Partial Differential Equations

Lecture notes

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Contents

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Some preliminary information

Literature. The lecture will be self-sufficient, but some textbooks will be very helpful. Below are my suggestions (from the most basic to more advanced).

1. *S.J.Farlow, Partial Differential Equations for Scientists and Engineers, Dover Publications, 1993.* This is a very light introduction to partial equations. It contains a lot of exercises and interestingly lined material.

2. *R.Haberman, Applied Partial Differential Equations with Fourier Series and Boundary Value Problems, Pearson, 2012.* Great book that discusses the subject in great detail. The exact solutions of many examples deserve attention.

3. *J. D. Logan, An Introduction to Nonlinear Partial Differential Equations, Wiley, 2008.* A very good applied book. For us, it will be sufficient to read only chapters concerning quasilinear equations.

4. *Samarski, Tichonov, Equations of mathematical physics.* One of my favourites. A classical book on PDEs. More difficult that the others but very deep and thorough.

5. *Ockendon, Howison, Lacey, Movchan, Applied Partial Differential Equations, Oxford University Press, 2003.* A lot of real-world examples along with with comprehensive theory written by experts in industrial mathematics. Caveat: its a little bit daunting and difficult read.

Grading. All the relevant information will be published on my webpage.

Preliminaries. In order to understand the material you will have to be proficient in Calculus, Algebra, and Ordinary Differential Equations.

1 Introduction

Partial differential equations (PDEs) are a central concept for many fields of science such as mathematics, physics, chemistry, biology, and even medicine. PDEs occur wherever there is a need to describe a change in certain quantities. As we already know, any change in time or space is described by derivatives. If a given phenomenon occurs in more than one dimension, the derivatives describing it are necessarily partial and are intertwined with equations modelling various real-world laws. During this lecture we will focus mostly on applications and abilities used in solving some important problems. However, please note that the theory for PDEs is extremely beautiful, deep, broad, interesting, and difficult. It is a subject of constant investigations of many world-class mathematicians all around the world.

Similarly to ordinary equations, we can classify PDEs. The basic types of classification are:

- with respect to the highest order of derivatives,
- with respect to linearity.

We will start discussing the topic with first-order equations, which can be both linear and non-linear (we will limit ourselves to quasilinear ones). Then we proceed to the analysis of second order linear equations. It turns out that almost all PDEs in mathematics and physics are at most of the second order. At the end of the semester, we will understand, among others, phenomena such as the propagation of all waves, heat, we will learn how the gravitational potential is distributed, and we will follow the shock wave propagation.

More concretely, we will focus on equations of the form

$$F(t, \mathbf{x}, u, u_t, u_t, u_x, ..., u_{xx}, u_{xy}, ...) = 0, \tag{1.1}$$

where F is some function, t and $\mathbf{x} = (x, y, z)$ are independent variables (temporal + spatial), $u = u(t, \mathbf{x})$ is the sought function, while partial derivatives are denoted with subscripts

$$u_t := \frac{\partial u}{\partial t}, \quad u_x := \frac{\partial u}{\partial x}, \quad u_{xx} := \frac{\partial^2 u}{\partial x^2}, \quad ... \tag{1.2}$$

Depending on the situation, we will mix traditional notation with ∂ and that with the subscript. In addition, in most cases we will limit ourselves to considering one spatial dimension (a notable exception will be the Poisson's equation). We also everywhere assume that the function u is *as regular as we need it to* (unless written differently). We will be primarily interested in *solving* partial equations under various conditions, but we will not refrain from purely theoretical results. Get ready!

2 Partial differential equations of the first order. Method of characteristics

Many natural or industrial phenomena occur while conserving some quantities. The simplest examples of these preserved variables are mass, momentum or energy. It

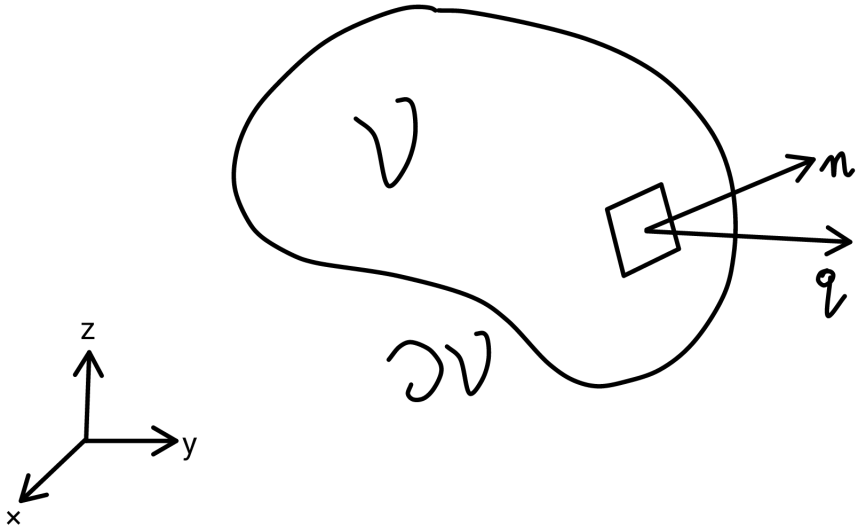

Figure 1: An arbitrary volume in space with its normal $\mathbf{n}$ and the flux vector $\mathbf{q}$.

turns out that we can write an equation that describes all possible laws in a very general way using a differential equation. This PDE describes the evolution over time of a quantity under study.

Example. (*Conservation law*)
Let us consider a certain quantity describing the concentration (for example, the density of a given substance, energy, momentum, charge, population size, ...), which can change both in time and space. Denote it by $u = u(\mathbf{x}, t)$, where t is the time while $\mathbf{x} = (x, y, z)$ the spatial variable. Let us now choose *any* volume $\mathcal{V}$ located in the space $\mathbb{R}^3$ (see Fig. 1). Denote the volume area of $\mathcal{V}$ by $\partial\mathcal{V}$. The total change in time of u in $\mathcal{V}$ can then be calculated with the formula

$$\text{Total change of } u \text{ in time} = \frac{d}{dt} \underbrace{\iiint_{\mathcal{V}} u(\mathbf{x}, t)\, d\mathbf{x}}_{\text{Total amount of } u \text{ in } \mathcal{V}} \tag{2.1}$$

Now, let us define the *flux* $\mathbf{q}(\mathbf{x}, t)$, i.e. the amount of u per unit time and per unit area that flows at the moment t in the direction $\mathbf{q}/|\mathbf{q}|$ at the point in space $\mathbf{x}$ (see Fig 1). The flux is positive if the volume flows towards the normal vector (i.e. *flows inside-out*) and negative when it goes in the opposite direction. Thus, the resultant flow of u through the whole volume $\partial\mathcal{V}$ is

$$\text{Net flow thorugh } \partial\mathcal{V} = -\iint_{\partial\mathcal{V}} \mathbf{q}(\mathbf{x}, t) \cdot \mathbf{n}\, dS, \tag{2.2}$$

where the integral is over the whole closed surface. Note that in the above we are calculating the dot product of the flux with the normal to the surface $\mathbf{q} \cdot \mathbf{n}$. This is because we measure how much of u is actually moving outside of $\mathcal{V}$ (recall the interpretation of the dot product).

Finally, assume that u can be created or destroyed (for example, if it is a chemical compound it could be formed as a result of a certain chemical reaction; if u represents a population size it could increase / decrease through births / deaths) . Let us define the *source function* $f = f(\mathbf{x}, t, u)$ as the rate of u (positive or negative) being created/annihilated. That is, $f(\mathbf{x}, t, u)$ describes how much of u per unit time appears at the moment t and the spatial point $\mathbf{x}$. Note that f may depend on u. We can see that the resultant change in the amount of u created in the volume $\mathcal{V}$ is

$$\text{Rate of the production of } u = \iiint_{\mathcal{V}} f(\mathbf{x}, t, u) d\mathbf{x}. \tag{2.3}$$

Combining (2.1) with (2.2) and (2.3) yields

$$\frac{d}{dt} \iiint_{\mathcal{V}} u(\mathbf{x}, t) d\mathbf{x} = - \iint_{\partial\mathcal{V}} \mathbf{q}(\mathbf{x}, t) \cdot \mathbf{n} \, dS + \iiint_{\mathcal{V}} f(\mathbf{x}, t, u) d\mathbf{x}. \tag{2.4}$$

Thus, we see that the rate of change of u in the volume $\mathcal{V}$ is equal to the net flow summed up with the rate of production / destruction of this quantity. The formula (2.4) is the *integral conservation law* and is true even for functions u, $\mathbf{q}$ and f which are not differentiable.

If we now assume that u, $\mathbf{q}$ and f are of C^1 regularity, then we can differentiate under the first integral and use the divergence theorem (Gauss-Ostrogradski) [1] in the second integral. We then get

$$\iiint_{\mathcal{V}} (u_t(\mathbf{x}, t) + \operatorname{div} \mathbf{q}(\mathbf{x}, t) - f(\mathbf{x}, t, u)) \, d\mathbf{x} = 0. \tag{2.5}$$

Since the integrand is continuous and the volume $\mathcal{V}$ is *arbitrary* it follows that

$$u_t + \operatorname{div} \mathbf{q} = f(\mathbf{x}, t, u), \tag{2.6}$$

where, similarly as for ordinary equations, we do not write arguments of the dependent variable. The above equation is called the *differential conservation law* or *continuity equation*. In one spatial dimension it has the form

$$u_t + q_x = f(x, t, u). \tag{2.7}$$

Note that if we do not know the flux $\mathbf{q}$ we have more variables than equations (2 vs. 1). This is because the conservation law is a very general equation that describes a magnitude of different phenomena. This is why we need yet another equation that gives us the relationship $\mathbf{q}(u)$. This is different for each physical situation and is called the *constitutive relation*. □

[1] Reminder. The divergence (or G-O) theorem shows the relationship between the oriented surface integral of $\mathbf{q} = (q_1, q_2, q_3)$ and the volume integral. It has the form $\iint_{\partial\mathcal{V}} \mathbf{q} \cdot \mathbf{n} dS = \iiint_{\mathcal{V}} \operatorname{div} \mathbf{q} \, d\mathbf{x}$, where the *divergence* is defined by $\operatorname{div} \mathbf{q} = \frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} + \frac{\partial q_3}{\partial z}$.

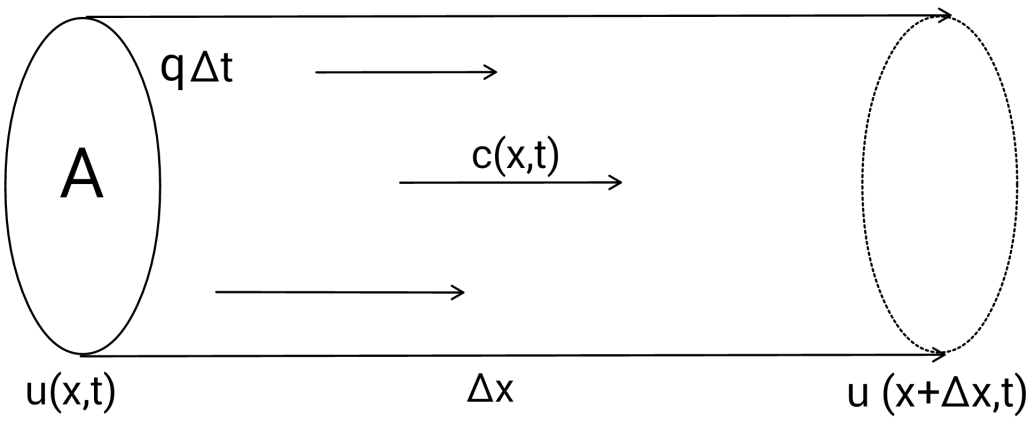

Figure 2: One dimensional advection.

Example. (*Convection/Advection*)
One of the simplest and most important constitutive relation is the one describing convection, i.e. transport of a given quantity caused by the movement of the medium. Examples are the spread of pollution in the river, smoke from a chimney or cars on the highway, to name only a few. Without the loss of generality, we will derive the form of $\mathbf{q}$ in one spatial dimension.

Let us assume that our substance with a density u is carried by the medium towards the x-axis at the speed of $c = c(\mathbf{x}, t)$. Let's consider a short period of time Δt, in which the medium travels a distance of Δx, of course

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = c(\mathbf{x}, t). \tag{2.8}$$

Imagine any surface A in the $y - z$ plane (see Fig 2). Then, the total amount of u that flows through A during Δt is $A\Delta t\ q$ (from the definition of flux). On the other hand, the same amount of substance travels Δx, so by the definition u (amount of substance per unit volume) we have

$$A\Delta t\ q = A\Delta x\ u. \tag{2.9}$$

Now, we divide the above equation and go to the limit using (2.8). We obtain

$$q = c(\mathbf{x}, t)u. \tag{2.10}$$

If we assume that the medium advects our substance at the speed of $\mathbf{c} = \mathbf{c}(\mathbf{x}, t)$ in any direction, then

$$\mathbf{q} = \mathbf{c}(\mathbf{x}, t)u. \tag{2.11}$$

By combining the above constitutive relation with the conservation law (**??**) we get *advection equation*

$$u_t + \operatorname{div}(\mathbf{c}(\mathbf{x}, t)u) = f(\mathbf{x}, t, u). \tag{2.12}$$

In an important case for a constant velocity we have

$$u_t + \mathbf{c} \cdot \mathbf{grad}\ u = f(\mathbf{x}, t, u), \tag{2.13}$$

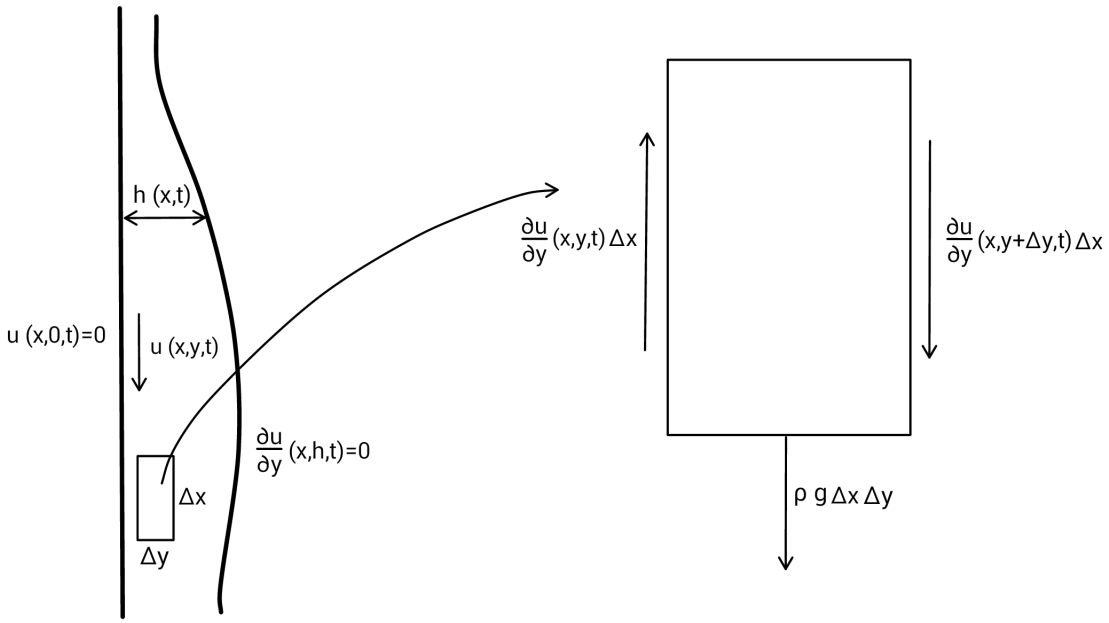

Figure 3: A cross section through a layer of paint.

while in one dimension this yields

$$u_t + cu_x = f(x, t, u). \tag{2.14}$$

□

Example. (*Paint flow*)
Many industrial-related situations can be modelled by first-order equations. Let us consider a layer of paint that flows down the wall. In industry, it is very important that the paint covers the surfaces evenly, without any heterogeneity. It is therefore important to choose the right viscosity and pressure to ensure the best coverage.

Now, let $u = u(x, y, t)$ be the velocity of the paint layer in the x direction vertically downwards and $h = h(x, t)$ be its thickness (see Fig. 3). The entire flow is governed by the balance of gravity and viscous forces. The experiment shows that for most substances encountered (Newtonian fluids), viscous forces are proportional to the fluid velocity gradient. In our case

$$\text{Visous force per unit length } = \mu \frac{\partial u}{\partial y}, \tag{2.15}$$

where the coefficient μ is called the *viscosity* of fluid. Let us now choose a tiny rectangle of paint with sides Δx and Δy (Fig. 3). The resultant viscous force acts in the vertical direction and has a value

$$\begin{gathered}\text{Net viscous force acting of the rectangle } = \\ \Delta x \left(\mu \frac{\partial u}{\partial y}(x, y, t) - \mu \frac{\partial u}{\partial y}(x, y + \Delta y, t) \right).\end{gathered} \tag{2.16}$$

The above force is (almost) balanced with the gravity

$$\mu\Delta x\left(\mu\frac{\partial u}{\partial y}(x,y,t)-\mu\frac{\partial u}{\partial y}(x,y+\Delta y,t)\right)=\rho g\Delta x\Delta y, \tag{2.17}$$

where ρ is the paint density. Dividing by Δx i and taking the limit Δx, $\Delta y \to 0$ we obtain

$$\frac{\partial^2 u}{\partial y^2}(x,y,t)=-\alpha, \tag{2.18}$$

where $\alpha := \rho g/\mu$ is a constant. The above equation can immediately be integrated yielding

$$u(x,y,t)=-\frac{1}{2}\alpha y^2+A(x,t)y+B(x,t), \tag{2.19}$$

where $A = A(x,t)$ and $B(x,t)$ are unknown functions, which can be determined from boundary conditions

$$u(x,0,t)=0, \quad \frac{\partial u}{\partial y}(x,h,t)=0. \tag{2.20}$$

The first condition states that the paint adheres to the wall and the second is a formulation of disappearance of viscous forces on the surface of the paint. We finally obtain

$$u(x,y,t)=\frac{1}{2}\alpha y\left(2h(x,t)-y\right). \tag{2.21}$$

As we can see the velocity has a *parabolic* profile.

To obtain an equation describing the dynamics of our fluid we must refer to the conservation law. In our case, the mass of paint is preserved, i.e. analogously to (2.4) we have

$$\frac{d}{dt}\int_0^h \rho dy+\frac{\partial}{\partial x}\int_0^h \rho u(x,y,t)dy=0. \tag{2.22}$$

The form of the second integral results from the definition of the flux and from what we said when deriving the equation of convection (2.12). Substituting (2.21) to the above equation we get

$$0=\frac{\partial h}{\partial t}+\frac{1}{2}\alpha\frac{\partial}{\partial x}\int_0^h y(2h(x,t)-y)dy=\frac{\partial h}{\partial t}+\frac{1}{2}\alpha\frac{\partial}{\partial x}\left(\frac{2}{3}h^3\right). \tag{2.23}$$

Therefore, we finally obtain a first order equation

$$\frac{\partial h}{\partial t}+\alpha h^2\frac{\partial h}{\partial x}=0. \tag{2.24}$$

□

Example. (*River flow*) Looking at large spatial scales, the river's water movement is almost one-dimensional and it is governed by the force of gravity. However, this flow is very turbulent, which forces us to adopt some averaged models. Despite many simplifications, they are extremely accurate - especially in describing flood waves.

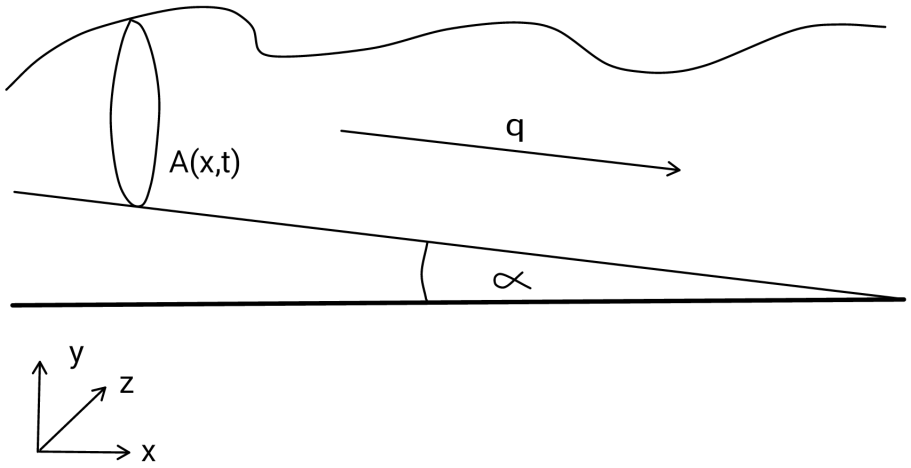

Figure 4: A crude cross-section through a river bed.

Imagine a river bed with a cross-sectional area $A = A(x,t)$ and a slope angle α (Fig. 4). Let the water flow in the x direction. Because the amount of water is preserved it follows that

$$\frac{d}{dt}\int_a^b \rho A(x,t)dx + \rho(q(b,t) - q(a,t)) = 0, \tag{2.25}$$

where $[a,b]$ is any interval along the direction of flow and $q = q(x,t)$ is the flow rate per unit mass (similar to flux). Going with the derivative under the integral, writing the flux as an integral of the derivative, we get the conservation law

$$\frac{\partial A}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{2.26}$$

Observations of the water level in different channels allow us to propose an empirical relationship between the flux and the cross section of the river $q = q(A,t)$. One of the simplest models is created if we assume that the water flow is caused by the balance of gravity and the friction force acting on the bottom of the bed. As in the convection equation, we can show that the average flow velocity is

$$v = \frac{q}{A}. \tag{2.27}$$

The friction force per unit of length, however, is proportional to the square of the velocity [2], That is $F_t = cv^2$. We thus get

$$cv^2 = \rho g A \sin\alpha. \tag{2.28}$$

We can now write the flux as

$$q = vA = A\sqrt{\frac{\rho g}{c}A\sin\alpha} = CA^{\frac{3}{2}}. \tag{2.29}$$

[2]Exactly as in the formula for air resistance (drag).

It is the so-called *Chézy's Law*. Plugging the flux into the conservation law we obtain

$$\frac{\partial A}{\partial t} + \frac{3}{2} C\sqrt{A}\frac{\partial A}{\partial x} = 0. \tag{2.30}$$

With some more advanced considerations we can obtain a more general expression (*Manning's Law*)

$$\frac{\partial A}{\partial t} + CA^n\frac{\partial A}{\partial x} = 0, \tag{2.31}$$

for some empirical constants C and n. □

Example. (*Demographic model with age structure*) Demography aims to determine the laws of evolution of the human population along with the age distribution of individual units. Let $u = u(a,t)$ be the distribution of the number of women with age a at the time t (i.e. $u(a,t)\Delta a$ is the approximate number of women aged from a to $a + \Delta a$). The total number of women is thus

$$N(t) = \int_0^\infty u(a,t)da. \tag{2.32}$$

(Why can we take the infinite limits of integration?) Our goal is to write an equation saying how N changes over time under the influence of births and deaths. Let $\Delta a > 0$. With the first order accuracy Δa we can write the conservation law

$$\frac{\partial}{\partial t}(u(a,t)\Delta a) = u(a,t) - u(a+\Delta a,t) - m(a)u(a,t)\Delta a. \tag{2.33}$$

Starting from left to right, the following terms describe: change in the total number of women aged from a to $a + \Delta a$, change due to aging, and mortality. The function $m = m(a)$ is the relative mortality density, i.e. $m(a)u(a,t)\Delta a$ is the number of women aged a to $a + \Delta a$ who die. By dividing by Δa and taking the limit we get

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = -m(a)u. \tag{2.34}$$

It is the so-called *McKendrick-von Foerster model*. As an initial condition we can take

$$u(a,0) = f(a). \tag{2.35}$$

We must now consider births. Let $b = b(a,t)$ be the relative distribution of children born to women with age a at time t. Then the total number of newborns is

$$B(t) = \int_0^\infty b(a,t)u(a,t)da. \tag{2.36}$$

Note that $u(0,t) = B(t)$, therefore births enter the model as a boundary condition. □

Example. (*Chemotherapy*) This model comes from Bischoff and describes the development of leukemia cancer cells. Let $0 \le x \le 1$ be a physiological variable describing the state of the cancer cell: 0 for the newly formed and 1 for the mature one, ready for

division. Let the cells mature at a constant rate, i.e. $dx/dt = v = \text{const}$. In addition, by $u = u(x,t)$ we denote the density of the number of cancer cells with maturity of x at time t. If $c = c(t)$ is the concentration of the chemotherapeutic drug, then a sensible model of cancer cell mortality is

$$m(t) = \frac{\alpha c(t)}{\beta + c(t)}, \tag{2.37}$$

where α and β are coefficients. We can see that the mortality is zero when the drug has not been given and saturates to the level of α when it was given a lot. This is the so-called *Michealis-Menten* model. As above, we can show that

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = -m(t)u, \quad 0 < x < 1, \quad t > 0, \tag{2.38}$$

with conditions

$$u(x,0) = u_0, \quad u(0,t) = 2u(1,t). \tag{2.39}$$

Note that the boundary condition describes the division of the mature cancer cell into two new ones. □

Example. (*Traffic flow*) We can also easily describe the basic dynamics of street traffic. Let $u = u(x,t)$ be the number of cars per unit of road length. If traffic is moving in one direction (as on highway lane), the number of cars is conserved according to the equation

$$u_t + q(u)_x = f(x,t,u), \tag{2.40}$$

where q is the flux and f is a source function (motorway entrances and exits). When there are few cars on the road, the flux should also be small. The other way round: when there are a lot of cars, the traffic is difficult and a jam is formed. In this situation, the flux also remains small. Therefore, the flux function should be defined in the range $[0, u_c]$, where u_c is the critical value of the density of cars for which the traffic is completely blocked. We have $q(0) = q(u_c) = 0$. The simplest model of this situation is $q(u) = Mu(u_c - u)$ parabola. Greenberg said that $q(u) = au\ln\frac{u_c}{u}$ for New York data. For the parabolic model, the equation takes the form

$$u_t + M(u_c - 2u)u_x = f(x,t,u). \tag{2.41}$$

It turns out that you can get a lot of interesting results based only on the qualitative features of q. □

2.1 Method of characteristics

It turns out that there is a very effective and elegantly geometric method of finding solutions to first order partial equations. To derive it, let us choose the class of equations for our study. From now on, we will be interested in equations in one spatial dimension.

Let us return to the conservation law (2.7). Suppose the flux is a function of time, space, and the dependent variable itself (e.g. concentration of substance). We thus have $q = q(x,t,u)$ which gives after calculating the derivative

$$u_t + q_u(x,t,u)u_x = f(x,t,u) - q_x(x,t,u). \tag{2.42}$$

This motivates us to formulate the following definition.

