FRACTIONAL POROUS MEDIUM EQUATION*

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Abstract. This paper deals with a construction and convergence analysis of a numerical scheme devised for solving the time-fractional porous medium equation with Dirichlet boundary conditions on the half line. The governing equation exhibits both nonlocal and nonlinear behavior making the numerical computations challenging. Our strategy is to reduce the problem into a single one-dimensional Volterra integral equation for the self-similar solution and then to apply a suitable discretization. The main difficulty arises due to the non-Lipschitzian behavior of the nonlinearity of the corresponding integral equation. By the analysis of the recurrence relation for the error, we are able to prove that there exists a family of schemes that is convergent for a large subset of the parameter space. More specifically, in the very slow regime of the subdiffusion, we require that the diffusivity parameter has to be sufficiently large to provide the convergence of the method. We illustrate our results with a concrete example of a method based on the midpoint quadrature.

Key words. porous medium equation, nonlinear diffusion, fractional derivative, finite difference method, Volterra equation

AMS subject classifications. 35R11, 45G10, 65R20

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1. Introduction.

The interest in nonlocal equations and models has flourished in the last few decades. The investigations has been conducted in a twofold way: experimental and theoretical. One of the most important real-world realizations of the nonlocal behavior is the anomalous diffusion [34, 27, 28], which is present in a multitude of various situations. To mention only a few examples, there is evidence of anomalous transport in quantum optics [44], physics of plasma [17], movement of bacteria and amoeba [31], G-protein on cell surface [48], and hydrology [15, 37, 38, 47]. In all of those applications diffusion is reported to be either slower (subdiffuion) or faster (superdiffusion) than in the classical case.

For the theoretical counterpart of the problem, the adequate model of the anomalous diffusion can be constructed with a use of the fractional derivative or some other nonlocal operator that takes into account temporal or spatial history of the process [34]. As mathematical entities these objects attracted a great interest of both pure and applied mathematicians. The prominent example is the fractional Laplacian, which is still a subject of vigorous investigations from many viewpoints [30, 11, 51].

The main motivation of our work is the subdiffusive moisture percolation in porous media such as building materials or minerals (for experimental results, see [14, 20, 29, 43, 54]). It appears that a decently accurate model for this anomalous phenomenon could be based on the time-fractional nonlinear diffusion equation. The nonlocality in the temporal operator models the waiting-time property of the medium. This is a consequence of the complex structure of the medium and, mathematically, is described

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by a nonlocal flux as was shown in [39]. Coupled with an assumption of power-law behavior of the waiting times produces the fractional derivative as the time-evolution operator.

In our previous investigations [40, 38, 39] we have considered the following time-fractional nonlinear diffusion equation (also known as the time-fractional porous medium equation)

$$\partial_t^\alpha u = \left(u^m u_x\right)_x, \quad 0 < \alpha < 1, \quad m > 1, \tag{1.1}$$

where the fractional derivative is of the Riemann–Liouville type

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} u(x,s) ds. \tag{1.2}$$

The boundary conditions that we impose are the following:

$$u(x,0) = 0, \quad u(0,t) = 1, \quad x > 0, \quad t > 0, \tag{1.3}$$

where the nondimensionalization has been implicitly assumed. The above problem is a description of an experiment where one measures the material properties of an essentially one-dimensional half-infinite medium under the water imbibition from the boundary. Note also that the vanishing initial condition forces the Riemann–Liouville derivative to be equivalent to the Caputo version, which would be present for nonhomogeneous initial data. Some fitting of the above model with experimental data has been conducted for example in [47, 38]. In the subdiffusive dynamics the obtained values of the parameters were $\alpha \approx 0.8 - 0.9$ and $m \approx 7 - 8$. The literature mentioned in [47] also suggests that α usually is reasonably close to 1 while m is smaller than 10.

There is a wealth of literature concerning numerical methods for fractional differential equations (both ODEs and PDEs)—some surveys can be found in [19, 5]. Concerning the linear fractional diffusion the reader can find relevant methods, for example, in [49, 32, 52]. The numerical methods for the main operator in the spatial diffusion, namely, the fractional Laplacian, can be found in [24, 13]. As for the nonlinear version of the anomalous diffusion, an interesting paper recently appeared in which the authors constructed a multigrid waveform method of fast convergence [22]. Moreover, a similar equation to ours has been solved in [4] in the context of the petroleum industry. Some other numerical approaches concern space-fractional nonlinear diffusion [21, 18], nonlinear source terms [55, 53, 6] and variable order diffusion [35].

Our approach is based on a transformation of the governing PDE to the equivalent nonlinear Volterra equation (for a recent survey of theory and numerical methods, see [7]). However, in our case the classical convergence theory cannot be applied since the nonlinearity of the equation is non-Lipschitzian. In the classical results [33], the Lipschitz condition is used to control the nonlinearity and, effectively, the proof of convergence is technically almost the same as in the linear case. According to our best knowledge, there is scarce literature concerning similar non-Lipschitzian problems. A very interesting paper is [8] where an iterative technique has been applied and convergence proofs given. Moreover, in [9] a short summary of the theoretical and numerical character has been published. Lastly, we mention our own work [41] from which the present considerations stem. In that paper we have given the convergence proof assuming the kernel's separation from zero. In the present discussion we relax this assumption.

The main result of this paper is a construction of a family of numerical schemes used to solve (1.1). This family is indexed with specific quadratures used to discretize the corresponding Volterra integral equation. That is to say, choosing a method of calculating the integral picks a member of that family. We then show that for a particular quadrature the constructed numerical scheme is convergent provided that m is constrained with respect to α. In practice m has to be sufficiently large only for small values of α. On the other hand, if we use the midpoint method as a quadrature and take $\alpha \geq 0.6$, the convergence of the method is guaranteed for all $m \geq 1$, which is the usual assumption for (1.1). Moreover, as experiments investigating construction materials found (see [20, 14, 29, 43]), the subdiffusion parameter α usually is close to 1. Therefore, according to our results the numerical method is convergent for any meaningful m in the experimental range of α.

The paper is structured as follows. In the second section we formulate the problem in terms of the Volterra setting starting from the self-similar form of the time-fractional porous medium equation. We proceed to the main results in the third section where a construction of a convergent method is given. We end the paper with numerical simulations illustrating our results.

2. Problem statement. The straightforward numerical approach is to start with (1.1) and, for example, apply a finite difference scheme supplemented with linearization to discretize it. This has been done in [38] (but see also the recent approach in [22]) but the authors noted a very large computational cost and stability issues caused by the interplay of two factors: nonlocality (of the fractional derivative) and nonlinearity (of the flux). A more sensible strategy is to transform (1.1) into its self-similar form and then derive an appropriate numerical scheme. We begin with necessary preparations.