Definition 1. *A* quasilinear *equation is a first order partial differential equation of the form*

$$u_t + c(x,t,u)u_x = g(x,t,u), \quad t > 0, \quad x \in \mathbb{R}, \tag{2.43}$$

where c *(propagation speed) and* f *are given functions.*

If $c = c(x,t)$ *then the equation is* semi-linear *and if further we have* $g = g(x,t)$*, then the equation is* linear*.*

The above is often called a *kinematic wave equation*. Just like ordinary equations, the PDE (2.43) must be provided with an initial condition in order to have a unique solution (although, as we will see, we also have other options)

$$u(x,0) = \phi(x), \quad x \in \mathbb{R}, \tag{2.44}$$

where ϕ is a function that describes the initial distribution of u. Note the huge difference between ordinary and partial equations - here the initial condition is a *function*, not a number. Also note that the non-linearity in the equation (2.43) can only involve the solution u, not its derivatives. This is the reason of the adjective - quasilinear.

To derive the method of characteristics, notice that the left side of the equation (2.43) resembles a derivative of a composite function. This can be seen by defining an auxiliary function $U(t) := u(X(t),t)$, where $x = X(t)$ is, for the time being, an arbitrary curve on the xt plane (the so-called *space-time*). We have then

$$\frac{dU}{dt} = u_t + \frac{dx}{dt}u_x. \tag{2.45}$$

The resemblance to (2.43) will be complete if we require that

$$\frac{dX}{dt} = c(X,t,U). \tag{2.46}$$

Then, the equation (2.43) takes the form

$$\frac{dU}{dt} = g(X,t,U), \tag{2.47}$$

Equations (2.46) - (2.47) constitute a system of *ordinary* nonlinear differential equations. Therefore, we replaced the first order PDE with a system of ordinary equations, which can still be very difficult to analyse. Fortunately, the equations found in applications often have a more specific form. The above system should be supplied with initial conditions, which usually have the form

$$X(0) = \xi, \quad U(0) = u(X(0),0) = u(\xi,0) = \phi(\xi), \tag{2.48}$$

where we used (2.44). The number ξ is a parameter saying on which curve $x = X(t,\xi)$ we are (from which point it emanates from the axis $t = 0$). So in general, as a solution to our system of equations, we obtain two families of curves $U = U(t,\xi)$ and $x = X(t,\xi)$. From the theory of ordinary differential equations we know that with appropriate

assumptions for c and g (e.g. when there are of class C^1) the system (2.46) - (2.47) has exactly one solution locally around $t = 0$. In addition, this solution depends smoothly on ξ and $X_\xi(0,t)$ $neq0$. According to the implicit function theorem, we are able to find a function $\xi = \xi(x,t)$, which gives us $U(t,\xi(x,t)) = u(x,t)$, i.e. we obtain our solution. In practice, however, we will not always be able to *explicitly* extract the ξ parameter from equations and the solution will be given in a parametric form.

Definition 2. *Any curve from the family $x = X(t,\xi)$ being a solution of (2.46) with initial condition $X(0) = \xi$ is called the* characteristic *of the equation (2.43).*

The method of characteristics has a very nice geometric interpretation, which we will learn when considering various special cases.

2.2 Semilinear equations

First, let $g \equiv 0$. Then, the equation (2.43) takes the form

$$\begin{cases} u_t + c(x,t)u_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = \phi(x), \quad x \in \mathbb{R}. \end{cases} \tag{2.49}$$

This means that (2.47) simplifies to $dU/dt = 0$. Therefore, U is a constant function

$$U(t) = U(0) = u(X(0),0) = u(\xi,0) = \phi(\xi). \tag{2.50}$$

Recall that by definition the function U is equal to u on characteristics. The above equation shows that in the case of $g \equiv 0$ the solution u is *constant on the characteristics*! [3] To find the value of u at an arbitrary point point (x_0,t_0) it is now sufficient to solve

$$\frac{dX}{dt} = c(X,t), \quad X(0) = \xi, \tag{2.51}$$

find the characteristic $x = X(t,\xi)$ passing through (x_0,t_0), and then go back along it to $t = 0$ finding at the same time $\xi = \xi(x_0,t_0)$. Since u is constant on characteristics we have $u(x_0,t_0) = u(\xi(x_0,t_0),0) = \phi(\xi(x_0,t_0))$. Figure 5 shows the situation.

Example. For advection with a constant speed we have

$$\begin{cases} u_t + cu_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = \phi(x), \quad x \in \mathbb{R}. \end{cases} \tag{2.52}$$

The equation $dX/dt = c$ can be solved immediately getting $X(t,\xi) = ct + \xi$, from where $\xi(x,t) = x - ct$. These are straight lines inclined to the Ox axis at the angle $\arctan(1/c)$ (see Fig. 6). Since the function u is constant on characteristics, we have

$$u(x,t) = u(\xi(x,t),0) = \phi(\xi(x,t)) = \phi(x - ct). \tag{2.53}$$

Let us check whether $\phi(x - ct)$ is actually a solution

$$u_t + cu_x = \frac{\partial}{\partial t}\phi(x - ct) + c\frac{\partial}{\partial x}\phi(x - ct) = -c\phi'(x - ct) + c\phi'(x - ct) = 0, \tag{2.54}$$

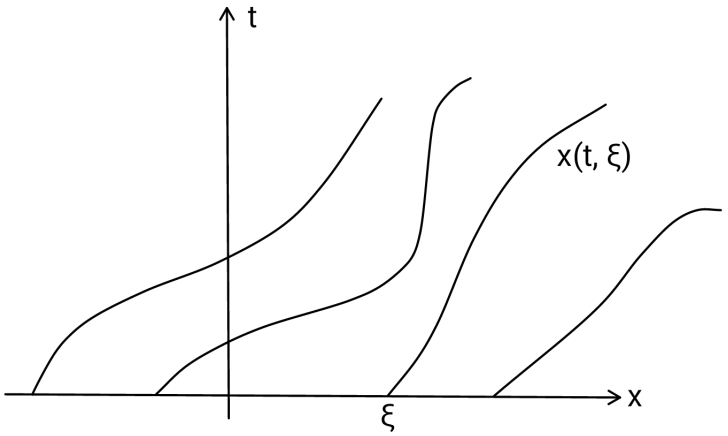

Figure 5: Exemplary characteristics.

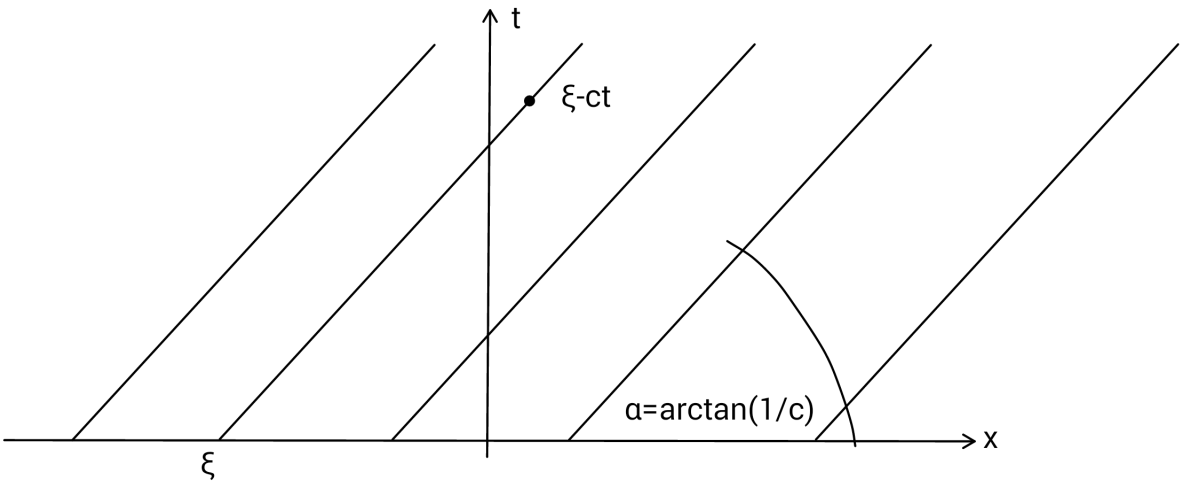

Figure 6: Characteristics for convection for constant speed of propagation.

and everything is correct. □

If, on the other hand, $g = g(x, t, u)$ then the system (2.46)-(2.47) has a form

$$\frac{dx}{dt} = c(x, t), \quad \frac{dU}{dt} = g(x, t, U). \tag{2.55}$$

The characteristics are still given by the same equation $dX/dt = c(X, t)$. We can therefore solve them exactly as before. Then from the equation $dU/dt = g(X, t, U)$ we will have to find U. This time u will not necessarily be constant on characteristics but will behave in a way predicted by its equation.

Example. Let us back to the convection equation but this time let us include the term that is responsible for generating the substance

$$\begin{cases} u_t + cu_x = -ku, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x, 0) = \phi(x), \quad x \in \mathbb{R}, \end{cases} \tag{2.56}$$

where $k > 0$ is a constant. Characteristics are still straight lines $X(t) = ct - \xi$ which implies that $\xi = x - ct$. The equation for U has the form

$$\frac{dU}{dt} = -kU, \quad U(t) = U(0)e^{-kt} = \phi(\xi)e^{-kt}. \tag{2.57}$$

Substituting $\xi = \xi(x, t)$ we obtain the solution $u(x, t) = \phi(x - ct)e^{-kt}$. □

The two examples above show that the solution to the equation of convection is a wave moving right. In other words, is a function of the $x - ct$ variable (see Fig. 6). This agrees with our intuition about the phenomenon of convection - the initial distribution of phi is lifted at a speed of c. In other words, *the initial condition moves on the characteristics*. In the case of variable speed $c = c(x, t)$, the situation will be different only in that the characteristics will not be straight lines.

2.3 Nonlinear speed of propagation

We will investigate the following problem of wave propagation

$$\begin{cases} u_t + c(u)u_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x, 0) = \phi(x), \quad x \in \mathbb{R}. \end{cases} \tag{2.58}$$

In this equation, the propagation speed depends on the *value* of u. If we return to the example with the conservation law for a flux only dependent on u then $c(u)u_x = (q(u))_x$, i.e. $c(u) = q_x(u)$. The propagation speed is therefore a derivative of the flux over space. We will later see that this has some very important implications. It turns out that our quasi-linear problem can be quite easily solved by the method of characteristics. First, let us see that $dU/dt = 0$ if

$$\frac{dX}{dt} = c(U(t)), \quad x(0) = \xi. \tag{2.59}$$

[3]On each of them separately.

Therefore, the function U is constant on the characteristics defined by the above equation. This implies that $U(t) = U(0) = u(X(0), 0) = u(\xi, 0) = \phi(\xi)$ on ξ-characteristic. It follows that on this curve $c(U(t)) = c(\phi(\xi))$. Hence, the characteristic is determined by the equation

$$\frac{dX}{dt} = c(\phi(\xi)), \quad X(0) = \xi, \tag{2.60}$$

which can immediately be solved yielding

$$X(t) = c(\phi(\xi))t + \xi. \tag{2.61}$$

Therefore, the characteristics are *straight line with a ξ-dependent speed* $c(\phi(\xi))$! Our problem (2.58) has then a solution given parametrically

$$u(X(t, \xi), t) = \phi(\xi), \quad X(t) = c(\phi(\xi))t + \xi, \quad \xi \in \mathbb{R}. \tag{2.62}$$

If we manage to extract the ξ parameter from the equation (2.61), that is, obtain the function $\xi(x, t)$, then substituting it in the above will give us an explicit form of the solution.

Example. One of the most important model equations is teh so-called *inviscid Burgers equation*

$$\begin{cases} u_t + uu_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x, 0) = x, \quad x \in \mathbb{R}, \end{cases} \tag{2.63}$$

where as an initial condition we have taken a linear function. Characteristics are straight lines (see Fig. 7)

$$X(t, \xi) = \xi t + \xi = \xi(1 + t) \quad \rightarrow \quad \xi = \frac{x}{1 + t}. \tag{2.64}$$

Therefore, the solution is constant on characteristics

$$u(x, t) = u(X(t, \xi), t) = u(\xi, 0) = \phi(\xi) = \phi\left(\frac{x}{1 + t}\right) = \frac{x}{1 + t}. \tag{2.65}$$

Note the changing angle of characteristics. □

For completeness let us check whether (2.62) satisfies (2.58). Let us start with computing the derivatives of u

$$u_t = \phi'(\xi)\xi_t, \quad u_x = \phi'(\xi)\xi_x. \tag{2.66}$$

The needed derivatives of ξ can be found from the second equation in (2.62), i.e. we have $\xi = \xi(x, t)$ which gives

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t}\left[c(\phi(\xi))t + \xi\right] \rightarrow \xi_t = -\frac{c(\phi(\xi))}{1 + c'(\phi(\xi))\phi'(\xi)t}, \tag{2.67}$$

and similarly

$$\frac{\partial x}{\partial x} = \frac{\partial}{\partial x}\left[c(\phi(\xi))t + \xi\right] \rightarrow \xi_x = \frac{1}{1 + c'(\phi(\xi))\phi'(\xi)t}. \tag{2.68}$$

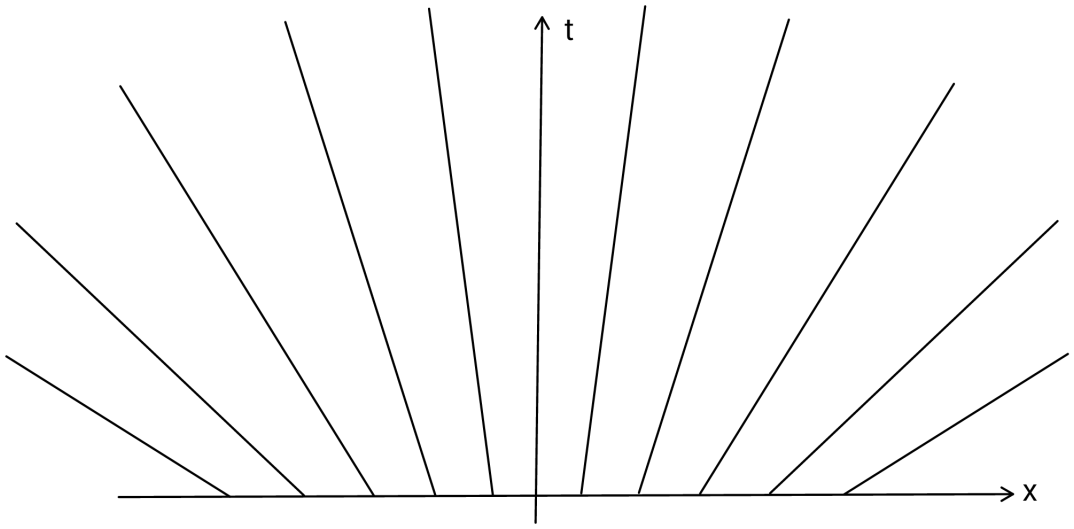

Figure 7: Characteristics for our problem with $c(u) = u$ and $\phi(x) = x$.

We thus obtain

$$u_t = -\frac{c(\phi(\xi))\phi'(\xi)}{1 + c'(\phi(\xi))\phi'(\xi)t}, \quad u_x = \frac{\phi'(\xi)}{1 + c'(\phi(\xi))\phi'(\xi)t}, \tag{2.69}$$

where it follows that $u_t + c(u)u_x = 0$. It is clear that for the solution to exist we must assume the denominator in (2.69) is positive. If $t > 0$, then the product of derivatives of c and phi must be positive so that the denominator always stays greater than zero. We have thus proved the below theorem. In what follows we will see the consequences of not meeting this assumption.

Theorem 1. *Let* $c = c(u)$ *and* $\phi = \phi(x)$ *be of* $C^1(\mathbb{R})$ *regularity class. Moreover, let them both be simultaneously either nondecreasing or nonincreasing. Then the problem (2.58) has a unique solution given by (2.62).*

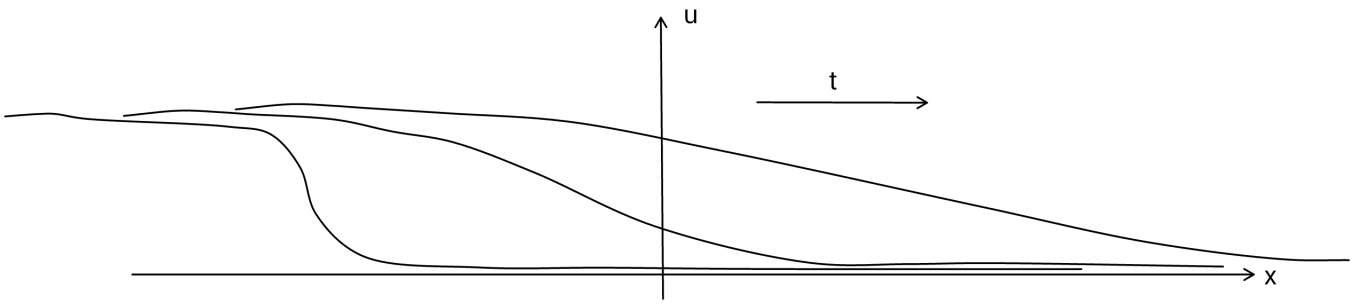

Figure 8: Evolution of the initial condition for $c'\phi' > 0$.

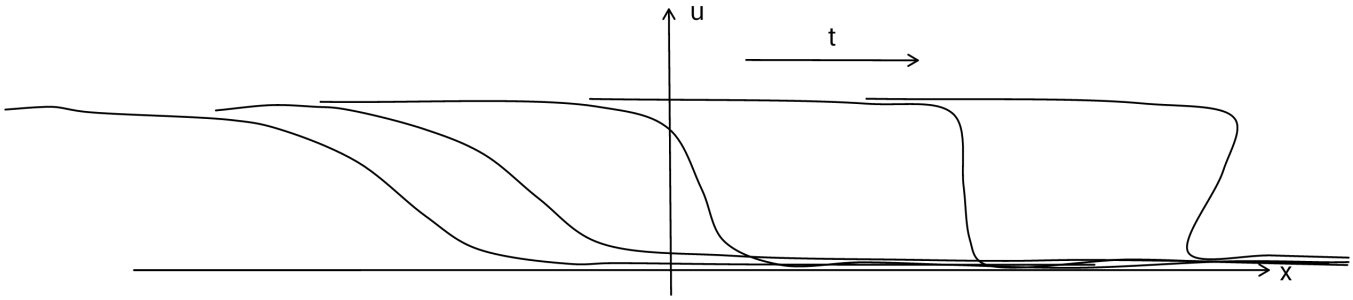

Figure 9: Evolution of the initial condition for $c'\phi' < 0$.

3 Shock waves

3.1 Rankine-Hugoniot's condition

Let us now return to the quasilinear problem (2.58) and carefully examine the structure of the characteristics, which in that case are straight lines. We know that $c = c(u)$ is the speed of wave propagation, i.e. it is directly related to the angle of inclination of the characteristics to $x-$axis. In the Theorem 1 we showed that this problem has a clearly defined and smooth solution as long as the speed c and the initial condition ϕ are exactly of the same monotonicity. Let us thus assume that both of these functions are nondecreasing. Then, *larger values* of ϕ move at a higher speed and the characteristics begin to be inclined at an increasingly smaller angle (Fig. 8). The graph of $x \mapsto \phi(x, t)$ for different t begins to "flatten".

Let us reverse the situation and assume that c is non-decreasing and ϕ is non-growing. Then the larger values of u are behind the smaller ones. But because $c(u)$ increases, larger values move faster than smaller ones and catch up with them at some point (Fig 9). The result is a triple-valued function. The characteristics are inclined at an ever greater angle to the Ox axis and, as a result, intersect. Surely something is wrong - what value should we assign to the function u at the intersection of straight lines?

Example. Let us consider the problem

$$\begin{cases} u_t + uu_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = \begin{cases} 1, & x \leq 0; \\ 0, & x > 0, \end{cases} \quad x \in \mathbb{R}. \end{cases} \tag{3.1}$$

For now, do not worry about the fact that the initial condition is not continuous. We can always smooth it out with some mollification such as convolution with a regular kernel, but it would disturb the transparency of the example. Characteristics have the form

$$X(t) = \begin{cases} t + \xi, & x \leq 0; \\ \xi, & x > 0, \end{cases} \tag{3.2}$$

and we can see that they intersect since the very beginning (Fig. 10).

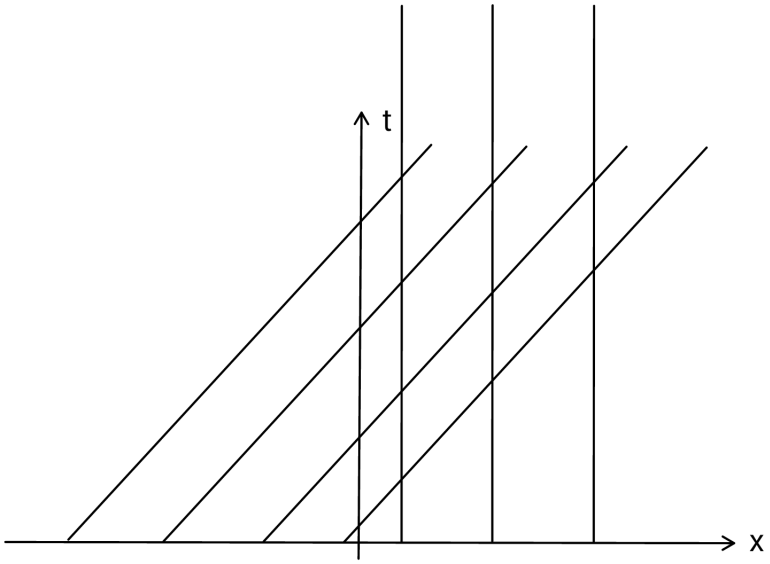

Figure 10: Characteristics for $c'\phi' < 0$.

The above discussion shows that in some situations the classic solution to our problem stops from being meaningful at some particular time - the spatial derivative becomes unbounded. This is called a *gradient catastrophe*. The intersection of the characteristics is not a contradiction with the basic conservation law, however, because until now we have always assumed that all the considered functions are smooth. Otherwise we could not talk about classical solutions of a differential equation. Let us recall that the archetype of our equation - the conservation law - was first derived in an integral form and only then we obtained the differential version of it. The integral relation (2.4) is therefore more fundamental because it does not require functions to be differentiable or even continuous!

A gradient catastrophe occurs when the u function stops being continuous. Suppose that this discontinuity moves on the curve $x = s(t)$. That is, for

$$\lim_{x \to s(t)^-} u(x,t) \neq \lim_{x \to s(t)^+} u(x,t), \tag{3.3}$$

and fixed $t > 0$. On each side of $s(t)$, the function u has continuous derivatives. We will see what must happen with u when passing the $x = s(t)$ curve. Conservation law (2.4) therefore has the form

$$\frac{d}{dt}\int_a^b u(x,t)dx = q(a,t) - q(b,t). \tag{3.4}$$

Here, we think of the flux as a function of the x and t variables, perhaps through the dependence on u, i.e. $q(x,t) = q(x,t,u(x,t))$. Let us take a and b such that $a < s(t) < b$, then

$$\frac{d}{dt}\left(\int_a^{s(t)} u(x,t)dx + \int_{s(t)}^b u(x,t)dx\right) = q(a,t) - q(b,t), \tag{3.5}$$

Since u is smooth on the left and right sides of $s(t)$ we can compute the derivative [4]

$$\int_a^{s(t)} u_t(x,t)dx + \int_{s(t)}^b u_t(x,t)dx + (u(s(t)^-,t) - u(s(t)^+,t))\,\frac{ds}{dt} = q(a,t) - q(b,t), \tag{3.6}$$

where we denoted

$$u(s(t))^{\pm} := \lim_{x\to s(t)^{\pm}} u(x,t). \tag{3.7}$$

Since u_t is by assumption continuous and we can take the limit $a, b \to s(t)^{\mp}$ and obtain *the speed of propagation of discontinuity*

$$\frac{ds}{dt} = \frac{[q]}{[u]}, \tag{3.8}$$

where we have introduced the notation for a change (a jump)

$$[f] := \lim_{x\to s(t)^+} f(x,t) - \lim_{x\to s(t)^-} f(x,t), \tag{3.9}$$

for an arbitrary function $f = f(x,t)$. The equation for a trajectory according to which the discontinuity moves (3.8) is called the *Rankine'a-Hugoniot's condition*.

Definition 3. *A propagation of a discontinuity along the curve $x = s(t)$ satisfying the Rankine-Hugoniot's condition (RH) (3.8) is called a* shock wave.

We have thus obtained a way to deal with the intersection of characteristics. Where the solution is continuous, the characteristics have exactly the same shape as they should while the discontinuity, in turn, moves along the trajectory given by the formula (3.8). The introduction of a shock wave ensures that the solution is unambiguous and meaningful. We still have a single-valued function.

Example. Let us return to the previous example and compute the flux

$$q(u) = \frac{1}{2}u^2, \tag{3.10}$$

[4] Recall the Leibnitz rule $\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} f_t(x,t)dx + f(b(t),t)b'(t) - f(a(t),t)a'(t)$.

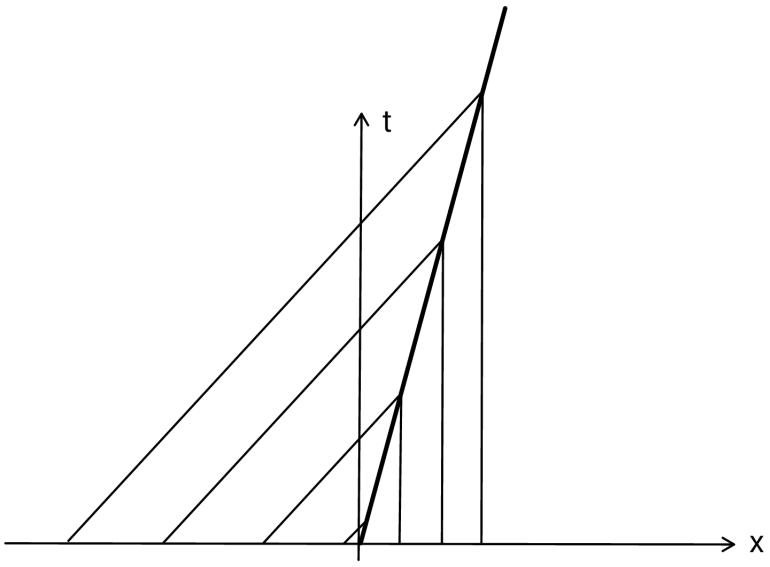

Figure 11: A shock wave.

since $q_x = q'(u)u_x = uu_x$. Now, the RH condition (3.8) gives us the shock speed

$$\frac{ds}{dt} = \frac{[q]}{[u]} = \frac{\frac{1}{2}1^2 - \frac{1}{2}0^2}{1-0} = \frac{1}{2}, \tag{3.11}$$

which results from the discontinuity of the initial condition. We can integrate the above equation immediately and obtain the shock wave trajectory

$$s(t) = \frac{1}{2}t, \tag{3.12}$$

where the integration constant was chosen so that the wave would agree with the initial condition. The space-time graph is presented on Fig 11. We see that the shock wave velocity is between the velocities of the other characteristics. The solution can be compactly written as

$$u(x,t) = \begin{cases} 1, & x \le \frac{1}{2}t; \\ 0, & x > \frac{1}{2}t, \end{cases} \tag{3.13}$$

which describes the simplest shock wave. □

3.2 Rarefaction waves

It is not the end with problems with quasilinear equations.

Example. Once again let us return to the previous example, but this time we invert the shape of the initial condition (see Fig. 12)

$$\begin{cases} u_t + uu_x = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = \begin{cases} 0, & x \le 0; \\ 1, & x > 0, \end{cases} & x \in \mathbb{R}. \end{cases} \tag{3.14}$$

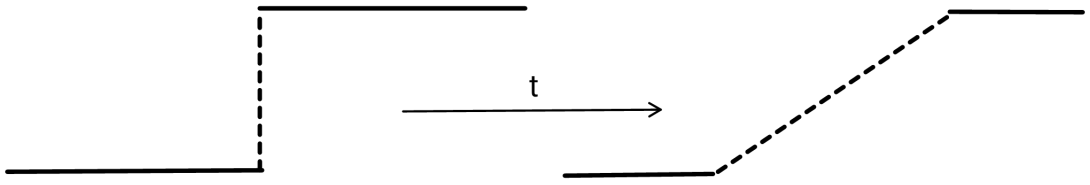

Figure 12: „Spreading" the initial condition.

We see that this time the characteristics do not intersect, but on the space-time diagram there is an area in which none of them arrives (the so-called *void*).

If we look at the example above, we will find the reason for the existence of the area without any characteristics - the initial condition is discontinuous and has the same monotonicity as the propagation speed. It turns out that we can easily find a solution specified on the missing subset of the field. If we look at Fig. 12, we can see that the values of initial condition diverge and between them there arises an area in which no value is set. The first and best idea is simply to continuously join the values of u for all times $t > 0$ and then move to the limit $t \to 0^+$. It turns out that there is exactly one way to do it.

Let us assume that $c'(u) > 0$ and the initial condition of the quasilinear problem (2.58) has a discontinuity at $x = \xi_0$, i.e. $\phi(\xi^-) = u_1 \neq u_2 = \phi(\xi^+)$. Then from the method of characteristics we have $x = c(\phi(\xi_0))t + \xi_0$ and $u(x,t) = \phi(\xi_0)$, where $c(u_1) < c(\phi(\xi_0)) < c(u_2)$ (plus modification if ϕ is one-sided continuous). This means that infinitely many characteristics emerge from ξ_0, which are continuously combined with the others. Specifically,

$$c(\phi(\xi_0)) = \frac{x-\xi_0}{t} \quad \to \quad u(x,t) = c^{-1}\left(\frac{x-\xi_0}{t}\right), \quad c(u_1) < \frac{x-\xi_0}{t} < c(u_2), \quad (3.15)$$

where the inverse of c exists from monotonicity. The characteristics form the so-called *fan* and the whole phenomenon is called a *rarefaction wave*.

Example. In our first example $c(u) = c^{-1}(u) = u$ and $\xi_0 = 0$, therefore the solution is

$$u(x,t) = \begin{cases} 0, & x < 0; \\ \frac{x}{t}, & 0 \leq x < t; \\ 1, & x \geq t, \end{cases} \quad (3.16)$$

which is the archetype for a rarefaction wave. □

Example. In general we can meet both rarefaction and shock waves. This is illustrated with the following example.

$$\begin{cases} u_t + u^2 u_x = 0, \quad t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = \begin{cases} -1, & x < -\frac{1}{2}; \\ 0, & -\frac{1}{2} \leq x < \frac{1}{2}; \\ 1, & x \geq \frac{1}{2} \end{cases} \quad x \in \mathbb{R}. \end{cases} \quad (3.17)$$

Characteristics are

$$x = \begin{cases} t + \xi, & \xi < -\frac{1}{2} \text{ and } \xi \geq \frac{1}{2}; \\ \xi, & -\frac{1}{2} \leq \xi < \frac{1}{2}. \end{cases} \tag{3.18}$$

The shock wave has the velocity

$$\frac{ds}{dt} = \frac{[q]}{[u]} = \frac{[\frac{1}{3}u^3]}{[u]} = \frac{0 + \frac{1}{3}}{0 + 1} = \frac{1}{3}, \quad -\frac{1}{2} < s(t) \leq \frac{1}{2}, \tag{3.19}$$

where we find that $s(t) = \frac{1}{3}t - \frac{1}{2}$, because the discontinuity causing the intersection of the characteristics is found at $\xi = -\frac{1}{2}$. The $\frac{1}{2} \leq x < \frac{1}{2} + t$ area is a void, so we fill it with a fan

$$u(x,t) = \sqrt{\frac{x - \frac{1}{2}}{t}}, \quad \frac{1}{2} \leq x < \frac{1}{2} + t. \tag{3.20}$$

Note that at $x = \frac{1}{2}$ and time $t = 3$, the shock wave enters the rarefaction wave region. That is why we get a new equation for a new shock wave

$$\frac{ds}{dt} = \frac{[\frac{1}{3}u^3]}{[u]} = \frac{1}{3}\frac{\left(\frac{s-\frac{1}{2}}{t}\right)^{\frac{3}{2}} - 1}{\left(\frac{s-\frac{1}{2}}{t}\right)^{\frac{1}{2}} - 1}. \tag{3.21}$$

Unfortunately, this equation cannot be solved explicitly. The final space-time graph is in Fig 13, where the solution is presented. □

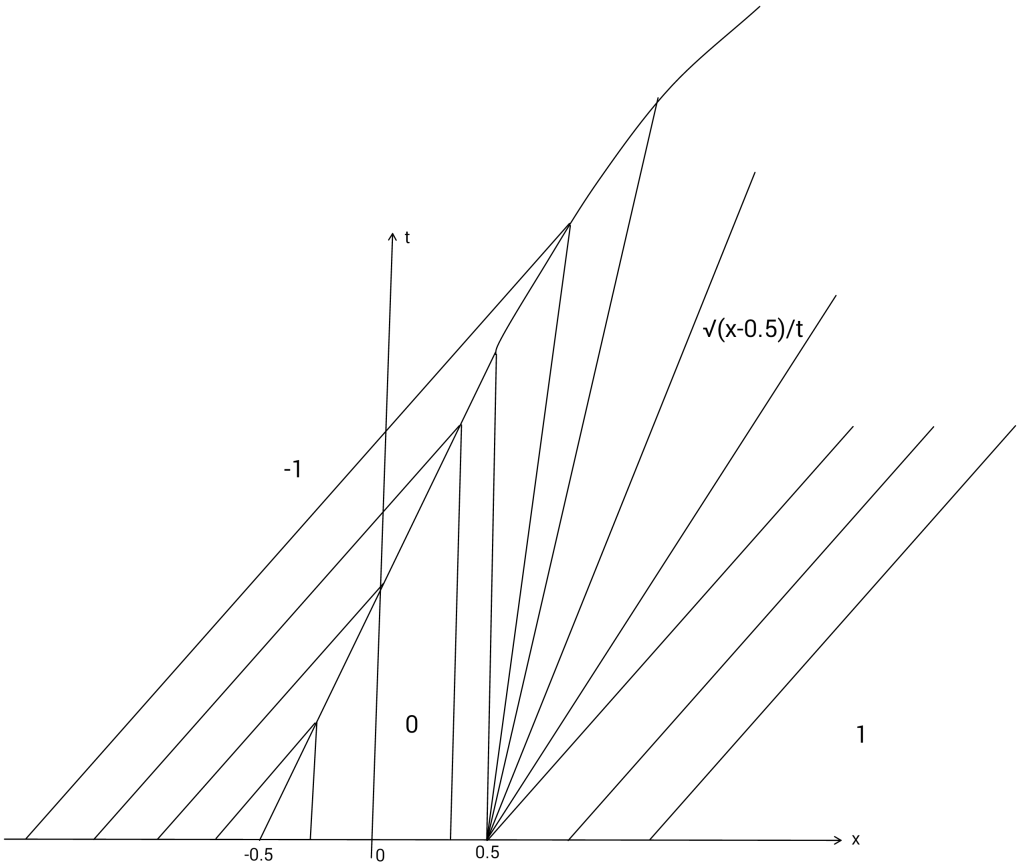

Figure 13: The space time diagram for an example with a rarefaction and shock wave.

4 Heat equation

We begin our review of second order PDEs with the most important parabolic equation - the heat equation (also known as the thermal conductivity equation or diffusion equation). As its name suggests, it should appear when describing heat distribution in a medium.

Example. (*Heat conduction*) Consider a medium which can conduct heat (in one dimension - we will traditionally consider an insulated rod). Because energy is conserved, the following equation has to be satisfied

$$E_t + \nabla \cdot \mathbf{q} = F, \tag{4.1}$$

where E is the energy per unit of volume, $\mathbf{q}$ is the thermal flux and F represents all sources, i.e. the amount of heat generated per unit time in a unit volume. Let $u = u(\mathbf{x}, t)$ denote the temperature at $\mathbf{x}$ and the time t. According to the definition of specific heat c we must then have

$$E = c\rho u, \tag{4.2}$$

where we assumed that c does not depend on the temperature [5]. The thermal flux $\mathbf{q}$ is given by *Fourier's Law* - heat flows from high temperature areas to domains that have a lower temperature. Mathematically, you can write it in the form

$$\mathbf{q} = -k(\mathbf{x}, t, u)\nabla u, \tag{4.3}$$

that is, heat flows along the (minus) gradient. Here, k is the *coefficient of thermal conductivity*, which is constant for homogeneous media. Combining the above equations we obtain the *heat equation*

$$c\rho u_t = \nabla \cdot (k(\mathbf{x}, t, u)\nabla u) + F, \tag{4.4}$$

which for constant k becomes

$$u_t = \alpha^2 \Delta u + f, \tag{4.5}$$

where $\Delta = \nabla^2$ is the Laplacian, while $\alpha^2 := k/(c\rho)$, and $f := F/(c\rho)$. In one dimension we have

$$u_t = \alpha^2 u_{xx} + f. \tag{4.6}$$

Two above equations will interest us the most. □

Example. (*Diffusion*) Suppose a substance has been sprayed into the air. According to Fick's law, it will diffuse to areas with lower concentrations. This can be written as

$$\mathbf{q} = -D(x, t, u)\nabla u. \tag{4.7}$$

The above flux is almost identical to that given by Fourier's law. Here, the proportionality factor D is called the diffusivity. According to the above, the heat equation is also

[5] In fact, both the density ρ and the specific heat c are slowly varying functions of the temperature. Then, the heat equation becomes nonlinear and its analysis is much more difficult ... and more interesting!

called the diffusion equation. □

Example. (*Brownian motion*) We will now give the derivation of the diffusion equation in stochastic setting. This reasoning has its origins in the works of Einstein and Smoluchowski. Let x be a fixed point on the straight line $\mathbb{R}$. Let us introduce a grid of points with the step h, that is, our particle initially placed at the point x, can only find itself only at $x + kh$, where $k \in \mathbb{Z}$. When the particle is at a certain point, then with probability p and during time τ it jumps to the right and with $q = 1 - p$ to the left. Let $u = u(x, t)$ denote the probability that at the moment $t > 0$ the particle is at $x \in \mathbb{R}$. To arrive there during $t + \tau$ the particle can come either from the right or from the left, i.e.

$$u(x, t + \tau) = pu(x - h, t) + qu(x + h, t). \tag{4.8}$$

We can now expand the above difference equation into Taylor series

$$\begin{aligned} u(x,t) + u_t(x,t)\tau + O(\tau^2) = p\left(u(x,t) - u_x(x,t)h + \frac{1}{2}u_{xx}(x,t)h^2\right) \\ + q\left(u(x,t) + u_x(x,t)h + \frac{1}{2}u_{xx}(x,t)h^2\right) + O(h^3). \end{aligned} \tag{4.9}$$

After simplification this becomes

$$u_t(x,t) = (q-p)\frac{h}{\tau}u_x(x,t) + \frac{h^2}{2\tau}u_{xx}(x,t) + O(\tau) + O(h^3), \tag{4.10}$$

when $\tau, h \to 0$. Now, we want to take the limit with the grid step going to zero. In order to do that we have to assume

$$\lim(q-p)\frac{h}{\tau} = c, \quad \lim\frac{h^2}{2\tau} = D, \tag{4.11}$$

where D and c are constant. This means that the probabilities of a jump over a finite step h decrease to zero just like h, i.e. $p - q = O(h)$. We obtain the convection-diffusion equation (Brownian motion with a drift)

$$u_t + cu_x = Du_{xx}. \tag{4.12}$$

When probabilities are identical, i.e. $p = q = \frac{1}{2}$ then the convection is absent

$$u_t = Du_{xx}, \tag{4.13}$$

which, one again, is the heat equation. □

4.1 Bounded domains and separation of variables

In this part, we will limit ourselves to considering a one-dimensional situation for which we will obtain a number of very important results. Despite the obvious simplification, the one-dimensional model describes many physical situations in which heat spreads in one dimension much faster than in others (e.g. a symmetrical rod). Let us consider the interval $[0, L]$.

The differential equation itself (4.6) describes the dynamics of thermal energy conduction. To complete the description of the physical situation, we need the appropriate initial and boundary conditions. The initial condition is the temperature distribution at the first moment, which we assume is $t = 0$. We therefore have

$$u(x, 0) = \phi(x), \tag{4.14}$$

for a given function ϕ. Boundary conditions, however, must be imposed on the boundary of $[0, L]$. It turns out that we have at least three possibilities corresponding to important physical situations.

1. *Dirichlet condition* (or a condition of the first kind) fixes the temperature at the boundary
$$u(0, t) = \mu(t) \quad \text{or} \quad u(L, t) = \nu(t), \tag{4.15}$$
where μ and ν are known functions. The Dirichlet condition can describe a situation where one end of a rod is kept at a fixed temperature (for example in a refrigerator).

2. *Neuman condition* (or a condition of the second order) fixes the value of the derivative
$$u_x(0, t) = \mu(t) \quad \text{lub} \quad u_x(L, t) = \nu(t). \tag{4.16}$$
Such a description is interpreted in terms of a given heat flux at one end of the interval. From Fourier's law (4.3) we have $q(x, t) = -ku_x(x, t)$, i.e.
$$\mu(t) = -\frac{q(0, t)}{k}. \tag{4.17}$$
This expression models a heat input through one end of the interval.

3. *Robin condition* (or a condition of the third kind) is a linear combination of the two above
$$u_x(0, t) = -\lambda_1 \left(u(0, t) - \theta_1(t)\right) \quad \text{lub} \quad u_x(L, t) = -\lambda_2 \left(u(L, t) - \theta_2(t)\right), \tag{4.18}$$
where $\theta_{1,2}$ are the given functions and $\lambda_{1,2}$ are constants. The physical interpretation of Robin's condition is derived from Newton's law of cooling bodies - the flux is proportional to the difference in body and ambient temperatures, i.e.
$$q(0, t) = -h(u(0, t) - \theta(t)). \tag{4.19}$$
Comparing this with Fourier's Law along with $\lambda = h/k$ we have
$$ku_x(0, t) = h(u(0, t) - \theta(t)). \tag{4.20}$$

Of course, the boundary conditions can be mixed, i.e. we can have a different type of condition at different ends of the interval. The above list is obviously not closed and there are many other possibilities. The conditions of Dirichlet, Neumann and Robin, however, are the most common and exhaust almost all of the standard situations. In the following, we will focus only on the Dirichlet condition and relegate the discussion of the others to the tutorial. Let us define what exactly we mean when we talk about a solution of the heat equation.

Definition 4. *A solution of the heat equation satisfying the Dirichlet boundary condition is a bounded function* $u = u(x,t)$ *defined over* $[0,L] \times [0,T]$ *satisfying the following PDE along with initial and boundary conditions*

$$\begin{cases} u_t = \alpha^2 u_{xx} + f(x,t), \quad (x,t) \in (0,L) \times (0,T), \\ u(x,0) = \phi(x), \quad x \in [0,L] \\ u(0,t) = \mu(t), \quad u(L,t) = \nu(t), \quad t \geq 0. \end{cases} \tag{4.21}$$

Furthermore, the consistency of the boundary and initial conditions have to take place: $\phi(0) = \mu(0)$, $\phi(L) = \nu(0)$.

Let us now make some very important observation that simplifies the analysis of the problem. Note that (4.21) contains three types of nonhomogeneity: source function f, initial condition ϕ, and boundary conditions μ and ν. Thanks to the linearity, we can break down the studied case and deal with one nonhomogeneity at a time. To this end, write

$$u(x,t) = u^{(1)}(x,t) + u^{(2)}(x,t) + u^{(3)}(x,t), \tag{4.22}$$

where $u^{(1)}$ satisfies

$$\begin{cases} u_t^{(1)} = \alpha^2 u_{xx}^{(1)}, \quad (x,t) \in (0,L) \times (0,T), \\ u^{(1)}(x,0) = \phi(x), \quad x \in [0,L] \\ u^{(1)}(0,t) = 0, \quad u^{(1)}(L,t) = 0, \quad t \geq 0, \end{cases} \tag{4.23}$$

$u^{(2)}$ is a solution to

$$\begin{cases} u_t^{(2)} = \alpha^2 u_{xx}^{(2)} + f(x,t), \quad (x,t) \in (0,L) \times (0,T), \\ u^{(2)}(x,0) = 0, \quad x \in [0,L] \\ u^{(2)}(0,t) = 0, \quad u^{(2)}(L,t) = 0, \quad t \geq 0, \end{cases} \tag{4.24}$$

while for $u^{(3)}$ we have

$$\begin{cases} u_t^{(3)} = \alpha^2 u_{xx}^{(3)}, \quad (x,t) \in (0,L) \times (0,T), \\ u^{(3)}(x,0) = 0, \quad x \in [0,L] \\ u^{(3)}(0,t) = \mu(t), \quad u^{(3)}(L,t) = \nu(t), \quad t \geq 0. \end{cases} \tag{4.25}$$

It is easy to show that u defined as above satisfies (4.21). Now, we will solve each of the above problems.

4.1.1 Initial condition

We will solve the following (we do not write the cumbersome parentheses in the superscript)

$$\begin{cases} u_t = \alpha^2 u_{xx}, \quad (x,t) \in (0,L) \times (0,T), \\ u(x,0) = \phi(x), \quad x \in [0,L] \\ u(0,t) = 0, \quad u(L,t) = 0, \quad t \geq 0, \end{cases} \tag{4.26}$$

where we prescribe the initial condition. This problem can be solved by using the elegant method originating from Fourier - the method of separation of variables. Let us look for solutions of (4.26) in the form

$$u(x,t) = X(x)T(t), \tag{4.27}$$

where each term is a function of only one argument. In addition, X must satisfy (because we are looking for a nonzero T)

$$X(0) = 0, \quad X(L) = 0. \tag{4.28}$$

Upon substitution of (4.27) into (4.26) we obtain

$$XT' = \alpha^2 X''T, \tag{4.29}$$

which yields

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}. \tag{4.30}$$

Since the left side of the above equation depends only on the x variable and the right side of the t, they must be equal to a constant. Otherwise, by changing x we would change the left side while the right would remain constant. This would lead to a contradiction. There exists, therefore, some real constant λ such that

$$X'' + \lambda X = 0, \quad T' + \alpha^2 \lambda T = 0. \tag{4.31}$$

We have reduced our problem to finding two functions, X and T, which are solutions to *ordinary* equations. Let us deal with the boundary value problem for X, in which boundary conditions are (**??**). As we know, the solution to this problem depends strongly on the sign of λ and to avoid a contradiction with the boundary conditions we must have $\lambda > 0$ (show it!). The general solution of the second order ordinary differential equation with constant coefficients will therefore be

$$X(x) = C\cos(\sqrt{\lambda}x) + D\sin(\sqrt{\lambda}x). \tag{4.32}$$

Because $X(0) = 0$ we immediately obtain $C = 0$. The second boundary condition $X(L) = 0$ leads to

$$\sin(\sqrt{\lambda}L) = 0, \tag{4.33}$$

that is $\sqrt{\lambda}L = n\pi$ for $n = 1, 2, 3, ...$, from which it follows that

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \tag{4.34}$$

where we have clearly indicated that λ depends on a natural number n. Therefore, our problem has solutions only for certain values of λ (this is typical for boundary value problems). Thus, we get a countably infinite number of solutions that we can write them in a form

$$X_n(x) = D_n \sin\left(\frac{n\pi}{L}x\right), \quad n \in \mathbb{N}. \tag{4.35}$$

Let us now return to (4.31) na T. We can readily solve it and obtain

$$T_n(t) = E_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}, \quad n \in \mathbb{N}, \tag{4.36}$$

where E_n are some integration constants. Note that the solution is bounded for $t \to \infty$ only for positive λ_n. Unbounded solutions are non-physical, therefore we reject them.

This gives another confirmation of the chosen sign of λ_n. By combining the the formulas for X_n and T_n and using (4.27) we have

$$u_n(x,t) = A_n \sin\left(\frac{n\pi}{L}x\right) e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}, \quad n \in \mathbb{N}, \tag{4.37}$$

where the product of D_n and E_n was renamed A_n. We see that for every natural number n we have exactly one solution u_n. Which one to choose? This is a clear lack of uniqueness but, fortunately, only superficially.

We have not used the initial condition $u(x,0) = \phi(x)$ yet. However, it is hard to expect that for a very general class of functions from which we can choose ϕ we will find $n \in \mathbb{N}$ such that $\phi(x) = u_n(x,0) = A_n \sin(n\pi x/L)$. We should remember that the heat equation is linear and this implies that any linear combination of solutions is also its solution. Because there are infinitely many such components u_n, let us construct a *formal* series[6], which we claim to be the general solution to the problem (4.26)

$$u(x,t) := \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}. \tag{4.38}$$

We now require that u satisfies the initial condition

$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right), \tag{4.39}$$

which is the sine Fourier series for ϕ. We know that the coefficients A_n of this expansion can be calculated from the Euler's formula

$$A_n = \frac{2}{L}\int_0^L \phi(x) \sin\left(\frac{n\pi}{L}x\right) dx. \tag{4.40}$$

We have thus a perfect candidate for a solution of the full problem (4.26). It is (4.38) with the constant specified in (4.40). We must therefore show that the function u given by the formula (4.38) is differentiable and satisfies the heat equation along with the initial and boundary conditions (4.26). It turns out that a sufficient condition for these requirements is only that ϕ be continuous and this is completely physically natural (however, this assumption can be greatly relaxed)

Theorem 2. *Let ϕ be a continuous function on $[0, L]$. Then $u = u(x,t)$ defined by (4.38) and (4.40) is a solution of the heat conduction problem (4.26). Moreover, u is then of $C^\infty[0,L]\times[0,T]$ class (it is infinitely differentiable!).*

Proof. First we will show that the defining series (4.38) converges uniformly with respect to x. To see this, note that the continuity of ϕ on $[0, L]$ implies its boundedness. Let $|\phi(x)| \leq M$ for some $M > 0$. Then from (4.40) we have

$$|A_n| \leq 2M, \tag{4.41}$$

[6] A linear combination is always finite in algebra. In analysis we deal with infinite combinations where a subtle question about convergence occurs.

which immediately leads to

$$\left|A_n \sin\left(\frac{n\pi}{L}x\right) e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\right| \leq 2Me^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}. \tag{4.42}$$

Let us take any number $0 < t_0 < T$. For $t_0 \leq t \leq T$ we have $e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \leq e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t_0}$. Therefore, we have found a majorant for the series (4.38), which has the form

$$2M \sum_{n=1}^{\infty} e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t_0} < \infty. \tag{4.43}$$

From the Weierstrass criterion (M-test), we have a uniform convergence of the series which defines u on $[0, L] \times [t_0, T]$. Since the functions u_n are continuous as functions of two variables, u will also be continuous at $[0, L] \times [t_0, T]$. But $t_0 > 0$ was arbitrarily chosen, so u is continuous at $[0, L] \times [0, T]$.

Exactly the same reasoning is sufficient to show that a series of derivatives (of any order!) of u_n converges uniformly at $[0, L] \times [t_0, T]$ for any $t_0 > 0$. So u is infinitely differentiable and the series can be differentiated term by term. From the fact that each u_n satisfies the heat equation, it follows that u is a solution of (4.26). This concludes the proof. □

The above theorem shows a very important feature of heat equation solutions - starting only from a continuous [7] initial condition, at any subsequent time $t > 0$ the solution u has all derivatives (it is infinitely smooth)! Homogeneous Neumann or Robin conditions can be treated in exactly the same way (tasks).

4.1.2 Green's function

The solution of the heat equation (4.38) can be transformed in a way that enhances physical interpretation. Remembering that the series specifying u converges uniformly we can integrate it term by term as follows

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \\ &= \sum_{n=1}^{\infty} \left(\frac{2}{L}\int_0^L \phi(\xi) \sin\left(\frac{n\pi}{L}\xi\right) d\xi\right) \sin\left(\frac{n\pi}{L}x\right) e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \\ &= \int_0^L \left(\frac{2}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}\xi\right) \sin\left(\frac{n\pi}{L}x\right) e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\right) \phi(\xi) d\xi. \end{aligned} \tag{4.44}$$

We define the *Green's function*

$$G(x, \xi, t) := \frac{2}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}\xi\right) \sin\left(\frac{n\pi}{L}x\right) e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}, \tag{4.45}$$

[7] Once again - this assumption can be seriously weakened.

then, the solution of the initial value problem for the heat equation takes a very simple form

$$u(x,t) = \int_0^L G(x,\xi,t)\phi(\xi)d\xi. \tag{4.46}$$

The Green's function defined in this way is also called an instantaneous point source, i.e. the temperature distribution in the rod $[0,L]$ and at the time t where suddenly in the moment $t = 0$ a unit quanta of heat is released at the point $\xi \in [0,L]$. Then the complete form of the solution (4.46) can be interpreted as a continuous superposition with the weight of $\phi(\ xi)$ of these point heat sources.