Since the initial-boundary conditions (1.3) are self-similar, we can substitute $\eta = xt^{-\alpha/2}$ for some $0 < \alpha < 1$, which by denoting $u(x,t) = U(xt^{-\alpha/2})$ yields

$$(U^m U')' = \left[(1-\alpha) - \frac{\alpha}{2}\eta\frac{d}{d\eta}\right] I^{0,1-\alpha}_{-\frac{2}{\alpha}} U, \quad 0 < \alpha < 1, \tag{2.1}$$

with the boundary conditions

$$U(0) = 1, \quad U(\infty) = 0. \tag{2.2}$$

Here, $I^{a,b}_c$ is the Erdélyi–Kober operator [25, 26, 45]

$$I^{a,b}_c U(\eta) = \frac{1}{\Gamma(b)} \int_0^1 (1-s)^{b-1} s^a U(s^{\frac{1}{c}}\eta) ds. \tag{2.3}$$

Notice that (2.1) is an ordinary integrodifferential equation which should be more tractable for numerical work than (1.1).

There is another transformation that can be done in order to simplify the matters even more. Notice that due to (2.2) our problem has a free boundary, which can cause difficulties to resolve it numerically (for some details see [12]). However, there exists a substitution that can take (2.1) into an equivalent initial-value problem. Physical reasons as well as theoretical considerations suggest that the solution of (2.1) has a compact support, i.e., there exists a point $\eta^* \geq 0$ such that

$$U(\eta) = 0 \quad \text{for} \quad \eta \geq \eta^*. \tag{2.4}$$

This simply means that the wetting front propagates at a finite speed that has a fundamental significance for the physical experiment [20]. The rigorous proof that the solution of the classical version of the porous medium equation has a compact support has been given for example in [3]. Similar results for the spatially nonlocal case are a subject of vigorous work and, for example, the compact support property has been proven in the last few years (see [46] and for a thorough survey [50]). As for our time-fractional case, in [42] we have shown that there exists exactly one self-similar solution of (1.1) possessing the compact support. The method that we used there was based on application of the sub- and supersolution technique to (2.1) (in particular, Bushell's theorem [10]). This method was powerful enough to overcome the difficulty of the non-Lipschitz nonlinearity when showing uniqueness. Along with that we have obtained some estimates on the size of the support. The method that we develop below can be used to numerically compute the approximation of this unique solution.

Having in mind the above discussion we are motivated to substitute

$$U(\eta) = \left(m(\eta^*)^2\right)^{\frac{1}{m}} y(z), \quad z = 1 - \frac{\eta}{\eta^*}, \tag{2.5}$$

which changes (2.1) into

$$m(y^m y')' = \left[(1-\alpha) + \frac{\alpha}{2}(1-z)\frac{d}{dz}\right] F_\alpha y, \quad 0 < \alpha < 1, \quad 0 \le z \le 1, \tag{2.6}$$

where the linear operator F_α is defined by

$$F_\alpha y(z) := \frac{1}{\Gamma(1-\alpha)} \int_{(1-z)^{\frac{\alpha}{2}}}^1 (1-s)^{-\alpha} y(1 - s^{-\frac{2}{\alpha}}(1-z)) ds. \tag{2.7}$$

The integrand above has two singular points: $s = 0$ and $s = 1$. Since $(1-z)^{\alpha/2} \le s \le 1$ only the latter one is relevant, and the integral is convergent because $0 < \alpha < 1$. Further, the transformation (2.5) yields a formula for the wetting front position

$$\eta^* = \frac{1}{\sqrt{m y(1)^m}}. \tag{2.8}$$

This formula can be used to calculate the front of the moisture at a particular time via the similarity transformation. That is to say, the first point at which the moisture vanishes is $\eta^* t^{\alpha/2}$.

Lastly, (2.6) can be integrated twice and transformed into a Volterra integral equation

$$y(z)^{m+1} = \frac{m+1}{m} \int_0^z \left(\frac{\alpha}{2} + \left(1 - \frac{\alpha}{2}\right) z - v\right) F_\alpha y(v) dv, \tag{2.9}$$

which is of the main interest for this paper. It is a Volterra equation in which the nonlinear term is non-Lipschitz.

We can manipulate the integrand of (2.9) in order to write the kernel explicitly. This would simplify the subsequent numerical implementation. To start, change the variable $u = 1 - s^{-\frac{2}{\alpha}}(1-z)$ in the definition of F_α given by (2.7) to obtain

$$F_\alpha y(z) = \frac{\alpha}{2} \frac{(1-z)^{\frac{\alpha}{2}}}{\Gamma(1-\alpha)} \int_0^z \left(1 - \left(\frac{1-z}{1-u}\right)^{\frac{\alpha}{2}}\right)^{-\alpha} (1-u)^{-\frac{\alpha}{2}-1} y(u) du. \tag{2.10}$$

The singular points of the above integrand are at $u = z$ and $u = 1$. Since $0 < u < z < 1$ the only meaningful singularity is the former. In that case

$$\left(1-\left(\frac{1-z}{1-u}\right)^{\frac{\alpha}{2}}\right)^{-\alpha} = \left(\frac{z-u}{1-z}\right)^{-\alpha}\left(\left(\frac{2}{\alpha}\right)^{\alpha}+O(z-u)\right) \quad \text{as} \quad u \to z^{-}, \tag{2.11}$$

and hence due to the fact that $0 < \alpha < 1$ the integral is convergent. Now, plugging the above formula into (2.9) and changing the order of integration (Fubini's theorem) we obtain

$$\begin{aligned} y(z)^{m+1} &= \frac{m+1}{m}\frac{\alpha}{2}\frac{1}{\Gamma(1-\alpha)} \\ &\times \int_0^z \left[\int_u^z \left(\frac{\alpha}{2}+\left(1-\frac{\alpha}{2}\right)z-v\right)(1-v)^{\frac{\alpha}{2}}\left(1-\left(\frac{1-v}{1-u}\right)^{\frac{\alpha}{2}}\right)^{-\alpha} dv\right]\frac{y(u)}{(1-u)^{\frac{\alpha}{2}+1}}du. \end{aligned} \tag{2.12}$$

To simplify further we can substitute $s = \left(\frac{1-v}{1-u}\right)^{\frac{\alpha}{2}}$ and arrive at

$$\begin{aligned} &y(z)^{m+1} \\ &= \int_0^z \left[\underbrace{\frac{m+1}{m}\frac{1}{\Gamma(1-\alpha)}\int_{\left(\frac{1-z}{1-u}\right)^{\frac{\alpha}{2}}}^1 \left(s^{\frac{2}{\alpha}}(1-u)-\left(1-\frac{\alpha}{2}\right)(1-z)\right)s^{\frac{2}{\alpha}}(1-s)^{-\alpha}\,ds}_{K(z,u)}\right] y(u)du. \end{aligned} \tag{2.13}$$