To strengthen our physical interpretation of Green's function let us fix $\epsilon > 0$. Suppose that the heat Q is released on the small interval $(\xi_0 - \epsilon, \xi_0 + \epsilon)$ at the time $t = 0$ and assume that the rest of the rod has a temperature equal to 0. By $\phi_\epsilon(x) \geq 0$ we denote the temperature distribution that will appear immediately after the release of thermal energy. By definition, we have heat

$$c\rho \int_{\xi_0-\epsilon}^{\xi_0+\epsilon} \phi_\epsilon(\xi)d\xi = Q. \tag{4.47}$$

Since the temperature distribution at time $t > 0$ is given by (4.46) we must have

$$u_\epsilon(x,t) = \int_0^L G(x,\xi,t)\phi_\epsilon(\xi)d\xi = \int_{\xi_0-\epsilon}^{\xi_0+\epsilon} G(x,\xi,t)\phi_\epsilon(\xi)d\xi = G(x,\xi^*,t)\int_{\xi_0-\epsilon}^{\xi_0+\epsilon}\phi_\epsilon(\xi)d\xi = \frac{Q}{c\rho}G(x,\xi^*,t), \tag{4.48}$$

where we used the vanishing of the function ϕ_ϵ outside the interval $(\xi_0-\epsilon, \xi_0+\ psilon)$ and we used the mean value theorem producing the point ξ^*. All we have to do now is to take a limit with $\epsilon \to 0$. From the continuity of Green's function we thus have

$$u(x,t) = \lim_{\epsilon\to 0} u_\epsilon(x,t) = \frac{Q}{c\rho}G(x,\xi_0,t). \tag{4.49}$$

So if $Q = c\ rho$ then the Green function is an immediate temperature response to the point release of heat. We will meet Green's functions many times when studying various other problems not necessarily related to the spread of heat[8].

4.1.3 Nonhomogeneous equation

Let us now turn to the problem of having nonhomogeneity only in the differential equation itself (a source of heat)

$$\begin{cases} u_t = \alpha^2 u_{xx} + f, \quad (x,t) \in (0,L)\times(0,T), \\ u(x,0) = 0, \quad x \in [0,L] \\ u(0,t) = 0, \quad u(L,t) = 0, \quad t \geq 0, \end{cases} \tag{4.50}$$

The method of separation of variables allowed us to obtain a solution to the initial value problem as the sum of the product of functions dependent on separate variables.

[8]Historically speaking, British mathematician George Green introduced this approach in the first half of the XIX century when describing electrostatic potential produced by a unit charge. We will talk about his approach soon.

To get the x dependent component we had to solve a homogeneous boundary problem. In the case of the non-zero function f we can get a similar result.

As the boundary conditions of our problem are exactly the same as in the case of the initial problem, it seems beneficial to look for solutions in the form of a series[9]

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi}{L}x\right), \tag{4.51}$$

where the time component $u_n(t)$ is unknown. We can immediately see that the boundary conditions for u are met since each term satisfies them. The f function must also be written in a similar way so that we can compare all the terms

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi}{L}x\right). \tag{4.52}$$

Note that the two formulas above are nothing but Fourier series for $u(x,t)$ and $f(x,t)$ as a function of x with fixed t. The functions $f_n(t)$ are Fourier coefficients calculated from the formula

$$f_n(t) = \frac{2}{L}\int_0^L f(x,t) \sin\left(\frac{n\pi}{L}x\right) dx. \tag{4.53}$$

Plugging into (4.50) gives us

$$\sum_{n=1}^{\infty} \left(u_n'(t) + \left(\frac{n\pi}{L}\right)^2 \alpha^2 u_n(t) - f_n(t)\right) \sin\left(\frac{n\pi}{L}x\right) = 0. \tag{4.54}$$

From the uniqueness of the Fourier series, each term in the above series must be zero which gives us an infinite set of equations

$$u_n'(t) + \left(\frac{n\pi}{L}\right)^2 \alpha^2 u_n(t) = f_n(t), \quad u_n(0) = 0, \quad n \in \mathbb{N}, \tag{4.55}$$

where we remembered about vanishing initial condition. Fortunately, each of these equations is uncoupled from the other, so we can solve them one by one. The solution (obtained for example by means of an integrating factor) is of course[10]

$$u_n(t) = \int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 (t-\tau)} f_n(\tau) d\tau, \tag{4.56}$$

which solves our problem. Combining (4.51) with (4.56) implies

$$u(x,t) = \sum_{n=1}^{\infty} \left(\int_0^t e^{-\left(\frac{n\pi}{L}\right)^2 \alpha^2 (t-\tau)} f_n(\tau) d\tau\right) \sin\left(\frac{n\pi}{L}x\right). \tag{4.57}$$

Showing that the above formula really determines the solution to the problem of heat with nonhomogeneity is similar to the pure initial condition case.

[9] This is called an eigenfunction expansion method.

[10] Please recall that from the ODE course.

To enhance the physical interpretation of the solution, we can again introduce the Green's function. To achieve this, let us use the formula (4.52) and write (4.57) in the form

$$u(x,t)=\int_0^t\int_0^L \frac{2}{L}\sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{L}\right)^2\alpha^2(t-\tau)}\sin\left(\frac{\pi n}{L}x\right)\sin\left(\frac{\pi n}{L}\xi\right)f(\xi,\tau)d\xi d\tau=\int_0^t\int_0^L G(x,\xi,t-\tau)f(\xi,\tau)d\xi d\tau, \tag{4.58}$$

where G is the Green's function defined in (4.45). This is a very elegant formula telling us that for zero initial and boundary conditions, the temperature distribution is the superposition of point heat sources located at $(\xi, t-\tau)$ with the weight $f(\xi,\tau)$ (i.e. in each point of the rod and at all previous times). This relation is sometimes called *Duhamel's formula*.

4.1.4 Nonzero boundary conditions

We are left with solving the nontrivial boundary condition case (4.25)

$$\begin{cases} u_t=\alpha^2 u_{xx}, \quad (x,t)\in(0,L)\times(0,T),\\ u(x,0)=0, \quad x\in[0,L]\\ u(0,t)=\mu(t), \quad u(L,t)=\nu(t), \quad \mu(0)=\nu(0)=0, \qquad t\geq 0. \end{cases} \tag{4.59}$$

A very useful technique to solve this problem is to consider a deviation from a certain carefully selected function. Let us substitute then

$$u(x,t)=U(x,t)+v(x,t), \tag{4.60}$$

where U will be chosen to satisfy the boundary conditions

$$U(0,t)=\mu, \quad U(L,t)=\nu(t). \tag{4.61}$$

Probably the simplest (you can think of an infinite number of candidates) choice is

$$U(x,t)=\mu(t)+\frac{x}{L}\left(\nu(t)-\mu(t)\right). \tag{4.62}$$

Since we have (4.59) the unknown v satisfies

$$\begin{cases} v_t=\alpha^2 v_{xx}-U_t, \quad (x,t)\in(0,L)\times(0,T),\\ v(x,0)=-U(x,0), \quad x\in[0,L]\\ v(0,t)=0, \quad v(L,t)=0, \quad \mu(0)=\nu(0)=0, \qquad t\geq 0. \end{cases} \tag{4.63}$$

and we have solved a similar problem before. We thus have

$$u(x,t)=U(x,t)-\int_0^L G(x,\xi,t)U(\xi,0)d\xi-\int_0^t\int_0^L G(x,\xi,t-\tau)U_t(\xi,\tau)d\xi d\tau. \tag{4.64}$$

Note that for *different* boundary conditions the function U will usually have a *different* form. Moreover, for time-independent boundary conditions the function U can be thought as a steady-state heat distribution for then $U_t\equiv 0$. This, as a matter of fact, is the very reason of using the technique of splitting the solution into the steady-state

and the so-called transient. The difficulty of resolving boundary conditions is then delegated into some simpler function while one can focus on the dynamics.

We can now combing (4.46), (4.57) and (4.64) in order to obtain a final and complete solution to (4.21)

$$u(x,t) = U(x,t) + \int_0^L G(x,\xi,t)(\phi(\xi) - U(\xi,0))d\xi + \int_0^t \int_0^L G(x,\xi,t-\tau)\left(f(\xi,\tau) - U_t(\xi,\tau)\right) d\xi d\tau, \tag{4.65}$$

where U is defined in (4.62).

4.1.5 Uniqueness

We have shown that the problem (4.21) has a solution. The obvious question arises - is it only one? The answer is yes and will prove this by means of a very elegant energy estimation method.

Theorem 3. *The heat equation with Dirichlet boundary conditions (4.21) has a unique bounded solution.*

Proof. Let (4.21) have two solutions u_1 and u_2. Denote their difference by $w := u_1 - u_2$. Then w satisfies the homogeneous heat equation with zero boundary and initial conditions (linearity!). Let us define the energy of the solution[11]

$$E(t) = \frac{1}{2}\int_0^L w(x,t)^2 dx. \tag{4.66}$$

Of course we have $E \geq 0$. Let us check how does the derivative looks like

$$\frac{d}{dt}E(t) = \int_0^L w(x,t)w_t(x,t)dx = \alpha^2 \int_0^L w(x,t)w_{xx}(x,t)dx, \tag{4.67}$$

where we used the fact that w satisfies the homogeneous heat equation. Integrating by parts and using boundary conditions we obtain

$$\frac{d}{dt}E(t) = w(x,t)w_x(x,t)|_0^L - \int_0^L w_x(x,t)^2 dx = -\int_0^L w_x(x,t)^2 dx \leq 0, \tag{4.68}$$

whence, the energy is not growing in time. Since w satisfies zero initial condition we also have $E(0) = 0$. Combining this with positivity we must have $E(t) \equiv 0$ for each time. Since the integral of a function is non-negative, we must have $w \equiv 0$, i.e. $u_1 \equiv u_2$. The solution is therefore unique. □

Note that the above proof uses a physical quantity - energy - and through reasoning shows that from zero initial and boundary conditions, a homogeneous heat equation cannot have a solution other than zero. The most important step here is to use the equation's linearity to get a homogeneous equation for w. For nonlinear equations, the matter would not be that simple.

[11] The word *energy* is used here very loosely following the pure mathematical tradition. Physically, the thermal energy is $c\rho u$ so that the appearing integral is related (up to a constant) to the energy squared for constant density ρ and specific heat c only. This is precisely our case so that everything works here.

4.2 Unbounded domains

We have seen how we can solve the heat equation for a one-dimensional rod (interval). We will now focus on the same problem set on an unbounded domains.

4.2.1 Real line $\mathbb{R}$

When we are dealing with a medium whose dimensions are much larger than the characteristic speed of thermal diffusion multiplied by a time scale, a very good model of such a problem is considering the heat equation on a straight line. We can also think about the fact that the rod is described by $[-L, L]$ where $L \to \infty$. Such a limit is a way to obtain a solution to the unbounded case from our previous considerations, however we will go another route.

Note that speaking about boundary conditions for an unbounded domain requires a little subtlety. The procedure of taking a limit suggests that we should impose some conditions at infinity. The most natural is the physical requirement saying that the temperature decreases to zero as we go to infinity (this is also required for the uniqueness of the solution). Hence, the main problem has the form

$$\begin{cases} u_t = \alpha^2 u_{xx}, \quad (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = \phi(x), \quad x \in \mathbb{R} \\ u(-\infty,t) = 0, \quad u(\infty,t) = 0, \qquad t \geq 0. \end{cases} \tag{4.69}$$

We are therefore investigating how the initial temperature distribution $\phi(x)$ will evolve over time in an infinite area. There are several ways to solve this problem: the mentioned limit of a bounded case, the Fourier transform method, the plane wave method, and self-similarity. We will use the latter technique because it is also very useful for studying nonlinear equations.

The method of self-similarity is based on the observation that if u satisfies the heat equation then for any λ function $v(x,t) := u(\lambda x, \lambda^2 t)$ is also a solution because

$$v_t = \lambda^2 u_t, \quad v_{xx} = \lambda^2 u_{xx}. \tag{4.70}$$

This suggests that the variable x^2 plays the same role as t. We will look for a self-similar solution satisfying the following condition

$$\frac{d}{dt} \int_{-\infty}^{\infty} u(x,t)\,dx = 0, \tag{4.71}$$

which is a statement of energy conservation[12]. Let us thus introduce the self-similar variable

$$z := \frac{x}{\sqrt{t}}, \quad u(x,t) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right). \tag{4.72}$$

We will show that a function of a *single* variable $U = U(z)$ is well-posed. The conservation of energy gives us

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} U\left(\frac{x}{\sqrt{t}}\right) dx = \int_{-\infty}^{\infty} U(z)\,dz = \text{const.}, \tag{4.73}$$

[12]One again: energy is equal to $c\rho u$, hence the integral over the whole domain is constant in time.

since heat is conserved. Note that this is the reason for $t^{-1/2}$ term in front of U in (4.72). We can now calculate the derivatives

$$u_t(x,t) = -\frac{1}{2}\frac{1}{\sqrt{t^3}}U(z) - \frac{1}{2}\frac{x}{t^2}U'(z), \quad u_{xx} = \frac{1}{\sqrt{t^3}}U''(z). \tag{4.74}$$

Whence, the heat equation takes the following self-similar form

$$-\frac{1}{2}U - \frac{1}{2}zU' = \alpha^2 U''. \tag{4.75}$$

The conditions at infinity are

$$U(\pm\infty) = 0. \tag{4.76}$$

We also add a extra condition for the heat flux to vanish (one again - we expect that from physics)

$$0 = q(\pm\infty, t) = -ku_x(\pm\infty, t) = -\frac{k}{t}U'(\pm\infty), \tag{4.77}$$

hence $U'(\pm\infty) = 0$. We have therefore reduced the partial equation to the ordinary equation, which we can even solve by observing that $U + zU' = (zU)'$, i.e. after integration

$$-\frac{1}{2}zU = \alpha^2 U' + C. \tag{4.78}$$

We choose the integration constant $C = 0$, which is equivalent to requiring that that the solution has $U'(0) = 0$ or $U'(z) \to 0$ when $z \to \pm\infty$. So we have $-\frac{1}{2}zU = \alpha^2 U'$ which we can easily integrate and get

$$U(z) = De^{-\frac{z^2}{4\alpha^2}}, \tag{4.79}$$

where the D constant is arbitrary. Therefore, it is convenient to take it so that the total heat energy is equal to 1, i.e. $D = \ sqrt4\pi\alpha^2$ (task). Ultimately, our solution has the form

$$u(x,t) = \frac{1}{\sqrt{4\pi\alpha^2 t}}e^{-\frac{x^2}{4\alpha^2 t}} =: G(x,t) \tag{4.80}$$

The G function is referred to as the *heat kernel* or *Gauss-Weierstrass kernel* or *fundamental solution of the heat equation*. We will soon show that this is a Green's function for the initial value problem (4.69)[13]. It can also be shown that the heat kernel for n -dimensional space is represented by the formula

$$u(\mathbf{x},t) = \frac{1}{(4\pi\alpha^2 t)^{\frac{n}{2}}}e^{-\frac{x^2}{4\alpha^2 t}}, \tag{4.81}$$

where $x^2 = \mathbf{x}\cdot\mathbf{x} = \sum_{i=1}^{n} x_i^2$.

Note that we have not used the initial condition anywhere but we can see that $u(x,0)$ does not exist (see Fig. 4.2.1). More precisely, $u(0,t)$ goes to infinity and $u(x,t)$, $x \neq 0$ goes to 0 for $t \to 0$. Remember, however, that the integral of $G(x,t)$ over x is equal to

[13]Note that this G has to be very important due to a multitude of its different names. It is! Compare it also with Brownian motion.

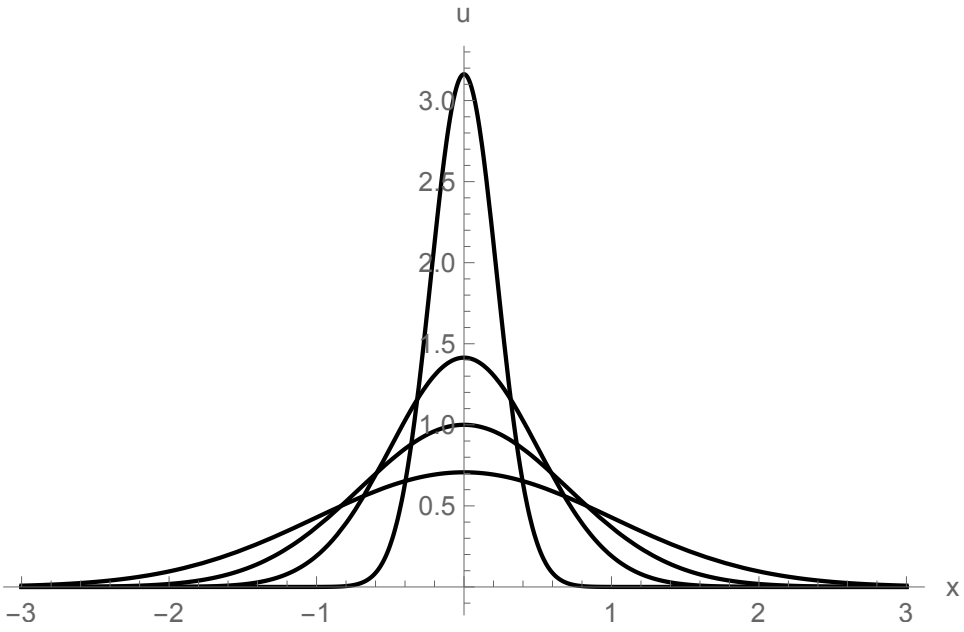

Figure 14: Heat kernel for $t = 2, 1, 0.5, 0.1$.

1 for every time. Physically, this means that the same amount of heat is concentrated closer and closer to $x = 0$ causing the temperature to rise. We can use this fact and deduce the solution to the problem (4.69). When t is very small $G(x - \xi, t)$ is zero almost everywhere except the tiny neighbourhood of $\xi = x$ and its integral is 1. We would like the temperature to be $\phi(x)$ there, thus after multiplication and integration we expect to have

$$\phi(x) \approx \int_{-\infty}^{\infty} G(x - \xi, t)\phi(\xi)d\xi, \quad t \to 0. \tag{4.82}$$

The above is true and we will show that a solution to (4.69) is given by

$$u(x, t) = \int_{-\infty}^{\infty} G(x - \xi, t)\phi(\xi)d\xi. \tag{4.83}$$

Therefore, G is actually a Green's function for our problem, i.e. an instantaneous point heat source. Note that thanks to Green's function our formulas for solution of the initial value problems are always integrals of the initial condition with a respective Green's function. What changes it the domain and boundary conditions. It is very important to keep that in mind.

Theorem 4. *Let ϕ be continuous and bounded on $\mathbb{R}$. Then the function u given by the formula (4.83) satisfies the heat equation, is bounded, and*

$$u(x, t) \to \phi(x) \quad gdy \quad t \to 0. \tag{4.84}$$

Proof. We leave the checking whether u satisfies the heat equation for exercises and here we will deal with the justification of the limit. The boundedness of u results immediately from the boundedness of $|\phi| \leq M$

$$|u(x, t)| \leq \int_{-\infty}^{\infty} G(x - \xi, t)|\phi(\xi)|d\xi \leq M, \tag{4.85}$$

since the integral of the kernel is equal to 1. Notice that due to the same reason we have

$$\phi(x) = \int_{-\infty}^{\infty} G(x-\xi,t)\phi(x)d\xi, \tag{4.86}$$

which yields

$$|u(x,t)-\phi(x)| \le \int_{-\infty}^{\infty} G(x-\xi,t)|\phi(\xi)-\phi(x)|d\xi. \tag{4.87}$$

Choose an arbitrary number $\epsilon > 0$ and separate the integral into three parts

$$|u(x,t)-\phi(x)| \le \int_{-\infty}^{x-\epsilon} + \int_{x-\epsilon}^{x+\epsilon} + \int_{x+\epsilon}^{\infty} = I_1 + I_2 + I_3. \tag{4.88}$$

We will show that each of them goes to 0 as $t \to 0$. The integrals I_1 and I_3 are analogous, so consider only one of them and immediately take advantage of the boundedness of ϕ

$$I_1 \le \frac{2M}{\sqrt{4\pi\alpha^2 t}} \int_{-\infty}^{x-\epsilon} e^{-\frac{(x-\xi)^2}{4\alpha^2 t}} d\xi = \frac{2M}{\sqrt{\pi}} \int_{-\infty}^{\frac{-\epsilon}{\sqrt{4\alpha^2 t}}} e^{-z^2} dz. \tag{4.89}$$

In the last integral we changed the variables. If now $t \to 0^+$ then the expression on the right-hand side tends to zero from the uniform convergence of the integral for fixed ϵ. Note also, that this statement is true for ϵ depending on t such that $\epsilon(t)/\sqrt{t} \to \infty$ for $t \to 0^+$. Similarly, $I_3 \to 0$. On the other hand, we estimate I_2 differently

$$I_2 \le \max_{\xi\in(x-\epsilon,x+\epsilon)} |\phi(\xi)-\phi(x)| \int_{x-\epsilon}^{x+\epsilon} G(x-\xi,t)d\xi \le \max_{\xi\in(x-\epsilon,x+\epsilon)} |\phi(\xi)-\phi(x)|. \tag{4.90}$$

From the uniform continuity of ϕ we can now see that with ϵ the expression on the right side of the above inequality becomes arbitrarily small (the maximum is taken over a smaller set). Take, for example, $\epsilon = t^{\frac{1}{4}}$ so that the integrals I_1 and I_3 will go to 0. □

We see that thanks to Green's function, the solution of a problem on the real line can be written in exactly the same form as the corresponding problem on the interval. Furthermore, essentially as above, we can show that the solution to the nonhomogeneous problem

$$\begin{cases} u_t = \alpha^2 u_{xx} + f, \quad (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = 0, \quad x \in \mathbb{R} \\ u(-\infty,t) = 0, \quad u(\infty,t) = 0, \qquad t \ge 0. \end{cases} \tag{4.91}$$

is

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-\xi,t-\tau)f(\xi,\tau)d\xi d\tau. \tag{4.92}$$

We can see that the Green's function is extremely useful.

Example. (*Diffusion of a cloud*) Imagine a cloud of dust that arises from a point charge explosion in the air. Let $u = u(\mathbf{x},t)$ denote the smoke concentration. We will consider the evolution of visibility of the cloud over time and the moment of its complete

disappearance. If we assume that dust particles evolve only by diffusion[14], their concentration is expressed by the formula (since the initial condition is a point source)

$$u(\mathbf{x}, t) = \frac{Q}{(4\pi Dt)^{\frac{3}{2}}} e^{-\frac{x^2+y^2+z^2}{4Dt}}, \tag{4.93}$$

where Q is the total amount of dust, while D is diffusivity. According to radiation theory, the light intensity I when passing through the medium can be calculated from the equation

$$\frac{dI}{ds} = -\kappa u(s) I, \tag{4.94}$$

where κ is a constant that depends on the type of medium. Here, the independent variable is the path s, which radiation passes along the way from the source to the observer. For constant κ[15] this equation can be immediately solved

$$I(s) = I_0 \exp\left(-\kappa \int u(s) ds\right). \tag{4.95}$$

Note that visibility, or the ratio $\frac{I}{I_0}$, depends only on the value of the integral above. Let us consider the projection of the cloud surface on the xy plane with the light source at infinity. In addition, we orient the light path according to the axis Oz. Then

$$\int_0^\infty u(\mathbf{x}, t) dt = \frac{Q}{(4\pi Dt)^{\frac{3}{2}}} e^{-\frac{x^2+y^2}{4Dt}} \int_0^\infty e^{-\frac{z^2}{4Dt}} dz = \frac{Q\sqrt{\pi Dt}}{(4\pi Dt)^{\frac{3}{2}}} e^{-\frac{x^2+y^2}{4Dt}} = \frac{Q}{8\pi Dt} e^{-\frac{x^2+y^2}{4Dt}}. \tag{4.96}$$

If our optical instrument (eye) has sensitivity δ, then the cloud is no longer visible for $\frac{I}{I_0} = e^{-\delta}$, whence

$$\delta = \frac{Q\kappa}{8\pi Dt} e^{-\frac{x^2+y^2}{4Dt}}, \tag{4.97}$$

determines the rim of the cloud for t. Therefore, the boundary of the cloud $r = x^2 + y^2$ is given by the formula

$$r(t) = 2\sqrt{-Dt \ln \frac{8\pi D\delta t}{Q\kappa}}. \tag{4.98}$$

The above function first increases from zero to its maximum, and then decreases back to zero. Easy calculation allows you to compute the exact values of these characteristic points (see Fig. 15).

4.2.2 Half-line $\mathbb{R}_+$

The phenomenon of heat conduction on a semi-infinite line often occurs in situations where we are interested in the behaviour of the system around one of the boundaries.

[14] Of course this is an oversimplification - usually convection is also present.

[15] Usually it is not constant.

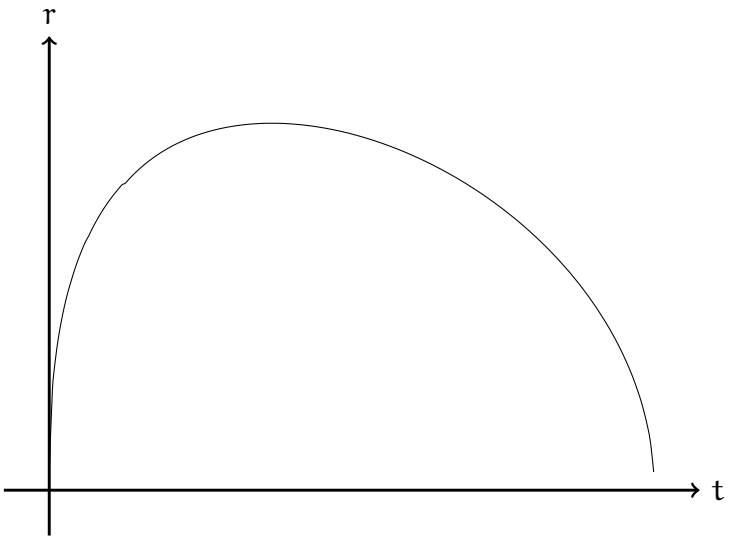

Figure 15: The boundary of the cloud of dust with time.

The other end is very far away (infinity) and its impact on the dynamics is negligible. In addition to the initial condition, we must also prescribe one boundary condition.

$$\begin{cases} u_t = \alpha^2 u_{xx} + f, & (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = \phi(x), & x \in \mathbb{R} \\ u(0,t) = \mu(t), & t \geq 0. \end{cases} \tag{4.99}$$

As usual, the above problem can be broken into three separate and considered all in turn. Let us look first at a homogeneous initial case

$$\begin{cases} u_t = \alpha^2 u_{xx}, & (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = \phi(x), & x \in \mathbb{R} \\ u(0,t) = 0, & t \geq 0. \end{cases} \tag{4.100}$$

In order to solve the problem, we can use the results regarding the real line. Note that if we define an auxiliary function defined on $\mathbb{R}$

$$\psi(x) := \begin{cases} \phi(x), & x \geq 0; \\ -\phi(-x), & x < 0, \end{cases} \tag{4.101}$$

then of course $\psi(x) = \phi(x)$ for $x \geq 0$. Also note that the above definition is well-posed because for $x < 0$ we have $-x > 0$. We say that ψ is an *odd extension* of ϕ. This extension can be used as the initial condition (matching ϕ on a half-line) for the problem (4.69). The solution is of course

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t)\psi(\xi)d\xi. \tag{4.102}$$

From the definition of the heat kernel (4.80) it follows that $G(x,t)$ is an even function of x, thus

$$u(0,t) = \int_{-\infty}^{\infty} G(-\xi,t)\psi(\xi)d\xi = \int_{-\infty}^{\infty} G(\xi,t)\psi(\xi)d\xi = 0, \tag{4.103}$$

since an integral over a symmetric interval of an odd function vanishes. Thus defined u satisfies a homogeneous boundary condition and, of course, $u(x,t) \to \phi(x)$ when $t \to 0^+$ for $x \geq 0$. So u is the solution to the problem (4.100). We can express them in terms of a given initial condition by writing

$$u(x,t) = \int_{-\infty}^{0} G(x-\xi,t)(-\phi(-\xi))d\xi + \int_{0}^{\infty} G(x-\xi,t)(\phi(\xi))d\xi. \tag{4.104}$$

Changing the variable in the first integral we arrive at

$$u(x,t) = \int_{0}^{\infty} \left(G(x-\xi,t) - G(x+\xi,t)\right)\phi(\xi)d\xi. \tag{4.105}$$

It is a solution to a homogeneous heat equation on a half-line with an initial condition. The idea of defining the auxiliary function ψ is called the *method of reflections*. In a similar way, we can also obtain solutions to other problems on the half-line or higher dimensional half-infinite domains.