Finally, we can notice the kernel K can be written in terms of the incomplete beta function defined by

$$B(z,a,b) := \int_0^z (1-s)^{a-1}s^{b-1}ds. \tag{2.14}$$

Hence,

$$y(z)^{m+1} = \int_0^z K(z,u)y(u)du, \tag{2.15}$$

where for $0 \le u \le z$ we have

$$\begin{aligned} K(z,u) &:= \frac{m+1}{m}\frac{1}{\Gamma(1-\alpha)} \\ &\times \left[(1-u)\left(B\left(\frac{4}{\alpha}+1,1-\alpha\right)-B\left(\left(\frac{1-z}{1-u}\right)^{\frac{\alpha}{2}},\frac{4}{\alpha}+1,1-\alpha\right)\right)\right. \\ &\left.-\left(1-\frac{\alpha}{2}\right)(1-z)\left(B\left(\frac{2}{\alpha}+1,1-\alpha\right)-B\left(\left(\frac{1-z}{1-u}\right)^{\frac{\alpha}{2}},\frac{2}{\alpha}+1,1-\alpha\right)\right)\right]. \end{aligned} \tag{2.16}$$

It is evident that the kernel is positive and continuous with a singular derivative. A class of equations similar to the above has been a subject of very active investigations. The non-Lipschitzian character of (2.15) is manifestly clear if we write it in an equivalent form by substituting $w = y^{m+1}$ and obtaining

$$w(z) = \int_0^z K(z,u) w(u)^{\frac{1}{m+1}} du. \tag{2.17}$$

Since the nonlinearity is of a root type for $m > 0$ it does not satisfy the Lipschitz condition. For readability, in further studies we will analyze the equation in the previous form (2.15).

Notice that according to the existence and uniqueness result in the previously cited paper [42] there is no loss of generality when going from (1.1) with (1.3) to (2.15) with (2.16). And hence, the latter problem is equivalent to the former. The inverse transformation is based on first calculating the wetting front from the formula (2.8) and then using (2.5) with the self-similar transformation to have

$$u(x,t) = \frac{y\left(1 - \sqrt{my(1)^m}\, x t^{-\frac{\alpha}{2}}\right)}{y(1)}, \quad \text{where} \quad 0 \le x \le \frac{t^{\frac{\alpha}{2}}}{\sqrt{my(1)^m}}, \quad t > 0. \tag{2.18}$$

For important results stating the existence and uniqueness for (2.15) for the general case the reader is invited to consult [10, 23, 36].

3. Numerical method. Let us begin with introducing the uniform grid

$$z_n := \frac{n}{N}, \quad h := \frac{1}{N}, \quad n = 0, 1, \ldots, N, \tag{3.1}$$

where the total number of grid points N is fixed. We further discretize the integral equation (2.15) by applying an arbitrary open quadrature to its right-hand side

$$\int_0^{z_n} K(z_n, t) y(t) dt = h \sum_{i=1}^{n-1} w_{n,i} K(z_n, z_i) y(z_i) + \delta_n(h), \tag{3.2}$$

where $\delta_n(h)$ is the local consistency error. Furthermore, we define the maximal error

$$\delta(h) := \max_{1 \le n \le N} |\delta_n(h)|, \tag{3.3}$$

and for the weights we assume their positivity and boundedness

$$0 < w_{n,i} \le W \quad \text{for} \quad 1 \le i \le n \le N. \tag{3.4}$$

Finally, by truncating the local consistency error we can propose the following scheme for solving (2.15)

$$y_n^{m+1} = h \sum_{i=1}^{n-1} w_{n,i} K_{n,i}\, y_i, \quad n = 2, 3, \ldots, N, \tag{3.5}$$

where $K_{n,i} := K(z_n, z_i)$.

Overall, the scheme (3.5) is a family of numerical methods indexed by the choice of the weights. These, in turn, are taken from the corresponding quadrature for calculating the integral (3.2). For example, the rectangle method can be implemented

by taking $w_{n,i} = 1$. In the sequel we use the midpoint quadrature in which the first step is calculated by the trapezoidal rule

$$w_{n,i} = \begin{cases} 2 & \text{if} \quad (-1)^{n+i} = -1 \text{ and } i > 1, \\ \frac{1}{2} & \text{if} \quad (-1)^n = -1 \text{ and } i \in \{0, 1\}, \\ 0, & \text{otherwise,} \end{cases} \qquad n > 1. \tag{3.6}$$

In general, the formula for $w_{n,i}$ can be taken from any open Newton–Cotes rule for approximating integrals (see [1], sections 25.4.21–27) or a combination of them. The main result of this paper states that, provided that certain technical conditions are met (to be stated in Theorem 3.5), there is a subfamily of methods (3.5) that are convergent to the solution of (2.15).

The objective of the following is to prove that (3.5) is convergent provided we choose an appropriate starting value y_1 for possibly the widest choice of quadratures. From (2.16) we notice that $K(1, 1) = 0$ and thus the kernel is not separated from zero. Hence, we cannot use our previous results from [41]. It is instructive, however, to see what are the differences between the usual case of Lipschitzian nonlinearity and ours. In the former, we are quickly able to write the estimate on the scheme's error $e_n = y_n - y(nh)$ in the form (see [33])

$$|e_n| \leq Ah \sum_{i=1}^{n-1} |e_i| + B, \tag{3.7}$$

where A and $B = B(h)$ are independent of n. The above recurrence inequality can be thought as a discrete version of the Gronwall–Bellman's lemma (G-B). It can be easily solved giving

$$|e_n| \leq B(h) e^{Anh}. \tag{3.8}$$

In our case $B(h)$ is associated with the quadrature's local consistency error and hence vanishes as $h \to 0$. If, additionally, we require that $nh \to$ const. to account for a comparison at a fixed point, the above estimate shows that $e_n \to 0$ as we refine the grid. The method is thus convergent.

The point when Lipschitz condition comes into play in (3.7) is the form of the terms $|e_i|$ inside the sum. Thanks to the Lipschitzian nonlinearity the estimate is essentially the same as in the linear case. When we relax this requirement the form of the recurrence estimate quickly gets complicated and relies on specific estimates used for a particular nonlinearity. For example, in general the summand can be a nonlinear function of n, i, or $|e_i|$. It is then impossible to apply the classical version of the G-B lemma to solve the recurrence. However, it is possible, to obtain a result analogous to (3.7) by a careful exploitation of the root-type nonlinearity of (2.15). Then, a more complex variation of G-B lemma can be proved to close the recurrence. We will see the details below.

Several ingredients from our developing theory will be useful. The first is a simple consequence of the integral equation structure and discretization scheme.

PROPOSITION 3.1. *Let y be the nontrivial solution of* (2.15) *and y_n are constructed via the iteration* (3.5). *If $\delta_n(h)$ in* (3.2) *is nonnegative (nonpositive) for all $n = 1, 2, \ldots, N$, then $y(nh) \geq y_n$ ($y(nh) \leq y_n$) provided that $y(h) \geq y_1$ ($y(h) \leq y_1$).*

This monotonicity result has been proved in [41]. The second piece of information about the solution of (2.15) concerns the *global* estimates (for the proofs of the next two lemmas, see [42]).

LEMMA 3.2. *Let y be the nontrivial solution of* (2.15). *Then, the following estimates take place*

$$C_1 z^{\frac{2-\alpha}{m}} \le y(z) \le C_2 z^{\frac{2-\alpha}{m}}, \quad 0 \le z \le 1, \quad 0 < \alpha \le 1, \tag{3.9}$$

where $C_{1,2}$ are given by

$$C_1 = \begin{cases} \left(\left(\frac{\alpha}{2}\right)^{1-\alpha} \frac{\Gamma\left(\frac{2-\alpha}{m}\right)}{\Gamma\left(2-\alpha+\frac{2-\alpha}{m}\right)} \frac{1}{2-\alpha+m(3-\alpha)} \right)^{\frac{1}{m+1}}, & 0 < \alpha \le 1 - \frac{1}{m+1}; \\ \left(\left(\frac{\alpha}{2}\right)^{2-\alpha} \frac{\Gamma\left(1+\frac{2-\alpha}{m}\right)}{\Gamma\left(2-\alpha+\frac{2-\alpha}{m}\right)} \frac{1}{2-\alpha} \right)^{\frac{1}{m+1}}, & 1 - \frac{1}{m+1} < \alpha \le 1; \end{cases} \tag{3.10}$$

$$C_2 = \Gamma(3-\alpha)^{-\frac{1}{m+1}}.$$

Our solution is thus bounded from below and above by a power function of the same class. As it appears, something more detailed can be said about the behavior of y at the origin.

LEMMA 3.3. *The solution $y = y(z)$ of* (2.15) *satisfies*

$$y(z) \sim \left(\frac{\alpha}{2}\right)^{2-\alpha} \frac{\Gamma\left(\frac{2-\alpha}{m}\right)}{\Gamma\left(1-\alpha+\frac{2-\alpha}{m}\right)} \frac{z^{\frac{2-\alpha}{m}}}{(2-\alpha)\left(1+\frac{1}{m}\right)-1} \quad \text{as} \quad z \to 0^+, \quad 0 < \alpha \le 1. \tag{3.11}$$

The asymptotics is thus of power type and the constant of proportionality is known. As a quick application of the above lemma we propose a sensible choice of the starting value for the numerical method

$$y_1 := \left(\frac{\alpha}{2}\right)^{2-\alpha} \frac{\Gamma\left(\frac{2-\alpha}{m}\right)}{\Gamma\left(1-\alpha+\frac{2-\alpha}{m}\right)} \frac{h^{\frac{2-\alpha}{m}}}{(2-\alpha)\left(1+\frac{1}{m}\right)-1}, \tag{3.12}$$

which introduces the starting error of higher order than $h^{\frac{2-\alpha}{m}}$ as $h \to 0^+$.

Now, we proceed to the proof of the fact that (3.5) is convergent. First, we state an auxiliary lemma which can be thought as a generalization of the discrete G-B (for a thorough review of similar results, see [2]).

LEMMA 3.4. *Let $\{e_n\}$, $n = 1, 2, \ldots$ be a sequence of positive numbers satisfying*

$$e_n \le \frac{1}{n^\beta} \left(A \sum_{i=1}^{n-1} (n-i)^\gamma \, e_i + B \right), \quad n \ge 2, \tag{3.13}$$

where A, B are positive constants, $\beta \ge 1$, and $\gamma \ge 0$. Then, provided that $e_1 \le B$ we have

$$e_n \le B \, \frac{f(n)}{n^\beta}, \tag{3.14}$$

where $f(1) = 1$ and

$$f(n) = 1 + An^\gamma \left[\prod_{j=2}^{n-1} \left(1 + Aj^{\gamma-\beta}\right) + \sum_{i=2}^{n-1} \frac{1}{i^\beta} \prod_{j=i+1}^{n-1} \left(1 + Aj^{\gamma-\beta}\right) \right], \quad n \ge 2. \tag{3.15}$$

Proof. The proof proceeds by mathematical induction. The right-hand side of (3.14) reduces to B for $n = 1$ (with the usual convention for the product: $\prod_{i=1}^{0} = 1$) which yields $e_1 \le B$. The initial inductive step is thus satisfied by the assumption.

Assume now that (3.14) is satisfied for $(n-1)$th term. We will show that this also is the case for e_n. To this end, use the inductive assumption to obtain

$$e_n \leq \frac{B}{n^\beta}\left(A\sum_{i=1}^{n-1}(n-i)^\gamma\ \frac{f(i)}{i^\beta}+1\right). \tag{3.16}$$

We can immediately estimate the sum to obtain

$$e_n \leq \frac{B}{n^\beta}\left(An^\gamma\sum_{i=1}^{n-1}\frac{f(i)}{i^\beta}+1\right). \tag{3.17}$$

In order to make the above inequality to satisfy the assertion we require that

$$An^\gamma\sum_{i=1}^{n-1}\frac{f(i)}{i^\beta}+1=f(n),\quad f(1)=1. \tag{3.18}$$

We will show that the solution of this nonlocal recurrence equation is equal to (3.15). Define $g(n):=\sum_{i=1}^{n-1}f(i)i^{-\beta}$, and notice that

$$g(n+1)-g(n)=\frac{f(n)}{n^\beta},\quad g(2)=1. \tag{3.19}$$

Hence, thanks to (3.18) we have

$$An^\gamma g(n)+1=n^\beta\left(g(n+1)-g(n)\right),\quad g(2)=1. \tag{3.20}$$

After rearranging we obtain the following nonhomogeneous recurrence relation

$$g(n+1)=n^{-\beta}+\left(1+An^{\gamma-\beta}\right)g(n),\quad g(2)=1. \tag{3.21}$$

To simplify the notation we temporarily introduce $a(n):=n^{-\beta}$ and $b(n):=1+An^{\gamma-\beta}$ and solve the following equation by the successive iteration

(3.22)
$$\begin{aligned} g(n+1) &= a(n)+b(n)g(n)=a(n)+a(n-1)b(n)+b(n)b(n-1)g(n-1)\\ &= a(n)+a(n-1)b(n)+a(n-2)b(n-1)b(n)+b(n)b(n-1)b(n-2)g(n-2)\\ &= \cdots \end{aligned}$$

Continuing in this inductive fashion we can show that

$$g(n+1)=\prod_{j=2}^{n}b(j)+\sum_{i=2}^{n}a(i)\prod_{j=i+1}^{n}b(j), \tag{3.23}$$

where the convention $\prod_{j=2}^{1}=1$ is used. Going back to the original variables we have

$$g(n+1)=\prod_{j=2}^{n}\left(1+Aj^{\gamma-\beta}\right)+\sum_{i=2}^{n}\frac{1}{i^\beta}\prod_{j=i+1}^{n}\left(1+Aj^{\gamma-\beta}\right),\quad n\geq 1, \tag{3.24}$$

and if we use (3.18) and the definition of $g(n)$, we can state the result in terms of the $f(n)$ function

$$f(n)=1+An^\gamma\left[\prod_{j=2}^{n-1}\left(1+Aj^{\gamma-\beta}\right)+\sum_{i=2}^{n-1}\frac{1}{i^\beta}\prod_{j=i+1}^{n-1}\left(1+Aj^{\gamma-\beta}\right)\right],\quad n\geq 2. \tag{3.25}$$

This concludes the proof. □

Having the above in hand we can state the main result.