Example. (*Temperature inside Earth*) Let $u = u(x,t)$ be the temperature inside our planet at depth x and time t. A simple model is

$$u_t = \alpha^2 u_{xx} + f, \tag{4.106}$$

where f is the rate of heat generation by nuclear reactions (this is the main source of thermal energy on our planet because solar radiation is almost completely radiated back into extremely cold space). The most popular, highly accurate, and broadly verified hypothesis is the assumption that heat producing radioactive elements are found in the lithosphere or the upper layer of the Earth, i.e.

$$f(x) = \begin{cases} \frac{A}{c\rho}, & 0 < x \leq H, \\ 0, & x > H, \end{cases} \tag{4.107}$$

where $H > 0$ is a characteristic depth. If we assume that $u(x,0) = 0$ and $u(0,t) = 0$, i.e. we study the effect of the radioactive decay on temperature, we can immediately write the solution (see (4.105))

$$\begin{aligned} u(x,t) &= \int_0^t \int_0^\infty \left(G(x-\xi,t-\tau) - G(x+\xi,t-\tau)\right) f(\xi)d\xi d\tau \\ &= \frac{A}{c\rho}\int_0^t \int_0^H \left(G(x-\xi,t-\tau) - G(x+\xi,t-\tau)\right) d\xi d\tau, \end{aligned} \tag{4.108}$$

where G is the heat kernel. The above integral can be expressed in terms of the error function. Without the radioactive elements, Earth would become as cold as outer space billions years ago. □

4.3 Maximum principle (optional)

The solution of the heat equation has a very important property called the *maximum principle*. Its physical interpretation is very clear and says that the maximum temperature can only occur at the edge of the domain or at the initial time. In the absence of sources it is impossible to raise the temperature to a value greater than that already present on the boundaries.

Theorem 5 (Maximum principle). *If the function* $u = u(x,t)$ *is bounded, continuous on* $[0,L] \times [0,T]$*, and satisfies the heat equation on* $(0,L) \times (0,T]$ *its maximum and minimum values can only occur on the sides* $t = 0$*,* $x = 0$ *or* $x = L$*.*

Proof. Let M denote the maximum of u on the corresponding sides of the rectangle, i.e. $t = 0$, $x = 0$ or $x = L$ (the minimum version can be tackled analogously). Let us assume indirectly that there is a point (x_0, t_0) belonging to the interior of the domain or its upper edge $t = T$, at which

$$u(x_0, t_0) = M + \epsilon. \tag{4.109}$$

From the necessary condition of the existence of the maximum, we must have $u_x(x_0, t_0) = 0$ and $u_{xx}(x_0, t_0) \leq 0$. Since the maximum occurs also with respect to time, we have $u_t(x_0, t_0) \geq 0$ where the strict inequality takes place for $t = T$. Of course, from the heat equation we have $u_t = \alpha^2 u_{xx}$, i.e. the contradiction occurs if $u_t > 0$. We must therefore show that there is such a point (x_1, t_1) for which the temporal derivative is positive.

Let us define an auxiliary function

$$v(x,t) := u(x,t) + k(t_0 - t), \tag{4.110}$$

where k is a fixed positive number satisfying the condition $k < \frac{\epsilon}{2T}$. Then of course we have $v(x_0, t_0) = u(x_0, t_0) = M + \epsilon$ and $k(t_0 - t) \leq kT$. Note that $v(x,0) = u(x,0) + kt_0 \leq M + kT < M + \frac{\epsilon}{2}$ and similarly for $u(0,t)$ and $u(L,t)$. Since the function v is continuous as the sum of two continuous functions, it must attain its maximum at some point (x_1, t_1). From what we have just noticed, we must have $v(x_1, t_1) \geq v(x_0, t_0) = M + \epsilon$. Because for $t = 0$, $x = 0$ and $x = L$ we have $v \leq M + \frac{\epsilon}{2}$ then (x_1, t_1) must belong to the inside of the rectangle or on its side $t = T$. At this point

$$v_{xx}(x_1, t_1) = u_{xx}(x_1, t_1) \geq 0, \tag{4.111}$$

and

$$v_t(x_1, t_1) = u_t(x_1, t_1) - k > 0, \tag{4.112}$$

which gives us a strict inequality $u_t(x_1, t_1) > 0$. Therefore, there exists a point at which the heat equation is not satisfied. A contradiction. □

As an important application we apply the maximum principle to showing that heat equation has a unique solution. Note that we have done that before with a use of energy methods. These are essentially two most useful techniques used to showing estimates on solution of parabolic and elliptic equations.

Theorem 6 (Uniqueness (once again)). *Let two functions* u_1 *and* u_2 *be defined on* $[0,L] \times [0,T]$ *and satisfy the heat equation*

$$u_t = \alpha^2 u_{xx} + f, \quad (x,t) \in (0,l) \times (0,\infty) \tag{4.113}$$

with identical Dirichlet boundary and initial conditions. Then $u_1 \equiv u_2$*, that is, the solution is unique.*

Proof. Let us define $v := u_2 - u_1$ and note, that v is a solution of the heat equations with vanishing boundary and initial conditions (linearity). Moreover, it is continuous and hence, attains its maximum at $t = 0$, $x = 0$ or $x = L$. But the conditions are zero which forces $v \equiv 0$, i.e. $u_1 \equiv u_2$. □

Example. (*Unbounded solutions*) Note that an important element of the proof of uniqueness is the boundedness of the solution. When we give up this requirement, the situation gets seriously complicated and non-physical. Let

$$w(x,t) := \frac{xe^{-\frac{x^2}{4t}}}{2\sqrt{\pi}t^{\frac{3}{2}}} = -2G_x(x,t), \tag{4.114}$$

where G is the heat kernel, while for simplicity and without any loss of generality, we assumed that $\alpha = 1$. Direct calculations can show that w satisfies the heat equation $w_t = w_{xx}$ on a the real line $\mathbb{R}$. Also,

$$\lim_{x\to\pm\infty} w(x,t) = 0,\ t > 0 \quad \text{and} \quad \lim_{t\to 0^+} w(x,t) = 0,\ x > 0, \tag{4.115}$$

therefore, both initial and boundary conditions are zero. Thus, if u is the solution of the heat equation with any initial condition (4.69) then for constant $C > 0$ the function $u + Cw$ is also a solution. What is wrong then? The reason is that w is really unbounded. We can see this by checking its value for $t \to 0^+$ following the curve $x = 2\sqrt{t}$

$$\lim_{t\to 0^+} w(2\sqrt{t},t) = \lim_{t\to 0^+} \frac{e^{-1}}{\sqrt{\pi}t} = \infty. \tag{4.116}$$

Therefore, if we do not listen to physics we can obtain very unintuitive results! □

4.4 Nonlinear diffusions (optional)

In many applications the diffusivity depends on the concentration itself. Let us see some important examples.

Example. (*Insects*) Suppose the insect population spreads diffusively. It makes sense to assume that the flow of such movement depends non-linearly on the size of the population (there is a population pressure to speed up the conquest of a new terrain). Then, the diffusion equation has the form

$$u_t = (D(u)u_x)_x. \tag{4.117}$$

It can also be written as

$$u_t = D'(u)u_x^2 + D(u)u_{xx}, \tag{4.118}$$

Notice that due to nonlinearity a new convective term arises. This model can predict insect outbreaks which is very useful in forestry (e.g. bark beetle). □

Example. (*Liquid helium*) At low temperatures, heat in liquid helium is transported according to a different law than Fourier's. The so-called Gorter-Mellink equation is more appropriate to model that phenomenon

$$T_t = \left(kT_x^{\frac{1}{3}}\right)_x. \tag{4.119}$$

Notice that the equation has a singularity for $T_x \to 0$. □

Example. (*Porous medium*) Let $u = u(x,t)$ is the soil moisture content at point x and time t. If we neglect gravity, then, according to Darcy's law, water in an unsaturated porous medium is transported along the capillary pressure gradient, i.e.

$$q = -k\frac{dp}{dx}, \tag{4.120}$$

where k is associated with pore distribution. Experiments show that capillary pressure is a monotone function of the moisture. Therefore,

$$q = -k\frac{dp}{du}\frac{\partial u}{\partial x}. \tag{4.121}$$

Conservation of mass gives us

$$u_t = (D(u)u_x)_x, \tag{4.122}$$

Where we have defined the diffusivity. In many cases a power-type function is a good model which yields

$$u_t = D_0(u^m u_x)_x, \tag{4.123}$$

where $m > 0$ and $D_0 > 0$. The above is the celebrated *porous medium equation*. Note that it is degenerate for $u \to 0$. □

Example. (*Glaciers*) Consider a huge mass of ice - a glacier or ice sheet. Let $h = h(x,t)$ be the height of the glacier at the point x and time t (as usual, the glacier moves mainly in one direction, so one-dimensional approximation is sensible and highly accurate). If v is the speed of the glacier and M is the rate of snowfall, mass conservation gives

$$h_t + (vh)_x = M. \tag{4.124}$$

The constitutive relation was experimentally found by Glen and Nye. It states that

$$v = kh\tau^n, \tag{4.125}$$

where k, n are constants and τ is the stress that causes glacier movement. Usually $n \approx 3$. Because the glacier is moving mainly due to its weight, the stress is given by the hydrostatic law (we neglect atmospheric pressure here)

$$\tau = -\rho g h h_x. \tag{4.126}$$

Plugging it into the conservation law gives us

$$h_t = K\left(h^{n+2}|h_x|^{n-1}h_x\right)_x + M, \tag{4.127}$$

where K is another constant. It is highly nonlinear doubly degenerate porous medium equation. □

Example. (*Reaction-diffusion*) Many phenomena in nature contain both diffusion and reaction elements (for example production of some chemical substance or population increase). This can be described by the equation

$$u_t - Du_{xx} = f(u), \tag{4.128}$$

where f is responsible for the reaction (a source). For example, if we assume that u describes the population size that diffuses in space and reproduces in accordance to logistic law, we get the *Fisher's equation*

$$u_t - Du_{xx} = ru\left(1 - \frac{u}{K}\right), \tag{4.129}$$

which is a prototype of spatial population modelling. □

Example. (*Burgers equation*) On the other hand, a prototype of a nonlinear convection-diffusion is the *Burgers' equation*

$$u_t + uu_x = Du_{xx}. \tag{4.130}$$

We have already encountered the first term of this equation earlier when investigating shock waves. It turns out that the diffusion element works against the non-linear convection and is able to mitigate the gradient catastrophe. The above equation arises as the simplest model of viscous fluid convection. Then, D is the kinematic viscosity of the fluid. There is an amazing formula called *Cole-Hopf transformation* that transforms the Burgers equation into a linear heat equation. Let us check it out.

First, put $u = w_x$ for some function w. Then (4.130) gives us

$$u_{xt} + w_x w_{xx} - Dw_{xxx} = 0, \tag{4.131}$$

which can immediately be integrated into

$$u_t + \frac{1}{2}w_x^2 - Dw_{xx} = 0, \tag{4.132}$$

where constant of integration vanishes because we assume that solutions disappear at infinity. Finally, we substitute

$$w = -2D \ln v. \tag{4.133}$$

By calculating partial derivatives we can write

$$0 = w_t + \frac{1}{2}w_x^2 - Dw_{xx} = -\frac{2D}{v}\left(v_t - Dv_{xx}\right). \tag{4.134}$$

Therefore, the substitution $u = -2Dv^{-1}v_x$ transforms (4.130) into

$$v_t = Dv_{xx}. \tag{4.135}$$

If u satisfies the initial condition $u(x,0) = \phi(x)$, then v has

$$v(x,0) = \exp\left(-\frac{1}{2D}\int_0^x \phi(y)dy\right) =: \psi(x). \tag{4.136}$$

Therefore, we can write the solution

$$v(x) = \int_{-\infty}^{\infty} G(x-\xi,t)\psi(\xi)d\xi, \tag{4.137}$$

or after returning to the original variables

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-\xi}{t} e^{-\frac{K(\xi,x,t)}{2D}} d\xi}{\int_{-\infty}^{\infty} e^{-\frac{K(\xi,x,t)}{2D}} d\xi}, \tag{4.138}$$

where

$$K(\xi,x,t) = \frac{(x-\xi)^2}{2t} + \int_0^x \phi(y)dy. \tag{4.139}$$

Therefore, we obtained the analytical formula for solving an initial problem for Burgers' equation. This is one of the few cases where a difficult nonlinear equation can be solved by explicit formula. However, one should not overestimate the fact that we have an analytical solution - after all, it is very complicated and at first glance does not tell us anything. However, there are a number of approximation techniques that allow us to examine the behaviour of the solution for different limits and specific initial conditions. The most important limiting case is of course $D \to 0$. We can then see how a smooth solution creates becomes a shock wave.

4.4.1 Self-similar solutions

Nonlinear parabolic equations are a huge branch of modern mathematics. There are no general methods for solving them or even investigating their solutions. However, the method of looking for self-similar solutions is very useful. It does not usually solve a generality of initial or boundary problems, but very often allows finding candidates for asymptotic forms of other solutions.

The method of self-similar solutions has its origins in the geometrical theory of Lie groups, but here we will see its practical version. Suppose we have an equation

$$u_t = (D(u)u_x)_x, \tag{4.140}$$

or any other nonlinear parabolic equation. For example, let us consider the porous medium equation $D(U) = U^m$[16]. We look for solutions is the form

$$u(x,t) = t^a U(\eta), \quad \eta = xt^b, \tag{4.141}$$

where a and b are to be found. Calculating the derivatives we can find that

$$u_t = at^{a-1}U(\eta) + bxt^{a+b-1}U'(\eta), \quad u_x = t^{a+b}U'(\eta). \tag{4.142}$$

[16]The familiar heat equation is retrieved for $m = 0$.

After substituting into the equation we obtain

$$a t^{a-1} U + b t^{a-1} \eta U' = t^{a+2b+ma} \left(U^m U'\right)'. \tag{4.143}$$

So for the equation to make sense, we must have $a - 1 = a + 2b + ma$. We still need to find the second equation to uniquely determine all the unknowns. This can be done with boundary conditions (whenever possible). For example, consider the familiar case when at the time $t = 0$ a fixed amount of substance is released at the point $x = 0$. This means that the total weight is constant

$$C = \int_{-\infty}^{\infty} u(x,t)dx = t^a \int_{-\infty}^{\infty} U\left(xt^b\right) dx = t^{a-b} \int_{-\infty}^{\infty} U(\eta)d\eta. \tag{4.144}$$

Therefore, for the above equation to be valid, we must have $a = b$. Moreover, the first equation for parameters gives us $a = -\,frac12 + m$. The self-similar solution thus has the form

$$u(x,t) = t^{-\frac{1}{2+m}} U\left(xt^{-\frac{1}{2+m}}\right), \tag{4.145}$$

where U satisfies

$$-\frac{1}{2+m}\left(U + \eta U'\right) = \left(U^m U'\right)'. \tag{4.146}$$

We can immediately write

$$-\frac{1}{2+m}\left(\eta U\right)' = \left(U^m U'\right)', \tag{4.147}$$

that is

$$-\frac{1}{2+m}\eta U = U^m U', \tag{4.148}$$

where we have set the integration constant to zero. We can immediately state the solution in the form

$$U(\eta) = \left(K - \frac{1}{2}\frac{m}{2+m}\eta^2\right)_+^{\frac{1}{m}}, \tag{4.149}$$

where $(a)_+ = \max a, 0$. This self-similar solution is the famous *Barrenblatt's profile* obtained in the middle of XX century when investigating a sudden release of energy[17]. We can compare it with the fundamental solution of the (linear) heat equation (4.80). The latter is positive for any $x \in \mathbb{R}$ and $t > 0$ while the former vanishes outside a bounded and closed set (in analysis we say that the solution has a compact support). This means that the velocity of information propagation is finite which is a required feature of many physical systems (since it does satisfy the second law of thermodynamics). Note also the Barrenblatt's profile is nondifferentiable at the junction point for which $u = 0$. We can see that solutions of nonlinear problems can behave completely different than their simpler counterpart in linear theory.

Example. (*Lake freezing*) Let us consider a lake and denote its temperature by $u(x,t)$ at the depth of x and time t. We know that initially $u(x,0) = \phi(x)$. Assume that there is a

[17] Apparently, the Cold War generated a lot of interesting mathematics...

certain temperature on the surface, i.e. $u(0,t) = u_0$ and the lake begins to freeze. After t, the boundary between ice and water is denoted by $s = s(t)$. Note that the point of contact between water and ice is unknown and must be found as part of the solution.

To complete the formulation of the problem we need to set a second boundary condition. During each phase transformation, latent heat L (per unit of mass) is exchanged with the environment while maintaining a constant temperature. When the boundary between phases changes by Δs during Δt, heat is released in an amount $\rho \Delta s L$. For the energy to be conserved, the above heat must be equal to the difference in heats between the phases

$$(k_1 u_x(s(t)^-, t) - k_2 u_x(s(t)^+, t))\,\Delta t = L\rho\Delta s, \tag{4.150}$$

after taking the limit we obtain the *Stefan's condition*

$$k_1 u_x(s(t)^-, t) - k_2 u_x(s(t)^+, t) = L\rho s'(t), \tag{4.151}$$

where the prime denotes the derivative with respect to time. Here, k_i are heat conduction coefficients for different phases.

The complete problem can be now formulated. The temperature satisfies the heat equation

$$u_t = \alpha(x, s(t))^2 u_{xx} \quad \text{gdzie} \quad \alpha(x, s(t)) = \begin{cases} \alpha_1, & 0 < x < s(t) \\ \alpha_2, & x \geq s(t) \end{cases}. \tag{4.152}$$

We have an initial and boundary conditions

$$u(x, 0) = \phi(x), \quad u(0, t) = u_0. \tag{4.153}$$

On the boundary between phases the temperature must be continuous

$$u(s(t)^-, t) = u(s(t)^+, t) = 0, \tag{4.154}$$

and the latent heat is exchanged (Stefan's condition)

$$k_1 u_x(s(t)^-, t) - k_2 u_x(s(t)^+, t) = L\rho s'(t). \tag{4.155}$$

Let us emphasize that $s(t)$ is not known from the very beginning and must be determined as a part of the solution. The boundary condition is therefore imposed at an unknown interface. This is one of the *free-boundary problems*.

Let us try to solve the Stefan's problem by using the method of self-similar solutions. Put $u(x, 0) = c$. If our case were a problem with only one phase, we would know from previous considerations that the solution of the heat equation on a half-line was a function of $x/\sqrt{t}$. Therefore, we try to look for

$$u(x, t) = U(\eta), \quad \eta = \frac{x}{\sqrt{t}}. \tag{4.156}$$

At the same time we also should have

$$s(t) = s_0\sqrt{t}, \tag{4.157}$$

for some $s_0 > 0$. The heat equation then becomes

$$-\frac{1}{2}\eta U' = \alpha(\eta)^2 U'' \quad \text{gdzie} \quad \alpha(\eta) = \begin{cases} \alpha_1, & 0 < \eta < s_0 \\ \alpha_2, & \eta \geq s_0 \end{cases}. \tag{4.158}$$

Note that in this formulation, the unknown boundary between phases is constant. The ordinary equation can be immediately solved (since $\alpha(\eta)$ is piecewise constant)

$$U(\eta) = A_{1,2} + B_{1,2}\,\mathrm{Erf}\left(\frac{\eta}{2\alpha_{1,2}}\right), \tag{4.159}$$

where subscripts $1, 2$ mean that for each domain $0 < \eta < s_0$ and $\eta \geq s_0$ we have different integration constants. Initial condition is now $c = u(x, 0) = U(\infty)$ while the boundary condition at $x = 0$ gives $u_0 = u(0, t) = U(0)$. In order to satisfy them we must have

$$A_1 = u_0, \quad A_1 + B_2 = c. \tag{4.160}$$

Continuity of u at $\eta = s_0$ forces

$$A_{1,2} + B_{1,2}\mathrm{Erf}\left(\frac{s_0}{2\alpha_{1,2}}\right) = 0. \tag{4.161}$$

The above system of algebraic equations is now

$$A_1 = u_0, \quad A_2 = -\frac{c\mathrm{Erf}\left(\frac{s_0}{2\alpha_2}\right)}{1 - \mathrm{Erf}\left(\frac{s_0}{2\alpha_2}\right)}, \quad B_1 = -\frac{u_0}{\mathrm{Erf}\left(\frac{s_0}{2\alpha_1}\right)}, \quad B_2 = \frac{c}{1 - \mathrm{Erf}\left(\frac{s_0}{2\alpha_2}\right)}. \tag{4.162}$$

It remains to find the constant s_0 determined from Stefan's condition. When we do this we obtain a transcendental equation that has to be solved numerically.

5 Laplace's and Poisson's equations

We will now deal with another very important partial differential equations: Laplace's and Poisson's. These are elliptical equations and we will see that their solutions exhibit completely different properties from the solutions of parabolic equations. First, however, let us see where such equations can arise.

Example. (*Steady-state heat distribution*) We already know that the evolution of temperature in a certain area is governed by the heat equation

$$u_t = \alpha^2 \Delta u + f, \tag{5.1}$$

where Δ is the Laplacian. If we wait long enough, then the state of our domain should be fixed and all temperature changes should stop, that is, $u_t \approx 0$. As an example, we can imagine heating a cold room with radiators on. First, the temperature in the whole room begins to rise until it reaches the one that is set on the thermostat. We can then specify two phases: *transient* and *steady-state*. With a vanishing temporal derivative the heat equation becomes

$$-\Delta u = \frac{f}{\alpha^2}. \tag{5.2}$$

This is the so-called *Poisson's equation*, and in the homogeneous case *Laplace's equation*

$$\Delta u = 0. \tag{5.3}$$

We can see immediately that the above two equations are only interesting for dimensions greater than or equal to 2, because on a real line $\mathbb{R}$ direct integration gives us the result. Functions that satisfy the Laplace's equation are called textit harmonic functions. □

Example. (*Potential*) A very important application of elliptic equations occurs in the theory of both gravitational and electrostatic potential but also describes the potential of velocity in fluid mechanics. Here we will focus on the gravitational field.

According to Newton's law of universal gravitation, the two point masses m and M attract each other with a gravitational force proportional to the inverse square of their distance

$$\mathbf{F} = G\frac{Mm}{r^2}\frac{\mathbf{r}}{r}, \tag{5.4}$$

where $r\mathbf{r}$ is a vector connecting two masses and $r := |\mathbf{r}|$ is its length. Since force is a vector, we had to include the $\mathbf{r}/r$ term in the above formula to indicate the direction. Imagine that M is a mass of a large body and m is the test mass. To focus only on the field created by this large body, it is worth defining *gravitational field* as a vector

$$\mathbf{g} := \frac{\mathbf{F}}{m}. \tag{5.5}$$

It is nothing else but a gravitational acceleration created by the mass M. Since $\mathbf{F}$ is a conservative force, there exists a potential $u = u(\mathbf{x})$ such that

$$\mathbf{g} = -\nabla u, \tag{5.6}$$

and it is not hard to see that

$$\phi(\mathbf{x}) = \frac{GM}{|\mathbf{x}|} = \frac{GM}{\sqrt{x^2+y^2+z^2}} = \frac{GM}{r}, \quad r \neq 0. \tag{5.7}$$

If we would like to find a field generated by two masses, the resultant $\mathbf{g}$ would be the sum of two components, each of which would come from every mass separately. In general, for a discrete mass distribution we have

$$\mathbf{g}_j = \sum_{i \neq j} G m_i \frac{\mathbf{r}_{ij}}{(r_i - r_j)^3}, \tag{5.8}$$

which gives us a gravitational field acting on j -th mass created by all other bodies. It is convenient to introduce the continuous volume density of the mass distribution $\rho = \rho(\mathbf{x})$ giving the field at the point $\mathbf{x}$

$$\mathbf{g}(\mathbf{x}) = G \iiint_{\mathbb{R}^3} \rho(\mathbf{y}) \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} d\mathbf{y}. \tag{5.9}$$

We expect that there must also be a gravitational potential for such continuous distribution (a smeared-out mass). Let us first assume that ρ disappears in the area containing $\mathbf{x}$. Then

$$\mathbf{g}(\mathbf{x}) = -G \iiint_{\mathbb{R}^3} \rho(\mathbf{y}) \nabla \left(\frac{1}{|\mathbf{x}-\mathbf{y}|} \right) d\mathbf{y} = -\nabla \left(\iiint_{\mathbb{R}^3} \frac{G\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} d\mathbf{y} \right) =: -\nabla u, \tag{5.10}$$

where the gradient is taken over the variable $\mathbf{x}$. On the other hand, we can see that

$$\nabla \cdot \mathbf{g} = G \iiint_{\mathbb{R}^3} \rho(\mathbf{y}) \nabla \cdot \left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} \right) d\mathbf{y} = 0. \tag{5.11}$$

Comparing the above two equations we get

$$0 = \nabla \cdot (-\nabla u) = -\Delta u, \tag{5.12}$$

Therefore, the potential for empty regions (a vacuum) satisfies Laplace's equation.