THEOREM 3.5. *Assume that* $0 \le z \le X < 1$, *and let* y_n *be the calculated from* (3.5) *approximation of the solution of* (2.15). *Moreover, define the error* $e_n := y_n - y(nh)$, *and assume that* $|e_1| \le \frac{1}{(m+1)C_1^m}\frac{\delta(h)}{h^{2-\alpha}}$, *where* C_1 *is from Lemma* 3.2. *Then, for a quadrature with* $\delta_n(h) \le 0$ *for every* $n \in \mathbb{N}$ *we have*

$$|e_n| \le const. \times \delta(h) h^{\alpha - A - 1} \quad as \quad h \to 0^+ \quad with \quad nh \to z_n, \tag{3.26}$$

with $A = \frac{WD}{(m+1)C_1^m}$, *where* W *is from* (3.4), *and* $D = \frac{m+1}{2m}\frac{1}{\Gamma(2-\alpha)}\left(\frac{1}{1-X}\right)^{1-\alpha}$.

Proof. Notice that by the assumption and Proposition 3.1 we have $e_n := y_n - y(nh) \ge 0$. Using Lagrange's mean-value theorem we obtain

$$y_n^{m+1} - y(nh)^{m+1} = (m+1)\xi_n^m e_n, \tag{3.27}$$

where $y(nh) \le \xi_n \le y_n$. On the other hand, from (2.15) and (3.2)

$$y_n^{m+1} - y(nh)^{m+1} = h\sum_{i=1}^{n-1} w_{n,i} K_{n,i} e_i - \delta_n(h) \le h\sum_{i=1}^{n-1} w_{n,i} K_{n,i} e_i + \delta(h). \tag{3.28}$$

Therefore, from (3.4)

$$(m+1) y(nh)^m e_n \le Wh\sum_{i=1}^{n-1} K_{n,i} e_i + \delta(h). \tag{3.29}$$

Now, using Lemma 3.2 we can estimate the left-hand side

$$(m+1) C_1^m (nh)^{2-\alpha} e_n \le Wh\sum_{i=1}^{n-1} K_{n,i} e_i + \delta(h). \tag{3.30}$$

The next step is to find an appropriate bound for the kernel K. To this end we go back to (2.13) and notice that

$$s^{\frac{\alpha}{2}}(1-u) - \left(1 - \frac{\alpha}{2}\right)(1-z) = s^{\frac{\alpha}{2}}(1-u)\left[1 - s^{-\frac{\alpha}{2}}\left(1 - \frac{\alpha}{2}\right)\frac{1-z}{1-u}\right] \le \frac{1}{2}, \tag{3.31}$$

because $(\frac{1-z}{1-u})^{\alpha/2} \le s \le 1$. Hence,

$$\begin{aligned} K(z,u) &\le \frac{m+1}{2m}\frac{1}{\Gamma(1-\alpha)}\int_{(\frac{1-z}{1-u})^{\alpha/2}}^{1} (1-s)^{-\alpha} ds \\ &= \frac{m+1}{2m}\frac{1}{\Gamma(2-\alpha)}\left(1 - \left(\frac{1-z}{1-u}\right)^{\alpha/2}\right)^{1-\alpha}. \end{aligned} \tag{3.32}$$

Moreover, by convexity

$$\left(1 - \left(\frac{1-z}{1-u}\right)^{\alpha/2}\right)^{1-\alpha} = \left(1 - \left(1 - \frac{z-u}{1-u}\right)^{\alpha/2}\right)^{1-\alpha} \le \left(\frac{z-u}{1-u}\right)^{1-\alpha}. \tag{3.33}$$

Finally,

$$K(z,u) \le \frac{m+1}{2m}\frac{1}{\Gamma(2-\alpha)}\left(\frac{z-u}{1-X}\right)^{1-\alpha} =: D\,(z-u)^{1-\alpha}, \tag{3.34}$$

where we have used the assumption that $0 \le z \le X < 1$.

Now, going back to (3.30) and noticing that $z_n - z_i = h(n-i)$ we can write

$$\begin{aligned} e_n &\le \frac{1}{n^{2-\alpha}} \left(\frac{WD}{(m+1)C_1^m} \sum_{i=1}^{n-1} (n-i)^{1-\alpha} e_i + \frac{1}{(m+1)C_1^m} \frac{\delta(h)}{h^{2-\alpha}} \right) \\ &=: \frac{1}{n^{2-\alpha}} \left(A \sum_{i=1}^{n-1} (n-i)^{1-\alpha} e_i + B \right). \end{aligned} \tag{3.35}$$

This form of the inequality can be plugged into Lemma 3.4 to yield

$$e_n \le \frac{1}{(m+1)C_1^m} \frac{\delta(h)}{(nh)^{2-\alpha}} f(n). \tag{3.36}$$

It is interesting for us to learn how the function f behaves asymptotically as $n \to \infty$. To see the exact order we write it explicitly as

$$f(n) = 1 + An^{1-\alpha} \left[\prod_{j=2}^{n-1} \left(1 + \frac{A}{j}\right) + \sum_{i=2}^{n-1} \frac{1}{i^{2-\alpha}} \prod_{j=i+1}^{n-1} \left(1 + \frac{A}{j}\right) \right], \quad n \ge 2. \tag{3.37}$$

First, we can use Stirling's formula to obtain

$$\prod_{j=2}^{n-1} \left(1 + \frac{A}{j}\right) = \frac{\Gamma(n+A)}{\Gamma(n)\Gamma(2+A)} \sim \frac{n^A}{\Gamma(2+A)} \tag{3.38}$$

as $n \to \infty$. Similarly,

$$\begin{aligned} \sum_{i=2}^{n-1} \frac{1}{i^{2-\alpha}} \prod_{j=i+1}^{n-1} \left(1 + \frac{A}{j}\right) &= \frac{\Gamma(n+A)}{\Gamma(n)} \sum_{i=2}^{n-1} \frac{1}{i^{2-\alpha}} \frac{\Gamma(i+1)}{\Gamma(i+1+A)} \\ &\sim n^A \left(\frac{1}{\Gamma(2+A)} + \sum_{i=2}^{\infty} \frac{1}{i^{2-\alpha}} \frac{\Gamma(i+1)}{\Gamma(i+1+A)} \right) \end{aligned} \tag{3.39}$$

as $n \to \infty$. Therefore, because nh remains bounded for large n, we have

$$e_n \le \text{const.} \times \delta(h) h^{\alpha - A - 1} \quad \text{as} \quad n \to \infty. \tag{3.40}$$

This concludes the proof. ∎

Up to this point we have a result stating that a family of numerical methods (3.5) with $\delta_n(h) \le 0$ will be convergent provided that the constant A is sufficiently small. Notice that since by Lemma 3.3 the solution y is not regular at $z \to 0^+$ we do not have to (and ought to!) use a high-order quadrature. Moreover, since $y(z)^{m+1}$ is Lipschitz continuous for $X \le z \le 1$, the classical theory of numerical methods for integral equations works on that interval. Hence, we are only interested in solving (2.15) for the neighborhood of zero, i.e., $0 \le z < X$. Eventually, X can be made sufficiently small.