Now, suppose that ρ may be nonzero in the area containing $\mathbf{x}$. The problem, of course, is that the point mass potential is not specified at $\mathbf{y} = \mathbf{x}$. Let us isolate this point. To this end, let $B(\mathbf{x}, \epsilon)$ be a ball with a centre at $\mathbf{x}$ and radius ϵ while ∂B its surface (a sphere). We can then write

$$\mathbf{g}(\mathbf{x}) = G \iiint_{\mathbb{R}^3 - B(\mathbf{x},\epsilon)} \rho(\mathbf{y}) \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} d\mathbf{y} + G \iiint_{B(\mathbf{x},\epsilon)} \rho(\mathbf{y}) \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} d\mathbf{y}. \tag{5.13}$$

We know that the divergence of the first integral is zero, thus it is sufficient to consider the second one. Let us use the mean-value theorem for integrals

$$\iiint_{B(\mathbf{x},\epsilon)} \rho(\mathbf{y}) \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} d\mathbf{y} = \rho(\mathbf{y}^*) \iiint_{B(\mathbf{x},\epsilon)} \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} d\mathbf{y}. \tag{5.14}$$

Taking divergence and using Gauss-Ostrogradskii's theorem we further obtain

$$\nabla \cdot \iiint\limits_{B(\mathbf{x},\epsilon)} \rho(\mathbf{y}) \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} d\mathbf{y} = \rho(\mathbf{y}^*) \iiint\limits_{B(\mathbf{x},\epsilon)} \nabla \cdot \left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} \right) d\mathbf{y} = \rho(\mathbf{y}^*) \iint\limits_{\partial B(\mathbf{x},\epsilon)} \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} \cdot \mathbf{n} dS(\mathbf{y}). \quad (5.15)$$

Since in the last integral, $\mathbf{y}$ belongs to the surface of the sphere with the center at $\mathbf{x}$ and radius ϵ we have

$$\nabla \cdot \iiint\limits_{B(\mathbf{x},\epsilon)} \rho(\mathbf{y}) \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} d\mathbf{y} = \rho(\mathbf{y}^*) \iint\limits_{\partial B(\mathbf{x},\epsilon)} \frac{\epsilon}{\epsilon^3} dS(\mathbf{y}) = \rho(\mathbf{y}^*) \frac{\epsilon}{\epsilon^3} 4\pi\epsilon^2 = 4\pi\rho(\mathbf{y}^*). \quad (5.16)$$

We can now take the limit $\epsilon \to 0$ which gives

$$\nabla \cdot \mathbf{g}(\mathbf{x}) = 4\pi G\rho(\mathbf{x}), \quad (5.17)$$

which is called the *Gauss law*. In the language of the potential we therefore have

$$-\Delta u = 4\pi G\rho, \quad (5.18)$$

whence, the mass density ρ defines a potential given by Poisson's equation. □

Example. (*Potential flow*) Let $\mathbf{u} = \mathbf{u}(\mathbf{x})$ denote the velocity field of a non-viscous fluid at $\mathbf{x}$, which is not dependent on time (stationary flow). From the law of conservation of mass we have

$$\rho_t + \nabla \cdot (\rho\mathbf{u}) = 0, \quad (5.19)$$

where ρ is the fluid density. If it is incompressible[18] we have

$$\nabla \cdot \mathbf{u} = 0, \quad (5.20)$$

Therefore, the flow is divergenceless. If we additionally assume that it is also irrotational we have

$$\nabla \times \mathbf{u} = 0. \quad (5.21)$$

It follows that there exists a potential φ satisfying

$$\mathbf{u} = -\nabla\varphi. \quad (5.22)$$

Using the incompressibility condition we can obtain

$$0 = \nabla \cdot \mathbf{u} = -\nabla \cdot \nabla\varphi = -\Delta\varphi, \quad (5.23)$$

hence, φ satisfies Laplace's condition. Despite its extreme simplicity (Feynman, after von Neumann, called this flow „dry water"), the above equation is very useful for describing low viscosity fluid flows away from the boundaries of the domain. An important example is the description of the flow around an aircraft wing (aerofoil). □

[18] And to a very high accuracy we can assume that under normal conditions many liquids are essentially incompressible, e.g. water.

Example. (*Analytical functions*) In the theory of complex functions, a beautiful result states that for the differentiability of $f(x,y) = u(x,y) + iv(x,y)$ in the complex sense (so-called *analytic* or *holomorphic* functions) a necessary and sufficient condition has to be met and it takes the form of Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x. \tag{5.24}$$

Computing second derivatives we obtain

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy}, \tag{5.25}$$

Therefore, $\Delta u = 0$. Similarly $\Delta v = 0$ and we see that both real and imaginary part of the analytical functions are harmonic. □

5.1 Separation of variables

As in the case of the heat equation, also for elliptic PDEs the method of separating variables allows us to obtain explicit solutions. However, it should be noted that this approach will work only for simple domain, of which the rectangle is the easiest and we leave its analysis for the reader.

5.1.1 Laplace's equation of a circle

Another interesting domain for which the method works is the circle, i.e.

$$\begin{cases} \Delta u = 0, \\ u|_B = f, \end{cases} \tag{5.26}$$

where B is a unit circle with a center at 0. Of course, it is natural to express the problem in polar coordinates and write $u = u(r,\theta)$. The above, therefore, has the form

$$\begin{cases} \dfrac{1}{r}(ru_r)_r + \dfrac{1}{r^2}u_{\theta\theta} = 0, & 0 < r < 1,\ -\pi \le \theta \le \pi, \\ u(1,\theta) = f(\theta), & -\pi \le \theta \le \pi. \end{cases} \tag{5.27}$$

Since polar coordinates are singular at $r = 0$, we can expect that the solution of the equation will be unbounded. To be in accordance with physical intuition let us require that

$$|u(0,\theta)| \le C. \tag{5.28}$$

Periodicity of polar coordinates must also be taken into account, therefore we require that $u(r,\cdot)$ and $u_\theta(r,\cdot)$ are 2π - periodic. Let us separate the variables and write

$$u(r,\theta) = R(r)\Theta(\theta). \tag{5.29}$$

In follows that the boundary conditions for the Θ function are

$$\Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi). \tag{5.30}$$

The PDE has the form

$$\frac{1}{r}(rR')'\Theta + \frac{1}{r^2}R\Theta'' = 0, \tag{5.31}$$

and after division by $R\Theta$ we have

$$\frac{r}{R}(rR')' = -\frac{1}{\Theta}\Theta'' = \lambda, \tag{5.32}$$

where λ is a constant and we anticipate that it is positive. The Θ-equation is very simple

$$\Theta'' + \lambda\Theta = 0, \tag{5.33}$$

with conditions (5.30). A solution is therefore

$$\Theta(\theta) = C\cos\left(\sqrt{\lambda}\theta\right) + D\sin\left(\sqrt{\lambda}\theta\right), \tag{5.34}$$

along with

$$\lambda = n^2, \quad n \in \mathbb{N}. \tag{5.35}$$

Thanks to that the equation for R becomes

$$r^2R'' + rR' - n^2R = 0, \tag{5.36}$$

which is a second order equation with variable coefficients called *Euler's equation*. This is one of the few equations of this type for which we can provide analytical solutions. To guess them, notice that polynomial differentiation reduces its degree by one. Therefore, we expect the solution of (**??**) to be

$$R(r) = r^\alpha, \tag{5.37}$$

for some constant α. Assume that $n \neq 0$ and plug this *ansatz* into (5.36). Thanks to that we obtain a quadratic characteristic equation for α

$$\alpha(\alpha - 1) + \alpha - n^2 = 0. \tag{5.38}$$

The solution is

$$\alpha = \pm n, \tag{5.39}$$

whence, the general solution of (5.36) is

$$R(r) = A_n r^n + B_n r^{-n}, \quad n \neq 0. \tag{5.40}$$

For $n = 0$ Euler's equation takes the form

$$r^2R'' + rR' = 0, \tag{5.41}$$

hence $R(r) = A_0 + B_0 \ln r$. We can see that for u to be bounded at $(0, \theta)$ we must have $B_n = 0$ for $n \in \mathbb{N}$. Therefore,

$$u_n(r, \theta) = r^n\left(C_n\cos(n\theta) + D_n\sin(n\theta)\right), \quad n \in \mathbb{N}. \tag{5.42}$$

As the above function is a solution of the Laplace equation for each n, we expect that the general solution has the form

$$u(r, \theta) = \sum_{n=0}^{\infty}\left[C_n\cos(n\theta) + D_n\sin(n\theta)\right]r^n, \quad 0 < r < 1, \quad -\pi \leq \theta \leq \pi. \tag{5.43}$$

Coefficients C_n i D_n, as always, have to be determined from the boundary condition

$$f(\theta) = u(1,\theta) = \sum_{n=0}^{\infty} C_n \cos(n\theta) + D_n \sin(n\theta). \tag{5.44}$$

It is a usual Fourier series for a periodic function f, therefore the coefficients are given by

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta, \quad C_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(\theta)\,d\theta, \quad D_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(\theta)\,d\theta. \tag{5.45}$$

It can be shown that this series converges and defines a solution of the Laplace's equation on a circle with Dirichlet boundary condition.

As in the case of the heat equation, let us check whether we can transform the solution into an integral form with a use of the respective Green's function. Put (5.45) in (5.43) and interchange the order of summation and integration

$$u(r,\theta) = \int_{-\pi}^{\pi} \frac{1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} (\cos(n\phi)\cos(n\theta) + \sin(n\phi)\sin(n\theta))\, r^n\right] f(\phi)\,d\phi. \tag{5.46}$$

After simplification we obtain

$$(\cos(n\phi)\cos(n\theta) + \sin(n\phi)\sin(n\theta))\, r^n = \cos(n(\phi-\theta))\, r^n, \tag{5.47}$$

whence,

$$u(r,\theta) = \int_{-\pi}^{\pi} \frac{1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \cos(n(\phi-\theta))\, r^n\right] f(\phi)\,d\phi. \tag{5.48}$$

The sum of an infinite series can also be calculated explicitly. To do this in the easiest way let us use the complex representation of trigonometric functions

$$\frac{1}{2}+\sum_{n=1}^{\infty} \cos(n(\phi-\theta))\, r^n = \frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty} \left(e^{in(\phi-\theta)} + e^{-in(\phi-\theta)}\right) r^n = \frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty} \left(re^{i(\phi-\theta)}\right)^n + \left(re^{-i(\phi-\theta)}\right)^n. \tag{5.49}$$

Each is a geometric series

$$\frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \left(re^{i(\phi-\theta)}\right)^n + \left(re^{-i(\phi-\theta)}\right)^n = \frac{1}{2}\left[1 + \frac{1}{1-re^{i(\phi-\theta)}} + \frac{1}{1-re^{-i(\phi-\theta)}} - 2\right], \tag{5.50}$$

and hence

$$\frac{1}{2}+\sum_{n=1}^{\infty} \cos(n(\phi-\theta))\, r^n = \frac{1}{2}\frac{\left(1-re^{i(\phi-\theta)}\right) + \left(1-re^{-i(\phi-\theta)}\right) - \left(1-re^{i(\phi-\theta)}\right)\left(1-re^{-i(\phi-\theta)}\right)}{\left(1-re^{i(\phi-\theta)}\right)\left(1-re^{-i(\phi-\theta)}\right)}. \tag{5.51}$$

Which after simplification is

$$\frac{1}{2} + \sum_{n=1}^{\infty} \cos(n(\phi-\theta))\, r^n = \frac{1}{2}\frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)}, \tag{5.52}$$

what is called the *Poisson's kernel*. Returning to the integral we obtain

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)} f(\phi)\,d\phi. \tag{5.53}$$

Poisson's kernel can be thought as a derivative of the Green's function for Laplace's equation on a unit circle.

5.1.2 Mean-value property, maximum principle, and uniqueness (optional)

Notice that when we put $r = 0$ in (5.53) we obtain

$$u(r,\theta) = \frac{1}{2\pi}\int_{0}^{2\pi} f(\phi)\,d\phi, \tag{5.54}$$

which means that the value of solution of Laplace's equation at the centre is equal to its *average* over circle's circumference. This is extremely important fact that has many implications. It is revealing to prove this property without the use of the exact solution. Recall that a function satisfying Laplace's equation is called *harmonic*.

Theorem 7 (Mean-value property)**.** *Let u be harmonic on $\mathbb{R}^n$, that is $\Delta u = 0$. Then,*

$$u(\boldsymbol{x}) = \frac{1}{|\partial B(\boldsymbol{x},r)|}\oint_{\partial B(\boldsymbol{x},r)} u\,dS, \tag{5.55}$$

where $B(\boldsymbol{x},r)$ is a ball centred at $\boldsymbol{x}$ with radius $r > 0$.

Proof. We will focus on the two-dimensional case $n = 2$, however, the proof for a general case is not much more difficult. Assume that $u_{xx} + u_{yy} = 0$. Since $\mathbb{R}^2$ is invariant under translations we can assume that $(x,y) = (0,0)$. Define the mean

$$m(r) := \frac{1}{2\pi r}\oint_{\partial B} u\,ds, \tag{5.56}$$

where $B := B((0,0),1)$. The integral above is a line integral that can be parametrized with $(x,y) = (r\cos t, r\sin r)$, i.e.

$$m(r) = \frac{1}{2\pi}\int_{0}^{2\pi} u(r\cos t, r\sin t)\,dt. \tag{5.57}$$

Notice that $m(0) = u(0,0)$ and

$$m'(r) = \frac{1}{2\pi}\int_{0}^{2\pi} (u_x\cos t + u_y\sin t)\,dt. \tag{5.58}$$

Now, we transform the above integral into a line integral over a vector field ∇u by noting that $dx = -\sin t\,dt$ and $dy = \cos t\,dt$ (since $x(t) = \cos t$ and $y(t) = \sin t$)

$$m'(r) = \frac{1}{2\pi}\oint_{\partial B} u_x\,dy - u_y\,dx. \tag{5.59}$$

Finally, we apply Green's theorem[19]

$$m'(r) = \frac{1}{2\pi} \iint_B (u_{xx} + u_{yy})\, dxdy, \tag{5.60}$$

to see that by the assumption $m'(r) = 0$. Therefore, for any $r > 0$ we have $m(r) = u(0,0)$. □

Knowing the mean-value property we can expect that since u is always some average it should present a decent regularity. Averages smear out irregularities. In fact, one can prove that harmonic functions are infinitely differentiable and even analytic. Moreover, for Poisson's equation $-\Delta u = f$ the regularity of u is determined by smoothness of f. The proof are more intricate and beyond the scope of these notes on applications.

An important implication of the mean-value property is the maximum principle which is somewhat similar to the analogous result for heat equation. It says that the non-constant harmonic function cannot attain its extrema in the interior of its domain.

Theorem 8 ((Strong) Maximum principle)**.** *Let* $D \subset \mathbb{R}^n$ *be a open connected set and* $u \in C^2(D)$ *with* $\Delta u = 0$*. Then, if* u *attains its maximum and minimum in the interior of* D *then it is constant.*

For now, we omit the proof. The maximum principle in a simple way leads to the uniqueness result.

Corolary 1 (Uniqueness)**.** *Let* $D \subset \mathbb{R}^n$ *be a bounded open set. Then, if* $\Delta u = 0$ *with* $u|_{\partial D} = 0$ *then* $u \equiv 0$ *on* D*.*

Proof. Since the extrema of u are attained at ∂D then $u \leq 0$ and $u \geq 0$ on D. Therefore $u \equiv 0$. □

5.1.3 Laplace's equation in a cylinder (optional)

Another very important application of separation of variables is a solution of the Laplace's equation on a cylinder. Various applications emerge in many fields spanning from electrostatics to astrophysics. Consider the following problem

$$\begin{cases} \Delta u = \dfrac{1}{r}(ru_r)_r + \dfrac{1}{r^2}u_{\theta\theta} + u_{zz} = 0, & 0 < r < a,\ 0 < \theta < 2\pi,\ 0 < z < H, \\ u(r,\theta,H) = f(r,\theta), & \text{(top)}, \\ u(a,\theta,z) = g(\theta,z), & \text{(side)}, \\ u(r,\theta,0) = h(r,\theta), & \text{(bottom)}, \end{cases} \tag{5.61}$$

where we have utilized cylindrical coordinates[20]. Notice the similarity with polar frame (5.27). As we will see, the term u_{zz} causes the main difference between these two. Separation of variables is initialized with an ansatz

$$u(r,\theta,z) = R(r)\Theta(\theta)Z(z). \tag{5.62}$$

[19] Recall that it states that $\oint_{\partial D}(P,Q)\cdot d\mathbf{r} = \iint_D (Q_x - P_y)dxdy$.

[20] Recall the form of the Laplacian.

When we plug the above into the PDE (5.61) we obtain (after dividing by $R\Theta Z$)

$$\frac{(rR')'}{rR} + \frac{\Theta''}{r^2\Theta} + \frac{Z''}{Z} = 0. \tag{5.63}$$

This time, the separation of variables is done in turns. First, we can split the z-term and conclude that

$$\frac{(rR')'}{rR} + \frac{\Theta''}{r^2\Theta} = -\frac{Z''}{Z} = -\lambda, \tag{5.64}$$

where the separation constant $\lambda \in \mathbb{R}$. We immediately obtain the equation for Z

$$Z'' - \lambda Z = 0, \tag{5.65}$$

which has a solution

$$Z(z) = \begin{cases} C_1 \sinh(\sqrt{\lambda}z) + C_2 \cosh(\sqrt{\lambda}z), & \lambda \geq 0, \\ C_1 \sin(\sqrt{-\lambda}z) + C_2 \cosh(\sqrt{-\lambda}z), & \lambda < 0. \end{cases} \tag{5.66}$$

where we prefer to use hyperbolic functions instead of exponentials. Returning to (5.63) we can separate the second time with a constant n

$$\frac{r^2R'' + rR + \lambda r^2}{R} = -\frac{\theta''}{\theta} = n, \tag{5.67}$$

from where the Θ equation is

$$\Theta'' + n\Theta = 0. \tag{5.68}$$

Since the solution of the above has to be 2π-periodic, we must have $n \in \mathbb{N}$ since then the solutions will be combinations of trigonometric functions

$$\Theta(\theta) = D_1 \cos(\sqrt{n}\theta) + D_2 \sin(\sqrt{n}\theta) \tag{5.69}$$

The ODE for radial coordinate is something new to us

$$r^2R'' + rR' + \left(\lambda r^2 - n^2\right) R = 0. \tag{5.70}$$

This second order variable coefficient ODE is called the *Bessel's equation*. Notice that we have two separation constants $\lambda > 0$ and $n \in \mathbb{N}$. Clearly, it reduces to the previously solved Euler's equation for $\lambda = 0$. For $n \neq 0$ the solution is not given in terms of elementary functions. However, by a substitution of a power series it can easily be shown that the general solution of the above is

$$R(r) = E_1 J_n(\sqrt{\lambda}r) + E_2 Y_n(\sqrt{\lambda}r), \tag{5.71}$$

where the *Bessel functions* of the *first* J_n and *second* Y_n kind are given by

$$J_n(z) = \sum_{j=0} \frac{(-1)^j}{j!(j+n)!}\left(\frac{z}{2}\right)^{2j+n}, \quad Y_n(z) = \lim_{\alpha \to n} \frac{J_n(z)\cos(\alpha z) - J_{-n}(z)}{\sin(\alpha z)}. \tag{5.72}$$

A plot of Bessel function is depicted on Fig. 16. Notice that Y_n are singular at the origin and thus we neglect them from our bounded solution. Both of these functions

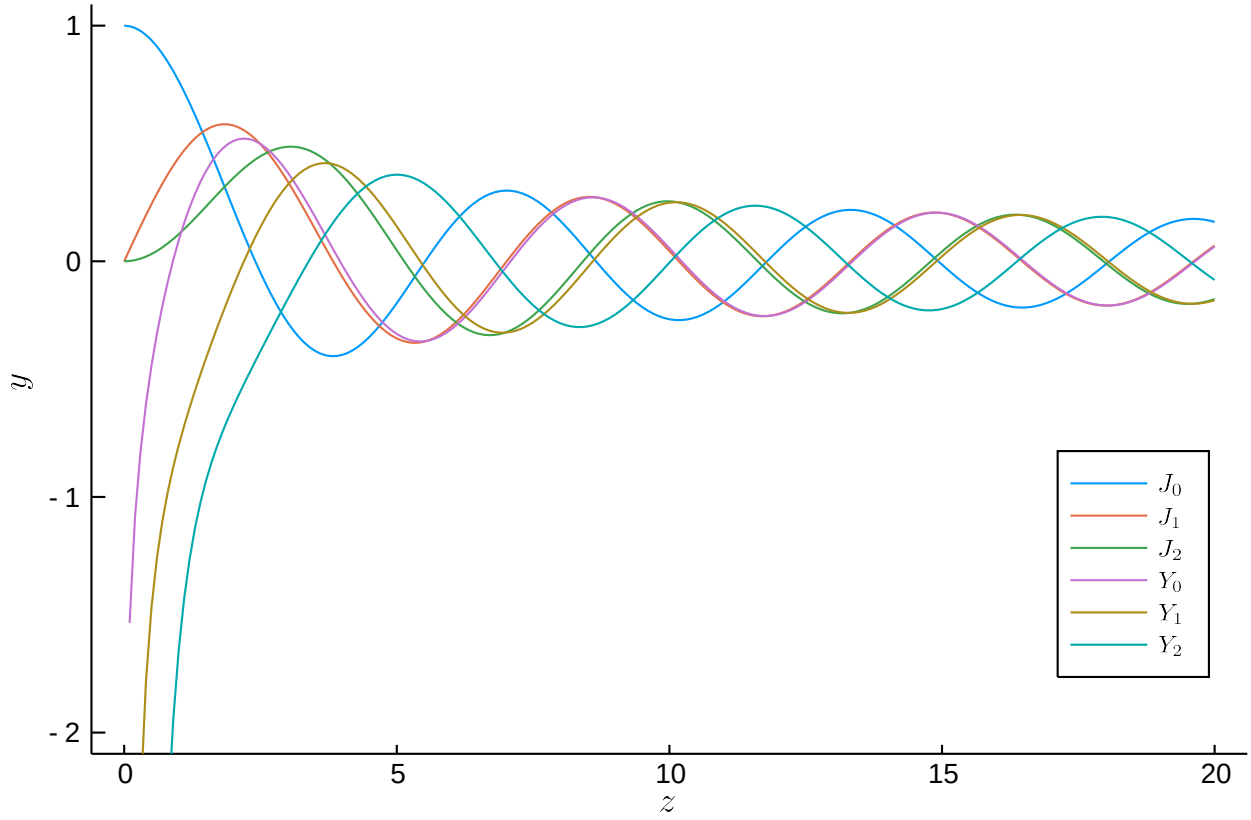

Figure 16: Bessel functions of the first J_n and second Y_n kind.

	$n=0$	$n=1$	$n=2$	$n=3$
$k=0$	2.4048	3.8317	5.1356	6.3802
$k=1$	5.5201	7.0156	8.4172	9.7610
$k=2$	8.6537	10.1735	11.6198	13.0152
$k=3$	11.7915	13.3237	14.7960	16.2235

Table 1: Zeros z_{nm} of Bessel functions.

oscillate and decay in amplitude. It can even be shown that that for large arguments they asymptotically are close to trigonometric functions with an amplitude decaying as $z^{-1/2}$.

Let us now return to the problem (5.61). Recall that due to linearity we can solve it in turns by splitting into three cases with exactly one nonhomogeneity each. To this end, start with $f = f(r,\theta)$ and $g = h \equiv 0$. The boundary conditions for our separated functions are then

$$Z(0) = 0, \quad R(0) \text{ - bounded}, \quad R(a) = 0. \tag{5.73}$$

Therefore, from (5.66) we have $C_2 = 0$ while from (5.71) we get $E_2 = 0$ and

$$0 = R(a) = J_n\left(\sqrt{\lambda}a\right). \tag{5.74}$$

Whence, $\sqrt{\lambda}a$ has to be a zero of J_n (compare the analogous situation for one dimension where here we would have some trigonometric function). Zeros of Bessel's function cannot be expressed in an elementary closed form. However, it can be proved that there are countably many of them and we usually denote them by z_{nm} meaning m-th zero of n-th Bessel function (see Tab. 5.1.3).

It follows that

$$\lambda = \left(\frac{z_{nm}}{a}\right)^2, \quad m \geq 1, \tag{5.75}$$

from which

$$R(r) = E_1 J_n\left(\frac{z_{nm}}{a}r\right), \quad n \in \mathbb{N}. \tag{5.76}$$

We can now make a superposition of all we have found. Summing up (5.66), (5.69), and (5.76) we obtain

$$u(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{z_{nm}}{a}r\right)\sinh\left(\frac{z_{nm}}{a}z\right)\left(a_{nm}\cos(\sqrt{n}\theta) + b_{nm}\sin(\sqrt{n}\theta)\right), \tag{5.77}$$

where we renamed the constants of integration. The coefficients of the above expansion a_{nm} and b_{nm} can be found with the use of orthogonality. First, by the boundary condition we have

$$f(r,\theta) = u(r,\theta,H) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n\left(\frac{z_{nm}}{a}r\right)\sinh\left(\frac{z_{nm}}{a}H\right)\left(a_{nm}\cos(\sqrt{n}\theta) + b_{nm}\sin(\sqrt{n}\theta)\right), / \tag{5.78}$$

Then, by the orthogonality of trigonometric functions we have

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)\,d\theta &= \sum_{m=1}^{\infty} a_{0m} J_n\left(\frac{z_{nm}}{a}r\right)\sinh\left(\frac{z_{nm}}{a}H\right),\\
\frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\cos(n\theta)\,d\theta &= \sum_{m=1}^{\infty} a_{nm} J_n\left(\frac{z_{nm}}{a}r\right)\sinh\left(\frac{z_{nm}}{a}H\right),\\
\frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\sin(n\theta)\,d\theta &= \sum_{m=1}^{\infty} b_{nm} J_n\left(\frac{z_{nm}}{a}r\right)\sinh\left(\frac{z_{nm}}{a}H\right),
\end{aligned} \tag{5.79}$$

for $n \geq 1$. Now, we would like to extract a_{nm} and b_{nm} from the sums. Fortunately, Bessel functions are also orthogonal with respect to a given scalar product, i.e.

$$\int_0^a J_n\left(\frac{z_{nm}}{a}r\right) J_n\left(\frac{z_{nk}}{a}r\right) r\,dr = \frac{a^2}{2}\left(J_{n+1}\left(\frac{z_{nk}}{a}r\right)\right)^2 \delta_{mk}, \tag{5.80}$$

where δ_{mk} is the Kronecker delta. When we apply this form of orthogonality we can calculate

$$\begin{aligned}
a_{0m} &= \frac{\frac{1}{2\pi}\int_0^{2\pi}\int_0^a f(r,\theta) J_0\left(\frac{z_{0k}}{a}r\right) r\,dr\,d\theta}{\sinh\left(\frac{z_{0k}}{a}H\right)\left(J_1\left(\frac{z_{0k}}{a}r\right)\right)^2},\\
a_{nm} &= \frac{\frac{1}{2\pi}\int_0^{2\pi}\int_0^a f(r,\theta) J_n\left(\frac{z_{nk}}{a}r\right)\cos(n\theta) r\,dr\,d\theta}{\sinh\left(\frac{z_{nk}}{a}H\right)\left(J_{n+1}\left(\frac{z_{nk}}{a}r\right)\right)^2},\\
b_{nm} &= \frac{\frac{1}{2\pi}\int_0^{2\pi}\int_0^a f(r,\theta) J_n\left(\frac{z_{nk}}{a}r\right)\sin(n\theta) r\,dr\,d\theta}{\sinh\left(\frac{z_{nk}}{a}H\right)\left(J_{n+1}\left(\frac{z_{nk}}{a}r\right)\right)^2},
\end{aligned} \tag{5.81}$$

where $n \geq 1$. Notice that the integrals in the numerators are calculated over the top of the cylinder, that is a circle. Usually, an orthogonal expansion with respect to Bessel functions is called *Fourier-Bessel series*. Having the above formulas we have solved the boundary value problem and we can proceed further.

The case with the conditions at the bottom, i.e. $h = h(r,\theta)$ and $f = g \equiv 0$ is completely analogous and we omit the details. For the lateral case $g = g(\theta, z)$ and $f = h \equiv 0$ the situation is a little bit different. The boundary conditions are now

$$Z(0) = Z(H) = 0, \quad R(0) \text{ - bounded.} \tag{5.82}$$

Therefore, it is convenient to start with Z-equation (5.65) which yields

$$Z(z) = C_1 \sin\left(\frac{m\pi}{H}z\right), \quad \text{with} \quad \lambda = -\left(\frac{m\pi}{H}\right)^2. \tag{5.83}$$

Now, Bessel equation (5.70) becomes

$$r^2 R'' + rR' - \left(\frac{m\pi}{H}r^2 + n^2\right) R = 0, \tag{5.84}$$

which differs from the previous by the minus sign. The solutions of the above are called *modified Bessel functions* of the first I_n and second K_n kind and can be found by an imaginary substitution $\rho = im\pi/H\, r$. They are the same for Bessel functions as hyperbolic are for trigonometric functions. Figure 17 depicts the behaviour of these functions. All of the second kind functions are singular at the origin and we exclude them from our solution. We leave the remaining details for the reader.

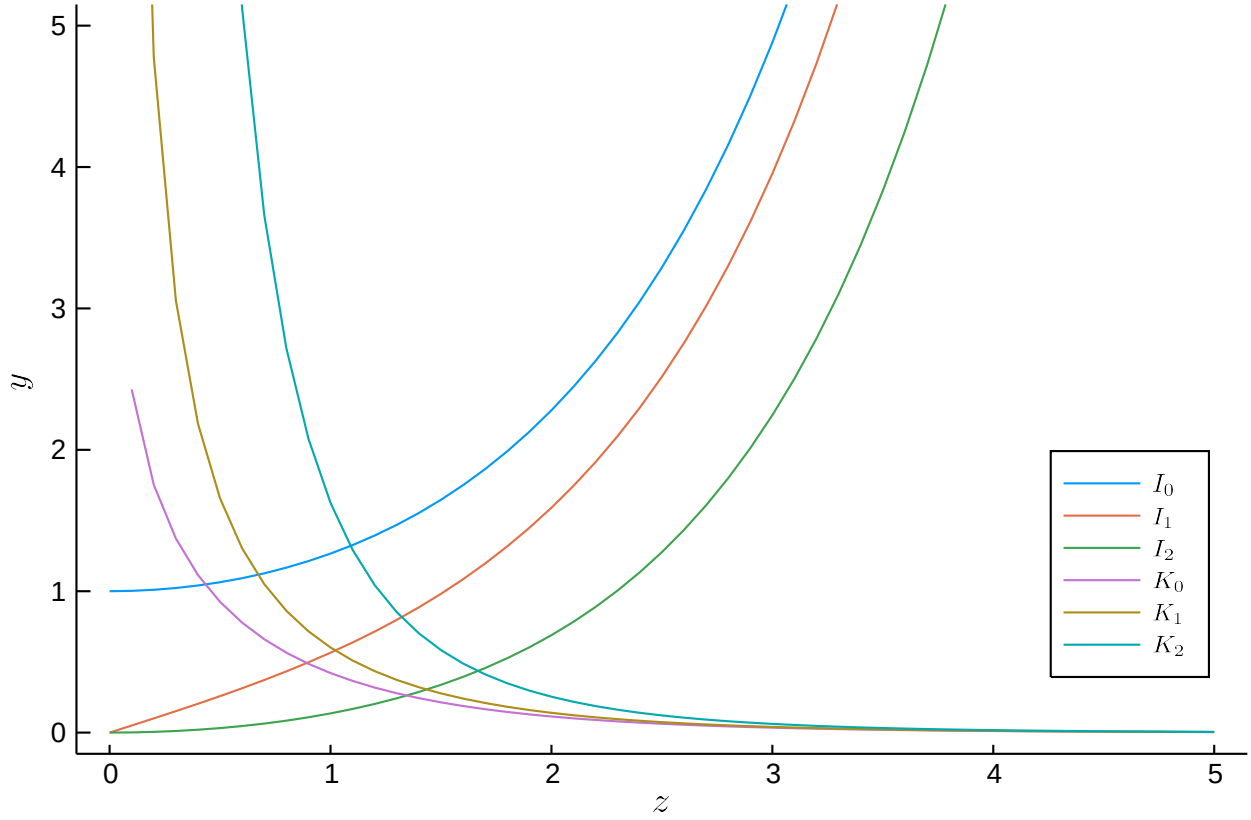

Figure 17: Modified Bessel functions of the first I_n and second K_n kind.

5.1.4 Laplace's equation in a ball (optional)

In many applications we are interested in solving Laplace's equation on a ball. Notable examples can be found in electrostatics and quantum mechanics but also in geophysics (seismology), geodesy, astrophysics, and geophysical fluid dynamics - these fields that are concerned in solving various problems on (almost) spherical planets. We will solve the following problem

$$\begin{cases} \Delta u = \frac{1}{r^2}\left(r^2 u_r\right)_r + \frac{1}{r^2\sin^2\phi}u_{\theta\theta} + \frac{1}{r^2\sin\phi}(\sin\phi\, u_\phi)_\phi = 0, \quad 0 < r < 1, \quad 0 < \phi < 2\pi, \quad 0 < \theta < \pi, \\ u(1,\phi,\theta) = f(\phi,\theta). \end{cases} \tag{5.85}$$

where we adopted spherical coordinates from the very beginning. We immediately separate the variables

$$u(r,\phi,\theta) = R(r)Y(\phi,\theta). \tag{5.86}$$

Using this ansatz and conducting a standard calculations we obtain two ODEs

$$\left(r^2R'\right)' = \lambda R, \quad \frac{1}{\sin\theta}\left(\sin\theta\, Y_\theta\right)_\theta + \frac{1}{\sin^2\theta}Y_{\phi\phi} = -\lambda Y, \tag{5.87}$$

where λ is the separation constant. The R-equation can immediately be written as

$$r^2R'' + 2rR' - \lambda R = 0, \quad R(0) \text{ - bounded}, \tag{5.88}$$

in which we recognize the familiar Euler's equation. On the other hand, by separating the second time $Y(\phi,\theta) = \Phi(\phi)\Theta(\theta)$ we can simplify the second equation

$$\Phi'' + m^2\Phi = 0, \quad \frac{1}{\sin\theta}\left(\sin\theta\,\Theta'\right)' + \left(\lambda - \frac{m^2}{\sin^2\theta}\right)\Theta = 0. \tag{5.89}$$

The first of the above has periodic solutions for $m \in \mathbb{N}$ (here we prefer exponentials)

$$\Phi(\phi) = D_1 e^{im\phi} + D_2 e^{-im\phi}, \tag{5.90}$$

while the second ODE is new to us. The latter can be further simplified with a substitution $x = \cos\theta$ and $P(x) = \Theta(\theta)$ which gives $\sin^2\theta = 1 - x^2$ and

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = \sin\theta\frac{d}{dx} = \sqrt{1-x^2}\frac{d}{dx}. \tag{5.91}$$

Notice that $-1 \le x \le 1$. Hence,

$$\left((1-x^2)P'\right)' + \left(\lambda - \frac{m^2}{1-x^2}\right)P = 0, \tag{5.92}$$

which is the *Legendre equation*. From the theory of special functions we know that the above has nonsingular solutions only when $\lambda = l(l+1)$, $0 \le |m| \le l$, and $l \in \mathbb{N}$. The solution of the above are in that case called *associated Legendre polynomials* and are denoted by P_l^m. Some first examples are depicted on Fig. 18. The product $Y_l^m(\phi,\theta) = P_l^m(\cos\theta)e^{im\phi}$ is called *spherical harmonics* and has many important applications when

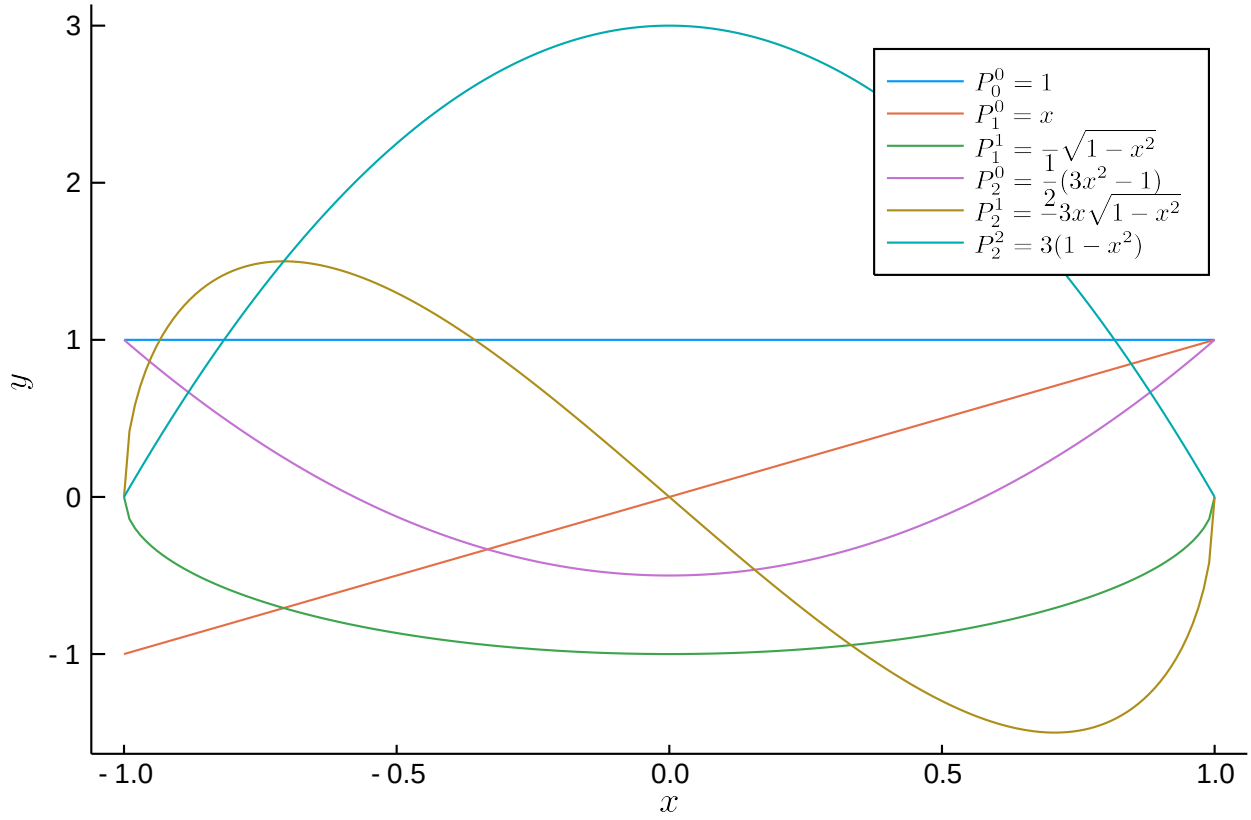

Figure 18: Some first associated Legendre polynomials.

solving various problems on a sphere. In quantum mechanics one can show that constants l and m are associated with momentum of the particle.

Finally, returning to the R-equation we obtain $R(r) = C_1 r^l + C_2 r^{-(l+1)}$ as a solution to Euler's equation. Due to regularity we have to take $C_2 = 0$ and hence

$$u(r, \phi, \theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{ml} r^l P_l^m(\cos\theta) e^{im\phi}. \tag{5.93}$$

The boundary condition gives us

$$f(\phi, \theta) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} c_{ml} P_l^m(\cos\theta) e^{im\phi}, \tag{5.94}$$

and in order to determine a_{ml} and b_{ml} we have to utilize orthogonality. Fortunately, associated Legendre polynomials are orthogonal with respect to a specific scalar product on a sphere, i.e.

$$\int_0^{2\pi} \int_0^{\pi} Y_l^m(\phi, \theta) Y_k^n(\phi, \theta) \sin\theta \, d\theta d\phi = \left(\int_0^{2\pi} \int_0^{\pi} Y_l^m(\phi, \theta)^2 \sin\theta \, d\theta d\phi \right) \delta_{mn} \delta_{lk}, \tag{5.95}$$

where δ is the Kronecker delta. Using this we immediately can find the formula for the coefficients

$$c_{ml} = \frac{\int_0^{2\pi} \int_0^{\pi} f(\phi, \theta) \sin\theta \, d\theta d\phi}{\int_0^{2\pi} \int_0^{\pi} Y_l^m(\phi, \theta)^2 \sin\theta \, d\theta d\phi}, \tag{5.96}$$

which is analogous to the Fourier series.

Example. (*Schrödinger equation for the hydrogen atom*) The energy spectrum of hydrogen atom has a profound meaning for both theoretical and experimental physics. When we consider the non-relativistic case the wave function ψ for the electron orbiting a stationary proton satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2\mu} \Delta\psi + V\psi = E\psi, \tag{5.97}$$

where $\hbar$ is the Planck constant, μ is the reduced of the electron, $E < 0$ is its energy in a bounded state, and the Coulomb potential V has the form

$$V(r) = -\frac{e^2}{r}, \tag{5.98}$$

where r is the distance between electron. Our goal is to find the energy state E and the corresponding wave function. After separating the variables in spherical coordinates

$$\psi(r, \phi, \theta) = R(r) Y(\phi, \theta), \tag{5.99}$$

in essentially the same way as before, we arrive at the radial and angular ODEs

$$\begin{aligned} (r^2 R')' - \left(\frac{2\mu r^2}{\hbar^2} (V(r) - E) + l(l+1) \right) R = 0, \\ \frac{1}{\sin\theta} (\sin\theta Y_\theta)_\theta + \frac{1}{\sin^2\theta} Y_{\phi\phi} + l(l+1) Y = 0, \end{aligned} \tag{5.100}$$

, where, having learned about spherical harmonics, we set the separation constant to be equal to $l(l+1)$ with $l \in \mathbb{N}$. The solution of the angular equation is of course

$$Y(\phi, \theta) = CP_l^m(\cos\theta)e^{im\phi}, \quad -l \leq m \leq l, \tag{5.101}$$

for the second separation constant $m \in \mathbb{Z}$ and the associated Legendre polynomial P_l^m for l being the angular momentum while m the magnetic momentum quantum numbers. Observe that with the radial equation some new challenges arise! Having a singular Coulomb potential complicates the problem and makes it much more interesting. As with many other variable coefficient second order ODEs our radial equation has already been investigated in the past by many mathematicians and physicists. We just have to determine the very kind of it[21].

We start with a substitution $y(r) = rR(r)$ which implies

$$(r^2R')' = ry'', \tag{5.102}$$

which eliminates the first derivative. Next, we substitute for the independent variable

$$x = r\sqrt{-\frac{8\mu E}{\hbar^2}}, \tag{5.103}$$

where we remember that in bounded state the electron has negative energy. After these, our radial ODE becomes

$$y'' + \left(-\frac{1}{4} + \frac{j+l+1}{x} - \frac{l(l+1)}{x^2}\right)y = 0, \quad j = \sqrt{-\frac{\mu e^4}{2\hbar^2 E}} - l - 1. \tag{5.104}$$

From here, we have the last step to reduce the above equation into a standard form. Notice that for $x \to \infty$ this ODE reduces to $y'' = y/4$ and hence has exponential solutions. It is therefore convenient to substitute

$$y(x) = e^{-x/2}x^{l+1}L(x), \tag{5.105}$$

which after some cumbersome algebra yields

$$xL'' + (1 - x + 2l + 1)L' + jL = 0, \tag{5.106}$$

which is the *associated Laguerre equation* with solutions being *associated Laguerre polynomials* L_j^{2l+1}. In general the solutions of the above are

$$y_j^l(x) = Dx^{l+1}e^{-\frac{x}{2}}L_j^{2l+1}(x). \tag{5.107}$$

Notice how often special functions arise in important physical problems. It is just a matter of simple exercise to gather all solutions R and Y to form a Fourier(-Laguerre) series (everything here, of course, is orthogonal by the Sturm-Liouville theory). We will omit the details and only comment on the important physical consequences of the various separation constants appearing in our calculations.

[21] Which is very difficult if you do not know what had already been done before.

Returning to the definition of j we have that

$$n := j + l + 1 = \sqrt{-\frac{\mu e^4}{2\hbar^2 E}}, \tag{5.108}$$

where we have defined the *principal quantum number* n. We thus see that the energy of the n-th state is

$$E_n = -\frac{13.6eV}{n^2}, \tag{5.109}$$

after plugging in the units. The ground state of the electron, when $n = 1$, is thus -13.6 eV. Note also that we were able to solve the difficult problem in an exact way! This is clearly a rare case and thus worth of careful examination. □

5.1.5 Eigenfunction expansions (optional)

So far we have considered many examples that have much in common. They all share orthogonality and we would like to explore this topics to unify and formalize the theory. Assume that would like to solve the Poisson's equation with homogeneous Dirichlet condition on some $\Omega \subseteq \mathbb{R}^n$, that is

$$\begin{cases} -\Delta u = f, \quad x \in \Omega \subseteq \mathbb{R}^d, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{5.110}$$

Suppose we can find a family of functions $\{\Phi_n\}_n$ and corresponding constants λ_n satisfying

$$-\Delta\Phi_n = \lambda_n \Phi_n, \tag{5.111}$$

that is, we can solve the eigenfunction problem for a (negative) Laplacian. The above is frequently met in many circumstances and is called the *Helmholtz equation*. We have seen above that in some simple domains Helmholtz equation can be solved by separation of variables. The general theory of PDEs on Hilbert spaces tells us then that $\{\Phi_n\}_n$ is orthonormal and, most importantly, constitutes a basis in Sobolev space $H^2(\Omega)$. We can then expand

$$f(x) = \sum_{n=0}^{\infty} f_n \Phi_n(x), \quad \text{where} \quad f_n = \int_\Omega f(x)\Phi_n(x)dx. \tag{5.112}$$

Now, the solution of (5.110) can also be expanded

$$u(x) = \sum_{n=0}^{\infty} u_n \Phi_n(x), \tag{5.113}$$

where u_n are to be found. Plugging the above into Poisson's equation we have

$$-\Delta \sum_{n=0}^{\infty} u_n \Phi_n(x) = \sum_{n=0}^{\infty} f_n \Phi_n(x). \tag{5.114}$$

Now, the theory tells us that we can differentiate the above series term by term in the L^2 sense

$$\sum_{n=0}^{\infty}(\lambda_n c_n - f_n)\Phi_n(x) = 0, \tag{5.115}$$

where we have used the fact that Φ_n is a eigenfunction of $-\Delta$. Since $\{phi_n\}$ is a basis it follows that

$$u_n = \frac{f_n}{\lambda_n}, \tag{5.116}$$

and our problem is solved. Of course, in practice this works only when we can solve Helmholtz equation. However, apart from separation of variables and numerical analysis, we have also the method of Green's function described further.

5.2 Green's function

We have already encountered the Green's function when investigating heat equation. We will now see that it is also a very useful tool for solving elliptic equations. As a matter of fact, it was with the Poisson's equation where George Green introduced his function. Our approach will be based on Dirac delta distribution, which we will define in a fairly heuristic way and postpone a more detailed analysis to the further section (optional).

Definition 5. Dirac delta distribution *is a mapping* δ*, which satisfies the following*

1. *Concentration at zero,*
$$\delta(x) = 0, \quad x \neq 0. \tag{5.117}$$
2. *Normalization,*
$$\int_{-\infty}^{\infty}\delta(x)dx = 1. \tag{5.118}$$

We can immediately show the filtering property

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = \int_{-\infty}^{\infty}(f(x)-f(x_0))\,\delta(x-x_0)dx + \int_{-\infty}^{\infty} f(x_0)\delta(x-x_0)dx. \tag{5.119}$$

From the definition of Dirac delta the last integral is equal to $f(x_0)$ since we can move this constant value of the function in front of the integral. For the other integral we have

$$\int_{-\infty}^{\infty}(f(x)-f(x_0))\,\delta(x-x_0)dx = \lim_{\epsilon\to 0}\int_{\mathbb{R}-(x_0-\epsilon,x_0+\epsilon)}(f(x)-f(x_0))\,\delta(x-x_0)dx = 0, \tag{5.120}$$

from the first property of the Dirac delta. Therefore,

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0), \tag{5.121}$$

what we had to show.

A multidimensional Dirac delta is defined as a product of one dimensional distributions

$$\delta(\mathbf{x}) = \delta(x_1)\delta(x_2)...\delta(x_n), \quad \mathbf{x} = (x_1, x_2, ..., x_n). \tag{5.122}$$

Apart from that we also will need the following Green's identities[22]

$$\iiint_D (u\Delta v - v\Delta u)\, d\mathbf{x} = \iint_{\partial D} (u\nabla v - v\nabla u)\cdot \mathbf{n}\, dS, \tag{5.123}$$

and

$$\iint_D (u\Delta v - v\Delta u)\, d\mathbf{x} = \oint_{\partial D} (u\nabla v - v\nabla u)\cdot \mathbf{n}\, dS. \tag{5.124}$$

We will now give a very general definition of the Green's function that, as we will see, is consistent with our intuition and previous results

Definition 6. Green's function $G = G(\boldsymbol{x}, \boldsymbol{x}_0)$ *for an elliptic equation* $Lu = f$ *with Dirichlet boundary condition* $u|_{\partial D} = g$ *is a function satisfying*

$$LG(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0), \tag{5.125}$$

where the differential operator L *acts on the* $\boldsymbol{x}$ *variable and* $G(\cdot, \boldsymbol{x}_0)|_{\partial D} = 0$ *for any* $\boldsymbol{x}_0$.

Of course, we will only deal with the case of $L = \Delta$, which gives the Poisson's equation. Albeit being a particular choice, it does not cloud the generality of reasoning. Thinking in terms of potential, the Green's function is the solution of Poisson's equation where mass (or electric charge) is concentrated at a single point $\mathbf{x}_0$. If we assume that the boundary condition is homogeneous $g \equiv 0$, then we can think that by adding (integrating over) up such point charges we can obtain a solution of our equation in the form

$$u(\mathbf{x}) := \iiint_D G(\mathbf{x}, \mathbf{x}_0) f(x_0)\, d\mathbf{x}_0. \tag{5.126}$$

Then, we can formally obtain

$$Lu(\mathbf{x}) = \iiint_D LG(\mathbf{x}, \mathbf{x}_0) f(x_0)\, d\mathbf{x}_0 = \iiint_D \delta(\mathbf{x} - \mathbf{x}_0) f(x_0)\, d\mathbf{x}_0 = f(\mathbf{x}). \tag{5.127}$$

Whence, u as defined above is the solution to the elliptical equation $Lu = f$. This is the main reason for introducing the Green's function. The above calculation is purely formal, however, with the help of Green's identities, we can justify this result rigorously. Let us take $L = \Delta$ and in the formula (5.123) put $v(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_0)$

$$\iiint_D (u\Delta G - G\Delta u)\, d\mathbf{x} = \iint_{\partial D} (u\nabla G - G\nabla u)\cdot \mathbf{n}\, dS. \tag{5.128}$$

[22] Please review Calculus 3.

From the definition of Green's function, Poisson's equation, and boundary condition we obtain

$$\iiint_D (u\delta(\mathbf{x}-\mathbf{x}_0) - G\Delta u)\,d\mathbf{x} = \iint_{\partial D} g\nabla G\cdot\mathbf{n}\,dS. \tag{5.129}$$

We have thus proved the following theorem[23]

Theorem 9. *If $G = G(\boldsymbol{x},\boldsymbol{x}_0)$ is a Green's function for the elliptic equation $Lu = f$ with Dirichlet boundary condition $u|_{\partial D} = g$, then its solution can be written as*

$$u(\boldsymbol{x}) = \iiint_D G(\boldsymbol{x}_0,\boldsymbol{x})f(\boldsymbol{x}_0)\,d\boldsymbol{x}_0 + \iint_{\partial D} g\nabla_{\boldsymbol{x}_0}G\cdot\boldsymbol{n}\,dS, \tag{5.130}$$

where $\nabla_{\boldsymbol{x}_0}$ is a gradient with respect to the $\boldsymbol{x}_0$ variable.

Therefore, we have solved the Poisson'a equation with a Dirichlet boundary condition. Of course, in order for this formula to be any useful we need to know the Green's function for the domain D , which is usually not easy to obtain. The analogous formula for two dimensional problems can be devised in a similar way.

5.2.1 The whole space $\mathbb{R}^3$

We will now find the Green's function for several specific domains. We will start with the whole space $D = \mathbb{R}^3$ (and leave the two-dimensional case to problems). Since no direction in $\mathbb{R}^3$ is special (the space is isotropic), we expect that the Green's function depends only on the distance from $\mathbf{x}_0$. Let us put $r := |\mathbf{x}-\mathbf{x}_0|$. Then

$$G(\mathbf{x},\mathbf{x}_0) = G(r), \tag{5.131}$$

and from the definition of Green's function

$$\frac{1}{r^2}\left(r^2G'\right)' = \delta(r). \tag{5.132}$$

Therefore, for $r \neq 0$ we have $\frac{1}{r}(rG')' = 0$, which has a solution

$$G(r) = \frac{C}{r} + E, \quad r \neq 0. \tag{5.133}$$

Constants C and E must be found from the second condition for Dirac delta, that is, its normalization. To do this, we proceed by exactly the same way as above in the example with gravitational potential. Let us select a small ball $B(\epsilon)$ with radius of ϵ and a centre at $\mathbf{x}_0$. We have

$$1 = \iiint_{\mathbb{R}^3}\delta(\mathbf{x}-\mathbf{x}_0)\,d\mathbf{x} = \iiint_{B(\epsilon)}\Delta G(\mathbf{x},\mathbf{x}_0)\,d\mathbf{x} = \iiint_{B(\epsilon)}\nabla\cdot\nabla G(\mathbf{x},\mathbf{x}_0)\,d\mathbf{x} = \iint_{\partial B(\epsilon)}\nabla G(\mathbf{x},\mathbf{x}_0)\cdot\mathbf{n}dS. \tag{5.134}$$

[23]Notice that we have *interchanged* the variable names $\mathbf{x} \leftrightarrow \mathbf{x}_0$ in order to write the formula more neatly. This is only an aesthetic procedure.