As an example of the above we choose the quadrature to be the midpoint method and formulate the result as a corollary.

COROLLARY 3.6. *Assume that $0 \le z \le X < 1$, and let y_n be the calculated from* (3.5) *approximation to the solution of* (2.15). *If the quadrature* (3.2) *is chosen to be the midpoint method* (3.6), *i.e., y_1 chosen according to* (3.12) *and*

$$y_{2n+k}^{m+1} = \frac{1}{2} h K_{2n+k,k}\, y_k + 2h \sum_{i=1}^{n} K_{2n+k,2i+k-1}\, y_{2i+k-1}, \quad k \in \{0,1\}, \quad n > 1, \tag{3.41}$$

then it is convergent provided that

$$m > 2 - \alpha \quad \textit{and} \quad A = \frac{2}{m} \frac{1}{C_1^m \Gamma(2-\alpha)(1-X)^{1-\alpha}} < \alpha + \frac{2-\alpha}{m}. \tag{3.42}$$

Moreover, the order of convergence is not smaller than

$$2 - (2-\alpha)\left(1 - \frac{1}{m}\right) - \frac{2}{m} \frac{1}{C_1^m \Gamma(2-\alpha)(1-X)^{1-\alpha}}. \tag{3.43}$$

Proof. We have to check whether the assumptions of Theorem 3.5 are satisfied. First, it is an easy geometrical reasoning to ascertain that $\delta_n(h) \le 0$ when y is locally a concave function (for details, see [41]). This is precisely the present case due to Lemma 3.3 and the assumption that $m > 2 - \alpha$ (since $0 < \alpha < 1$).

As was noted in [16] the midpoint quadrature for a nonsmooth function will, in our case, have an order of $\delta(h) \propto h^{1+(2-\alpha)/m}$ (in contrast with the second order for smooth functions). Since we are initializing the iteration from (3.12), which is an asymptotic form of the solution at $z \to 0^+$, the starting error will be $\mu = o(h^{(2-\alpha)/m})$. The number $(2-\alpha)/m$ is always greater than $1 + (2-\alpha)/m - 2 + \alpha$, and hence $|e_1|$ will be smaller than const. $\times\, \delta(h) h^{\alpha-2}$ for sufficiently small h. The assumption of Theorem 3.5 concerning the initial step is thus satisfied.

Finally, we can estimate the convergence error. From Theorem 3.5 we have

$$|e_n| \le \text{const.} \times h^{1-(2-\alpha)\left(1-\frac{1}{m}\right)-A+1}. \tag{3.44}$$

To prove the convergence we have to show that the exponent is a positive constant. To this end notice that by the assumption

$$\begin{aligned} 1 - (2-\alpha)\left(1 - \frac{1}{m}\right) - A + 1 &> 1 - (2-\alpha)\left(1 - \frac{1}{m}\right) \\ &\quad -\alpha - \frac{2-\alpha}{m} + 1 = 1 - 2 + \alpha - \alpha + 1 = 0. \end{aligned} \tag{3.45}$$

Therefore the method is convergent. ∎

From Lemma 3.2 we see that C_1^m is bounded for any $m \ge 1$, and hence, the last term in (3.43) can be arbitrarily close to zero for sufficiently large m. Moreover, we can see that the theoretical estimate on the convergence error converges to α for large m. Therefore, the method is convergent for any $0 < \alpha < 1$ for sufficiently large m.

4. Numerical calculations. We illustrate above results by a series of numerical examples. First, we will determine the empirical order of convergence by approximating the solution of (2.15) by (3.41) for different values of α and m. In each simulation we approximate the order with the use of the Aitken's method (it finds the order by extrapolation and refinement of the grid with n, $2n$, and $4n$ steps; see [33]) and compare it to the theoretical estimate (3.43). The results obtained in MATLAB are given in Table 1.

First, we have to remark that obtaining an accurate value of the order of convergence is demanding on the computer power. We have settled to choosing $N = 3 \times 10^3$ as the number of subdivisions of the $[0,1]$ interval (hence, the maximal division in Aitken's method is 1.2×10^4).

TABLE 1
Results of the simulations done for $n = 3000$. Theoretical estimate on the order has been written only when (3.43) gave a positive value.

α	m	Empirical order	Theoretical order (3.43)
$\alpha = 0.1$	$m = 1$	0.69	–
	$m = 10$	0.46	–
	$m = 100$	0.14	–
	$m = 1000$	0.68	–
	$m = 10000$	0.98	0.09
$\alpha = 0.2$	$m = 1$	0.68	–
	$m = 10$	0.51	–
	$m = 100$	0.39	–
	$m = 1000$	0.93	0.14
$\alpha = 0.3$	$m = 1$	0.65	–
	$m = 10$	0.55	–
	$m = 100$	0.62	0.05
	$m = 1000$	0.96	0.27
$\alpha = 0.4$	$m = 1$	0.54	–
	$m = 10$	0.60	–
	$m = 100$	0.78	0.26
	$m = 1000$	0.95	0.38
$\alpha = 0.5$	$m = 1$	1.02	–
	$m = 10$	0.66	–
	$m = 100$	0.87	0.41
	$m = 1000$	0.93	0.49
$\alpha = 0.6$	$m = 1$	1.01	0.15
	$m = 10$	0.73	0.12
	$m = 100$	0.92	0.55
	$m = 1000$	0.90	0.59
$\alpha = 0.7$	$m = 1$	0.90	0.44
	$m = 10$	0.81	0.39
	$m = 100$	0.95	0.66
	$m = 1000$	0.86	0.69
$\alpha = 0.8$	$m = 1$	0.86	0.68
	$m = 10$	0.89	0.59
	$m = 100$	0.96	0.77
	$m = 1000$	0.82	0.79
$\alpha = 0.9$	$m = 1$	0.84	0.88
	$m = 10$	0.96	0.76
	$m = 100$	0.96	0.88
	$m = 1000$	0.78	0.89
$\alpha = 0.99$	$m = 1$	0.82	1.04
	$m = 10$	1.00	1.07
	$m = 100$	0.96	0.98
	$m = 1000$	0.74	0.98

Notice that for $\alpha \to 1^-$ the method order approaches 1 both theoretically (for large m) and empirically. For smaller values of α the empirical order gets lower, but this fact can be due to the singularity of the kernel (2.16) at $\alpha = 0$, which is difficult to resolve numerically. In many cases the theoretical estimate is lower than the numerically found value of the order. This was anticipated since the value (3.43) is not optimal—it depends on the bound from Lemma 3.2, which might not be accurate enough. It also may be conjectured that, guessing from our simulations, the true

order of the method is equal to 1 for, at least, α close to unity. The verification of this claim requires more refined proof techniques and numerical simulations, which is the main goal of our future work. Overall, the numerical simulations show that (3.41) is convergent for any m and α that we have chosen with an indication that it may converge for the whole admissible range of these parameters.

In the second experiment we use (3.41) to calculate the wetting front position (2.8), which is of utmost importance in the physical experiment. It basically indicates how far the water has percolated at a given time. The results of our calculations are given on Figure 1. The graph presents η^* as a function of $1 \leq m \leq 100$ plotted on log-log scale for $\alpha \in \{0.1, 0.5, 0.9, 1\}$. The calculation for the classical diffusion with $\alpha = 1$ has been added for a comparison. There is a clear monotone relation between η^* and m for fixed α and vice versa. The power law when $m \to \infty$ can be easily obtained from Lemma 3.2 and (2.8). To this end notice that $y(1)^m$ is always bounded and hence from (2.8) we have

$$\frac{D_1}{\sqrt{m}} \leq \eta^* \leq \frac{D_2}{\sqrt{m}} \quad \text{as} \quad m \to \infty \tag{4.1}$$

for some constants $D_{1,2}$. Therefore, η^* for fixed α is bounded by power functions of the type $m^{-1/2}$. We can observe this fact on the log-log plot on Figure 1. For large m all curves have the same tangent equal to $-\frac{1}{2}$.

Some exemplary self-similar profiles of solutions to (1.1) are depicted on Figure 2. They have been calculated with the use of (2.5). Functions are monotone decreasing

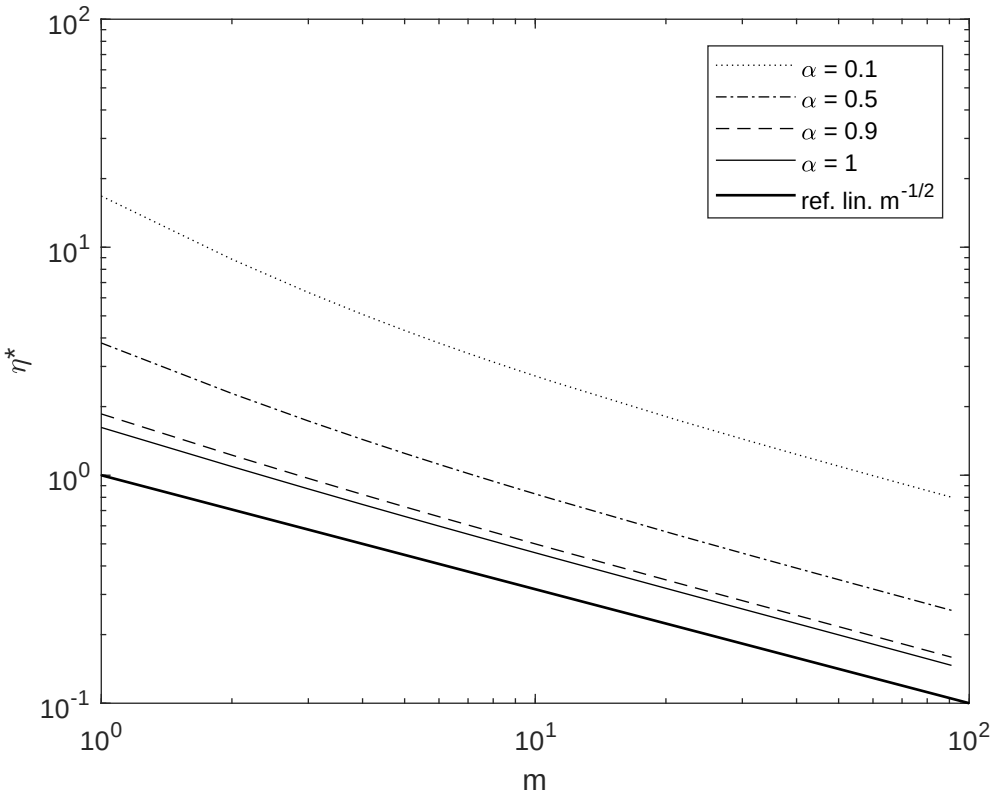

FIG. 1. *Wetting front position* (2.8) *for different* α *and* m *on a log-log scale. The reference line* $m^{-1/2}$ *plotted in bold has been added for comparison. Here, the number of divisions is* $n = 3000$.

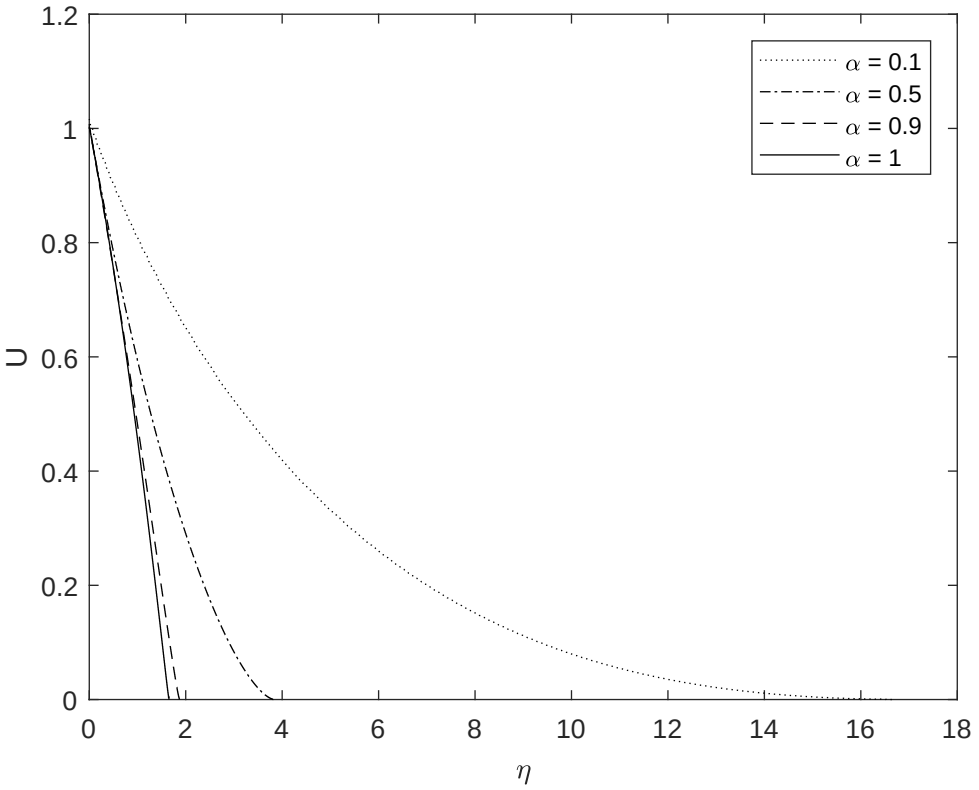

FIG. 2. *Self-similar solutions of* (1.1) *plotted for* $m = 1$ *and several values of* α. *Here, the number of divisions is* $n = 3000$.

and have a compact support which confirms our intuition and was rigorously proved in [42]. Once again, we can see that η^* decreases with increasing α.

Finally, we would like to present a comparison between our method and the one used in [38]. The latter is a finite difference discretization of the original PDE (1.1) with a linearization of the diffusivity. Notice that, intuitively, numerically solving (1.1) should be inferior to using (3.5) since it involves discretizing two independent variables rather than one. Linearization done in each step is also a factor that can introduce errors.

Let u_j^i denote the numerical approximation $u(jk, il)$ being the solution to (2.15) where k and l are the spatial and temporal steps. Also fix $X > 0$ and $T > 0$ to be the bounds of the space-time domain. The method used in [38] is the θ-weighted implicit finite difference

(4.2)
$$u_j^{i+1} - (1-\theta)\,\frac{h^\alpha}{k^2\Gamma(2-\alpha)}\left(D_{j-1/2}^{i+1}u_{j-1}^{i+1} + \left(D_{j-1/2}^{i+1} + D_{j+1/2}^{i+1}\right)u_j^{i+1} + D_{j+1/2}^{i+1}u_{j+1}^{i+1}\right)$$
$$= -\sum_{k=1}^{i} a_{k,i}u_j^k + \theta\frac{h^\alpha}{k^2\Gamma(2-\alpha)}\left(D_{j-1/2}^{i}u_{j-1}^{i} + \left(D_{j-1/2}^{i} + D_{j+1/2}^{i}\right)u_j^{i} + D_{j+1/2}^{i}u_{j+1}^{i}\right),$$

where the weights are defined by

$$a_{k,i} = (i-k+2)^{1-\alpha} - 2(i-k+1)^{1-\alpha} + (i-k)^{1-\alpha}, \tag{4.3}$$

and $D^i_{j\pm1/2}$ is the linearized value of the diffusion coefficient

$$D^i_{j\pm1/2} = \frac{1}{2}\left((u^m)^i_j + m(u^{m-1})^i_j\left(u^i_j - u^{i-1}_j\right) + (u^m)^i_{j\pm1/2} + m(u^{m-1})^i_{j\pm1/2}\left(u^i_{j\pm1/2} - u^{i-1}_{j\pm1/2}\right)\right). \tag{4.4}$$

For stability reasons we require that the spatial and temporal grids have to be interrelated

$$kl^{-\frac{\alpha}{2}} = \lambda = \text{const}. \tag{4.5}$$

For a benchmark of our method suppose that we would like to calculate the wetting front using (2.8) and (4.2). To compare those two methods we fix the grid spacing of (3.41), i.e., h, and have to set k and l to produce a similar accuracy for the self-similar variable $xt^{-\alpha/2}$. In order to calculate the wetting front with (4.2) we iterate the scheme up to the final time T, find the zero $x^*(T)$ of the function $u(\cdot, T)$, and compute

$$\eta^*_{FD} = x^*(T)T^{-\frac{\alpha}{2}}. \tag{4.6}$$

Now, we have to specify the correct values of k and l according to h in order to evaluate η^* with a similar accuracy for our two methods. For a fixed time $t = T$ the difference between two consecutive grid points of the self-similar variable is taken to be equal to h:

$$(j+1)kT^{-\frac{\alpha}{2}} - jkT^{-\frac{\alpha}{2}} = h. \tag{4.7}$$

Simplifying and using (4.5) we obtain

$$l = T\left(\frac{h}{\lambda}\right)^{\frac{2}{\alpha}}. \tag{4.8}$$

Immediately we notice one difficulty in numerical calculations. Since the exponent in the above formula is at least equal to 2, the time spacing l can quickly become prohibitively small (especially for small values of α). Therefore, it is much more convenient and less demanding on computer power to use the discretization of the integral equation (2.15) rather than the original PDE (1.1).

We illustrate the above remarks with an example. We calculate the reference value of η^*_{REF} with (3.41) using sufficiently large number of divisions, say, 3000. We fix $T = 0.1$ and $X = \gamma\eta^*_{REF}T^{-\alpha/2}$, where $\gamma > 1$ (here we take $\gamma = 1.1$). The choice of X is motivated by the fact that it will always be larger than the support of u at any time. We have $h = 1/n$ and further calculate k, l using (4.8) and (4.5). Now, compute the wetting front once from (2.8), call it η^*_{INT}, and once from (4.6). We further require that

$$\max\left\{\left|\eta^*_{REF} - \eta^*_{FD}\right|, \left|\eta^*_{REF} - \eta^*_{INT}\right|\right\} \le \epsilon, \tag{4.9}$$

where $\epsilon > 0$ is some arbitrarily chosen tolerance.

Results of numerical calculations are given in Table 2. The benchmark comparing two methods is the time factor

$$\tau = \frac{\text{calculation time with (4.2)}}{\text{calculation time with (3.41)}}. \tag{4.10}$$

Hence, the number τ shows how much the finite difference method is slower than the scheme based on the integral equation. Due to very high computational price of the finite difference method we have chosen to aim for only modest accuracy in calculating the wetting front. That is to say, we have put $m = 1$, $n = 100$, $\epsilon = 0.5$, $\lambda = 0.5$, and $\alpha \in \{0.75, 0.9, 0.99\}$. Immediately it is evident that the integral equation method (3.41) is far superior from the finite difference (4.2) in calculating the wetting front. The time factor for small α grows very fast and we have resigned from repeating calculations for $\alpha < 0.75$. However, we clearly see that the important parameter η^* can be readily and accurately computed with (3.41). Moreover, having the self-similar solution U it is just a matter of using (2.18) to obtain the space-time representation of the diffusion.

TABLE 2
Comparison between finite difference method (4.2) *and midpoint method* (3.41) *in calculating the wetting front. Here,* $m = 1$, $n = 100$, $\epsilon = 0.5$, *and* $\lambda = 0.5$.

α	0.75	0.9	0.99
τ	2.22×10^4	1.17×10^3	2.51×10^2

5. Conclusion. We have devised a convergent numerical method for solving the nonlocal nonlinear porous medium equation. Due to the non-Lipschitzian nonlinearity, the classical proof methods could have not been used. Our approach gives an estimate on the convergence error, but it is clear that it does not cover all of the admissible (α, m) parameter space and is not optimal. One of the reasons is the C_1 constant that enlarge the essential exponent A. The object of our future work will be to overcome this difficulty and obtain more strict bounds on the convergence order. On the other hand, as Table 1 indicates, the whole admissible range of m is allowed for $\alpha \geq 0.6$, which is consistent with the experimental results mentioned in the introduction. Moreover, numerical simulations show that the method should be convergent for $m > 1$ and every $\alpha \in (0, 1)$. In the future we are planning to prove this analytically. The comparison with the finite difference scheme (4.2) indicates that our method is superior in performance and is able to compute the solution of the subdiffusion diffusion equation very quickly.

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