Because the Green's function depends only on the distance between points $\mathbf{x}$ and $\mathbf{x}_0$ it follows that $\nabla G(\mathbf{x}, \mathbf{x}_0) \cdot \mathbf{n} = G'(r)$ and we have

$$1 = 4\pi\epsilon^2 G'(\epsilon), \tag{5.135}$$

which is true for every $\epsilon > 0$. Hence, an appropriate normalization condition is

$$\lim_{\epsilon \to 0} \epsilon^2 G(\epsilon) = \frac{1}{4\pi}. \tag{5.136}$$

From which we immediately find $C = -\frac{1}{4\pi}$ and $E = 0$. Finally,

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}_0|}. \tag{5.137}$$

This is of course the gravitational potential we have found earlier. Recall that we have devised the integral representation of the solution to Poisson's equation (5.130) assuming boundedness of the domain D. Otherwise the surface integral would be meaningless. To find a version for the whole space, we need to consider a growing family of balls that fill the entire space in the limit. Then the solution of $\Delta u = f$ with $u \to 0$ when $|\mathbf{x}| \to \infty$ is

$$u(\mathbf{x}) = -\frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{1}{|\mathbf{x} - \mathbf{x}_0|} f(\mathbf{x}_0) \, d\mathbf{x}_0, \tag{5.138}$$

which is identical to our previous physical result regarding the potential of continuous mass distribution. However, for the well-posedness of the above formula, we must impose an additional decay condition

$$\lim_{r \to \infty} (u + r u_r) = 0, \tag{5.139}$$

for are functions that solve the Poisson's equation throughout the space, which do not satisfy the above condition.

5.2.2 A unit ball - method of reflections

Finding the Green's function for bounded domains is not easy nor straightforward, however we can use the results that we obtained for the whole space. Usually the most difficult task in devising the form of Green's function is to ascertain that we can produce a sufficiently strong singularity, i.e. $\Delta G = \delta$. Because we have found a function that has just the right type of the behaviour at $\mathbf{x}_0$, i.e. $\Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0)$, we can slightly modify it to ensure it vanishes at the boundary of the domain. Let us thus put

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}_0|} + v(\mathbf{x}, \mathbf{x}_0), \tag{5.140}$$

where v is the sought correction representing the effect of the boundary. We, of course, would like to have

$$\Delta v = 0, \quad v|_{\partial D} = \frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{x}_0|}, \tag{5.141}$$

therefore v is the solution of the Laplace's equation with nonhomogeneous Dirichlet condition. Finding v is much simpler than searching for the function producing the Diract Delta after application of the Laplacian. For simple areas this can be accomplished by separation of variables. Here, we will demonstrate another technique called *method of reflections*[24] applied to the ball. Note that the Laplace's equation has already been solved for the two-dimensional circle and a ball with the use of separation of variables. The results were enlightening however, a little complicated. It is therefore useful to find some simpler representation of the solution.

Consider a 3-ball of radius 1. We would like to find such a function G, that $\Delta G(\mathbf{x}, \mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0)$ and $G(\mathbf{x}, \mathbf{x}_0) = 0$ for $|\mathbf{x}| = 1$ and arbitrary $\mathbf{x}_0$. Therefore, we have to require

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x} - \mathbf{x}_0|} + v(\mathbf{x}, \mathbf{x}_0), \tag{5.142}$$

where v is a solution of Laplace's equation. Recall that the Green's function is the potential at $\mathbf{x}$ due to the unit charge placed at $\mathbf{x}_0$. Let us try to select v so that the potential vanishes precisely on the sphere. We put another unit (negative) charge at some point $\mathbf{y}_0$ so that for $|\mathbf{x}| = 1$ the Green's function $G(\mathbf{x}, \mathbf{x}_0) = 0$. Of course, $\mathbf{y}_0$ should depend on $\texttt{textbf}x_0$ and hence, we have to find this dependence. Since we have put a negative charge we have

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x} - \mathbf{x}_0|} + \frac{1}{4\pi}\frac{C}{|\mathbf{x} - \mathbf{y}_0|}, \tag{5.143}$$

where C is a constant denoting the strength of this added charge. We will find it later. Since $G = 0$ on $|\mathbf{x}| = 1$ we have to choose $\mathbf{y}_0$ in order to satisfy

$$|\mathbf{x} - \mathbf{x}_0| = \frac{1}{C}|\mathbf{x} - \mathbf{y}_0|, \tag{5.144}$$

for $|x| = 1$. Let ϕ be the angle between $\mathbf{x}$ and $\mathbf{x}_0$. Note that this is the same angle as the one subtended between $\mathbf{x}$ and $\mathbf{y}_0$. We have[25]

$$|\mathbf{x} - \mathbf{x}_0|^2 = |\mathbf{x}|^2 + |\mathbf{x}_0|^2 - 2|\mathbf{x}||\mathbf{x}_0|\cos\phi, \tag{5.145}$$

and similarly

$$|\mathbf{x} - \mathbf{y}_0|^2 = |\mathbf{x}|^2 + |\mathbf{y}_0|^2 - 2|\mathbf{x}||\mathbf{y}_0|\cos\phi. \tag{5.146}$$

In order for (5.144) to hold we need

$$1 + |\mathbf{x}_0|^2 - 2|\mathbf{x}_0|\cos\phi = \frac{1}{C^2}\left(1 + |\mathbf{y}_0|^2 - 2|\mathbf{y}_0|\cos\phi\right). \tag{5.147}$$

Let us see whether there exists a point $\mathbf{y}_0$ which lies on the same line as $\mathbf{x}_0$ and 0, that is to say we propose the *ansatz* $\mathbf{y}_0 = \alpha\mathbf{x}_0$

$$1 + |\mathbf{x}_0|^2 - 2|\mathbf{x}_0|\cos\phi = \frac{1}{C^2}\left(1 + \alpha^2|\mathbf{x}_0|^2 - 2\alpha|\mathbf{x}_0|\cos\phi\right). \tag{5.148}$$

[24] Actually, we have used that method without even knowing when we investigated heat equation on the half-line.

[25] This is the elementary cosine theorem.

In order the above be satisfied for every angle ϕ (arbitrariness of $\mathbf{x}$ and $\mathbf{x}_0$) we must have

$$1 + |\mathbf{x}_0|^2 = \frac{1}{C^2}\left(1 + \alpha^2|\mathbf{x}_0|^2\right), \quad 1 = \frac{\alpha}{C^2}, \tag{5.149}$$

which gives us $\alpha = C^2$ and hence, $C = \frac{1}{|\mathbf{x}_0|}$. Therefore,

$$\mathbf{y}_0 = \frac{\mathbf{x}_0}{|\mathbf{x}_0|^2}, \tag{5.150}$$

in other words $|\mathbf{x}_0||\mathbf{y}_0| = 1$. You may recall from elementary geometry that in situation, $\mathbf{x}_0$ and $\mathbf{y}_0$ are called *conjugate with respect to an inversion in a sphere*. We thus have

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x} - \mathbf{x}_0|} + \frac{1}{4\pi}\frac{1}{|\mathbf{x}_0|}\frac{1}{|\mathbf{x} - \frac{\mathbf{x}_0}{|\mathbf{x}_0|^2}|}. \tag{5.151}$$

This is the famous Green's function for a ball. Note that, similarly as with the whole space $\mathbb{R}^3$, the function G is symmetric with respect to $\mathbf{x}$ and $\mathbf{x}_0$[26]. Thanks to our result we can readily write the solution of the Poisson's equation on a sphere

$$\Delta u = f, \quad u|_{\partial D} = g. \tag{5.152}$$

According to the fundamental integral representation formula (5.130) the solution is a integral over the Green's function. In particular, for homogeneous boundary condition, $g \equiv 0$, we have

$$u(\mathbf{x}) = -\frac{1}{4\pi}\int_B \left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|} - \frac{1}{|\mathbf{x}_0|}\frac{1}{|\mathbf{x} - \frac{\mathbf{x}_0}{|\mathbf{x}_0|^2}|}\right) f(\mathbf{x}_0)\, d\mathbf{x}_0. \tag{5.153}$$

Among other regular regions in which the method of reflections work is the half-space, rectangle (cube), cylinder, etc. These examples are extremely important in electronics where many RLC circuit elements have regular shapes. Then, solving Poisson's equation can help finding the electrostatic potential and hence, to design circuits and devices.

[26] This is a deeper result that will be proved during the tutorial.

6 Distributions

Now we move to a more advanced topic of distribution theory. This section is optional however, highly recommended in order to really understand Green's function which is a central object in parabolic and elliptic PDEs.

6.1 Motivation

Both in theory and applications it is very useful to be able to deal with objects which are more general than just functions. We have already seen that the heat kernel became more singular at $x = 0$ when $t \to 0^+$. At the same time, its integral has always been constant. The limit of such an expression is called the *Dirac delta distribution* or, with an abuse of nomenclature, *Dirac function* (it is not a function!). Physically, Dirac delta is a concentrated point mass or charge (why?).

Sometimes, Dirac delta is (incorrectly) defined as a *function* δ having the properties: $\delta(x) = 0$ for $x \neq 0$ and $\int \delta(x)dx = 1$. Of course, such a function does not exists since an integral of function equal to 0 almost everywhere is also equal to 0. In other places one can meet another even more risky "definition" which puts $\delta(0) = \infty$.

Aforementioned definitions are still in use in modern science and technology and show that it might be difficult to rigorously define what really Dirac delta is. Historically, ideas associated with "generalized functions" appeared in the second half of the nineteenth century. Later, they were formally used by Heaviside and Dirac. A proper mathematical treatment of these matters was given only in fifties of the last century by Laurent Schwartz. We will investigate how this was done.

The heat kernel is not the only family of functions that can approximate Dirac delta. At least when it comes to the so-called filtration property which sifts the value of a function from inside of the integral.

Theorem 10 (Approximate idenity)**.** *Let f be a positive integrable function such as*

$$\int_{\mathbb{R}} f(x)dx = 1. \tag{6.1}$$

Define $f_n(x) = \frac{1}{n}f(\frac{x}{n})$. If φ is differentiable on $\mathbb{R}$ and bounded by $M > 0$ then

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi(x)f_n(x)dx = \varphi(0). \tag{6.2}$$

Proof. We obviously have $\int_{\mathbb{R}} f_n(x)dx = 1$. We then have

$$\left| \int_{\mathbb{R}} \varphi(x)f_n(x)dx - \varphi(0) \right| \leq \frac{1}{n} \int_{\mathbb{R}} |\varphi(x) - \varphi(0)| \, f\left(\frac{x}{n}\right) dx, \tag{6.3}$$

because the integral of f_n equals 1. The idea of the proof is to separate the set of integration into several regions: where $\varphi(x)$ is close to $\varphi(0)$, and where the tail of the integral of f is small. Fix $A > 0$ and write

$$\frac{1}{n} \int_{\mathbb{R}} |\varphi(x) - \varphi(0)| \, f\left(\frac{x}{n}\right) dx = \frac{1}{n} \left(\int_{-\infty}^{-A} + \int_{-A}^{A} + \int_{A}^{\infty} \right). \tag{6.4}$$

The tail can be estimated with

$$\frac{1}{n}\int_A^\infty |\varphi(x)-\varphi(0)|\, f\left(\frac{x}{n}\right) dx \le 2M\int_{nA}^\infty f(y)dy, \tag{6.5}$$

where we have substituted $x = ny$. We do similarly with integral over $(-\infty, A]$. Further, we have

$$\frac{1}{n}\int_{-A}^A |\varphi(x)-\varphi(0)|\, f\left(\frac{x}{n}\right) dx \le \max_{x\in[-A,A]} |\varphi(x)-\varphi(0)|\, \frac{1}{n}\int_{-\infty}^\infty f\left(\frac{x}{n}\right) dx = \max_{x\in[-A,A]} |\varphi(x)-\varphi(0)|\,. \tag{6.6}$$

Using the mean value theorem we obtain

$$\frac{1}{n}\int_{-A}^A |\varphi(x)-\varphi(0)|\, f\left(\frac{x}{n}\right) dx \le \max_{x\in[-A,A]} |\varphi'(\xi(x))|\,|x| \le A \max_{x\in[-A,A]} |\varphi'(\xi(x))|\,, \tag{6.7}$$

where ξ is between 0 and x. If we now take $A = \frac{1}{\sqrt{n}}$ then according to (6.5) the tails go to zero, and from the uniform continuity of the derivative we have

$$\frac{1}{n}\int_{-A}^A |\varphi(x)-\varphi(0)|\, f\left(\frac{x}{n}\right) dx \le \frac{1}{\sqrt{n}} \max_{x\in[-\frac{1}{\sqrt{n}},\frac{1}{\sqrt{n}}]} |\varphi'(\xi(x))| \to 0. \tag{6.8}$$

Then f_n has the same filtering property as the heat kernel. □

We can thus see that there is an infinite number of functional series which limits can be candidates for Dirac delta. Notice also that in the previous theorem the local behaviour of φ is approximated with an integral over the whole domain. We can think that the integral is a smeared out version of φ. Further, if we *formally* write $f_n \to \delta$ then it seems sensible to introduce the following notation

$$\int_{\mathbb{R}} \delta(x)\varphi(x)dx = \varphi(0), \tag{6.9}$$

for sufficiently regular φ. This property can be made rigorous by considering the integral as a *funcional* mapping φ into some number.

6.2 Functionals and test functions

A functional has to act on some space and this has to be defined precisely. It must not be too small to include all meaningful objects and not too large in order to retain the regularity properties.

Definition 7. Test function φ *is function of* $C^\infty(\mathbb{R})$ *class vanishing outside a bounded interval. The space of all test functions is denoted by* $\mathcal{D}$.

Maybe it is not clear that the above definition gives a nonempty class. The following example constructs a concrete family of its members.

Example. The function

$$\varphi(x) = \exp\left(\frac{1}{x^2-1}\right)\chi_{(-1,1)}, \tag{6.10}$$

where χ is the characteristic function of an interval, is a test function. For the proof it suffices to show that φ is differentiable at $x = 1$ infinitely may times (the point $x = -1$ is completely analogous). Calculating the derivative we have

$$\varphi'(x) = -\frac{2x}{(x^2-1)^2}\exp\left(\frac{1}{x^2-1}\right), \tag{6.11}$$

for $x \in (-1, 1)$. Similarly, we can show that a derivative of arbitrary order is a linear combination of products of an exponential and rational functions having $(x^2-1)^n$ in their denominators. Because the exponential decay kills the algebraic growth (recall how to prove it!) we have

$$\lim_{x\to 1^-} \frac{1}{(x^2-1)^n}\exp\left(\frac{1}{x^2-1}\right) = 0. \tag{6.12}$$

Therefore, every derivative is continuous at $x = 1$ and vanishes there. □

Example. Other test functions can be constructed based on the above by translations and dilations. The set of all test functions is a vector space over $\mathbb{R}$. □

We will also need the notion of convergence in the space $\mathcal{D}$. That is to say, we have to define what it means that a sequence of functions from $\mathcal{D}$ converges (pointwise or uniform convergence is not strong enough for further purposes but they help to get used to different topologies).

Definition 8. *Let $\varphi_n \in \mathcal{D}$ be a sequence of functions vanishing outside a common closed interval. We say that φ_n converges to $\varphi \in \mathcal{D}$ in $\mathcal{D}$ if*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left|\varphi_n^{(k)}(x) - \varphi^{(k)}(x)\right| = 0, \quad k \in \mathbb{N}, \tag{6.13}$$

that is each derivative of φ_n converges uniformly *to the corresponding derivative of φ.*

We are finally able to give a rigorous definition of a distribution.

Definition 9. *A* functional f *over the space of test functions $\mathcal{D}$ is a mapping $f : \mathcal{D} \to \mathbb{R}$. Its action on a test function φ is denoted by (f, φ).*

Definition 10. *A* distribution *is a linear functional that maps $\mathcal{D}$ into $\mathbb{R}$ and is continuous in the following* weak *sense. Let φ_n be a convergent sequence of test functions vanishing outside a common interval. Denote its limit by φ. Then, the functional f is continuous if*

$$(f, \varphi_n) \to (f, \varphi) \quad \textit{when} \quad n \to \infty. \tag{6.14}$$

The set of all distributions is denoted by $\mathcal{D}'$.

Note that the above expression is a limit of *real numbers*. Finally, we can give a rigorous definition of the Dirac delta.

Definition 11. Dirac delta *(or Dirac distribution) is a distribution $\delta \in \mathcal{D}'$ defined with the following formula*

$$(\delta, \varphi) = \varphi(0), \quad \varphi \in \mathcal{D}. \tag{6.15}$$

Showing that the above definition is well-posed is a simple exercise.

Example. This is an important example. Other distributions can be constructed with a use of *locally integrable* functions f, i.e. those for which $\int_I |f(x)|dx < \infty$ where $I \subset \mathbb{R}$ is any bounded interval. Let f be locally integrable. Then we can define a distribution in the following way

$$(f, \varphi) = \int_{\mathbb{R}} f(x)\varphi(x)dx, \quad \varphi \in \mathcal{D}. \tag{6.16}$$

The proof that the above really defines a distribution is a problem on the list. □

Because many useful distributions are defined by functions, to keep the notation neat, we will write the following for Dirac delta

$$\int_{\mathbb{R}} \delta(x)\varphi(x)dx = \varphi(0). \tag{6.17}$$

We have to keep in mind that this is only a notation: so-called "useful fiction". Let us go back to the first example in this notes. We have shown there that if f is integrable then for every test function we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\varphi(x)dx = \varphi(0). \tag{6.18}$$

How to understand this limit? We know that the right-hand side of the above is the Dirac delta

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\varphi(x)dx = \int_{\mathbb{R}} \delta(x)\varphi(x)dx. \tag{6.19}$$

Since f_n defines a sequence of distributions we would like to write $f_n \to \delta$ in some sense. It appears that this is a good definition of the convergence for distributions (do not confuse it with the continuity of functionals)

Definition 12. *Let $f_n \in \mathcal{D}'$. If there exists a distribution f, for which it holds that*

$$\lim_{n\to\infty} (f_n, \varphi) = (f, \varphi), \quad \varphi \in \mathcal{D}, \tag{6.20}$$

we say that f_n converges to f in the distributional *(or weak*) sense.*

Notice that the above definition *assumes* existence of the limit. Showing that a given sequence of distributions has or does not have a limit is usually difficult. Moreover, distributional convergence is something completely different than the pointwise convergence.

Example. We know that $f_n(x) = \sin(nx)$ does not have a pointwise limit when $x \neq 0$. Since f_n is a locally integrable function it defines a distribution

$$(f_n, \varphi) = \int_{\mathbb{R}} \sin(nx)\varphi(x)dx. \tag{6.21}$$

Since φ is a test functions, it is also bounded, say by M. Assume that φ vanishes outside $[a, b]$. We now have

$$|(f_n, \varphi)| \leq M \left| \int_a^b \sin(nx) dx \right| \leq \frac{2M}{n}, \tag{6.22}$$

because $|\cos(nx)| \leq 1$. We can see that the right-hand side converges to zero as $n \to \infty$ and hence f_n converge to zero distribution. □

We can conduct some elementary operations on distributions. The below definitions follow from the properties of distributions defined by locally integrable functions (convince yourself about that!).

Definition 13. *Let $f \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$. We* define *the following operations.*

- *(Translation)* $(f(x - y), \varphi(x)) = (f(x), \varphi(x + y))$.
- *(Dilation)* $(f(\alpha x), \varphi(x)) = \frac{1}{|\alpha|} \left(f(x), \varphi\left(\frac{x}{\alpha}\right)\right)$ *for* $\alpha \neq 0$.
- *(Multiplication by functions) if* $\psi \in C^\infty(\mathbb{R})$ *then* $(\psi f, \varphi) = (f, \psi\varphi)$.

Notice also that a multiplication of two distributions is frequently undetermined. For example, δ^2 does not have any meaning.

6.3 Differentiation

Distributions have one more very useful property thanks to which they are commonly used in differential equations. It appears that every distribution can be differentiated arbitrary many times. To see this take any differentiable function f. It defines a distribution according to the standard formula

$$(f', \varphi) = \int_{\mathbb{R}} f'(x)\varphi(x) dx, \quad \varphi \in \mathcal{D}. \tag{6.23}$$

Now, integrating by parts we obtain

$$(f', \varphi) = -\int_{\mathbb{R}} f(x)\varphi'(x) dx = -(f, \varphi'), \tag{6.24}$$

where we used the fact that φ, as a test function, vanishes identically at infinity. We can thus see that in order to compute the derivative of a distribution we only have to know the values of the function f. This motivates the following definition

Definition 14. *A* derivative of a distribution f *is defined by the following formula*

$$\left(f^{(n)}, \varphi\right) = (-1)^n \left(f, \varphi^{(n)}\right), \quad \varphi \in \mathcal{D}. \tag{6.25}$$

The well-possessedness of the above definition, i.e. that the derivative is a distribution for itself, follows from linearity and properties of test functions.

Example. Let $H(x)$ be the Heaviside functions, that is a unit step at zero

$$H(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0. \end{cases} \tag{6.26}$$

this function generates a distribution in the usual sense. According to the distributional derivative we have

$$(H', \varphi) = -(H, \varphi') = -\int_{\mathbb{R}} H(x)\varphi(x)dx = -\int_0^\infty \varphi'(x) = \varphi(0) = (\delta, \varphi). \tag{6.27}$$

We can thus see that the derivative in the distributional sense of the Heaviside function is the Dirac delta. It is a rigorous justification of the intuitive fact known to us before. □

Example. We can compute similarly with the derivatives of Dirac delta

$$\left(\delta^{(n)}, \varphi\right) = (-1)^n \left(\delta, \varphi^{(n)}\right) = (-1)^n \varphi^{(n)}(0). \tag{6.28}$$

Notice that the differentiability of distributions is, as a matter of fact, a consequence of the smoothness of test functions. This is one of the reasons why we had chosen this function space. □

Example. Let L be a differential operator

$$L := a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + ... + a_1(x)\frac{d}{dx} + a_0(x), \tag{6.29}$$

where $a_i(x)$ are real functions. If $f \in \mathcal{D}'$ then

$$(Lf, \varphi) = \left(\sum_{i=0}^{n} a_0(x)\frac{d^i f}{dx^i}, \varphi\right) = \left(f, \sum_{i=0}^{n}(-1)^i \frac{d^i}{dx^i}\left(a_i(x)\varphi(x)\right)\right) = (f, L^*\varphi), \tag{6.30}$$

where the differential operator L^* is the so-called *formal adjoint* to L. The above formula is obvious if f is locally integrable. For an arbitrary distribution it follows from the definition. □

So far we have collected all the necessary ingredients that will let us use distributions in studying differential equations. In many applications the classical notion of the solution to an equation is too restrictive (recall conservation laws and shock waves). Thanks to distributions we can give the weakest definition what we mean by a solution to a differential equations. For simplicity, we will focus only on ordinary equations however, generalization to partial ones is relatively easy.

First, consider the simple equation

$$f' = 0. \tag{6.31}$$

We are thus looking for distributions having a vanishing derivative. This means that

$$0 = (f', \varphi) = -(f, \varphi'), \quad \varphi \in \mathcal{D}. \tag{6.32}$$

Therefore, a distribution f will be a solution of the above equation if and only if $(f, \psi) = 0$ for all test functions ψ which are a derivative of some other test function. It is an exercise on the list to show that this condition is equivalent to $\int_{\mathbb{R}} \psi(x)dx = 0$. Let now φ be any test function while φ_0 some other one with the property that $\int_{\mathbb{R}} \varphi_0(x)dx = 1$ (we can always rescale any integrable function to have a norm equal to 1). We can then write

$$\varphi(x) = \varphi_0(x) \int_{\mathbb{R}} \varphi(x)dx + \underbrace{\left[\varphi(x) - \varphi_0(x) \int_{\mathbb{R}} \varphi(x)dx\right]}_{\psi}. \tag{6.33}$$

We can see that ψ has a vanishing integral. If now f is the solution of the equation $f' = 0$ we must have

$$(f, \varphi) = \left(f, \varphi_0 \int_{\mathbb{R}} \varphi(x)dx\right) = \left(\int_{\mathbb{R}} \varphi(x)dx\right)(f, \varphi_0) = C \int_{\mathbb{R}} \varphi(x)dx = (C, \varphi), \tag{6.34}$$

where we have defined a constant C independent of φ. Above reasoning shows that the solution of $f' = 0$ is $f = C$ in the distributional sense. This result is, of course, in a complete agreement with the classical case.

Now, consider a general linear nonhomogeneous equation of order n

$$Lf = g, \quad L := \sum_{i=0}^{n} a_i(x) \frac{d^i}{dx^i}. \tag{6.35}$$

If we treat it in the distributional sense we have

$$(Lf, \varphi) = (g, \varphi), \quad \varphi \in \mathcal{D}. \tag{6.36}$$

We can use the formal adjoint to L to recast the above into

$$(f, L^*\varphi) = (g, \varphi). \tag{6.37}$$

A distribution f is the solution of (6.35) if the above equality is satisfied for every $\varphi \in \mathcal{D}$. This solution can belong to one of the three categories.

1. If the solution f is a n-times differentiable function we say that it is a **classical solution**.

2. If the solution f is a function which is less than n-times differentiable we say that it is a **weak solution**.

3. If the solution f is not a function we say that it is a **distributional solution**.

All the above cases are sequenced according to the decreasing regularity of the solution.

In particular we can take $g(x) = \delta(x - \xi)$ and look for solutions of

$$LG_\xi(x) = \delta(x - \xi). \tag{6.38}$$

If we find its solution then the distribution f defined by the formula

$$(f, \varphi) = \int_{\mathbb{R}} (G_\xi, \varphi)\, d\xi, \quad \varphi \in \mathcal{D}, \tag{6.39}$$

is a solution of the equation $Lf = g$. Distribution G_ξ is called the **Green's function** (since it is almost always a function). We see that the Green's function is just a solution forced by a point source both in space and time.

Example. A solution of $xf' = 0$ is the Heaviside function $H(x)$. To wit,

$$(xH', \varphi) = -\big(H, (x\varphi)'\big) = -\int_{\mathbb{R}} H(x)(x\varphi(x))'dx = \int_0^\infty (x\varphi(x))'dx = 0. \tag{6.40}$$

Because H is a non-smooth function it is a weak solution. If we looked for a classical solution, only a constant would be it. □

To conclude, we will see an example that the Green's function defined via the distributions is the same as the one that we used before. As an example we take the heat conduction problem

$$\begin{cases} u_t = \alpha^2 u_{xx} + f, \quad (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = 0, \quad x \in \mathbb{R} \\ u(-\infty,t) = 0, \quad u(\infty,t) = 0, \qquad t \geq 0. \end{cases} \tag{6.41}$$

We have shown that its solution is given by (4.80), i.e.

$$u(x,t) = \int_0^t \int_{-\infty}^\infty G(x-\xi, t-\tau) f(\xi,\tau) d\xi d\tau, \tag{6.42}$$

where G is the heat kernel. Notice that if $f(x,t) = \delta(x-x_0)\delta(t-t_0)$, and we think in the distributional sense, we have

$$u(x,t) = \int_0^t \int_{-\infty}^\infty G(x-\xi, t-\tau)\delta(\xi-x_0)\delta(\tau-t_0) d\xi d\tau = G(x-x_0, t-t-0). \tag{6.43}$$

Which confirms our intuition that the Green's function is the temperature at x and t caused by a point source at $x = 0$ and $t = 0$.

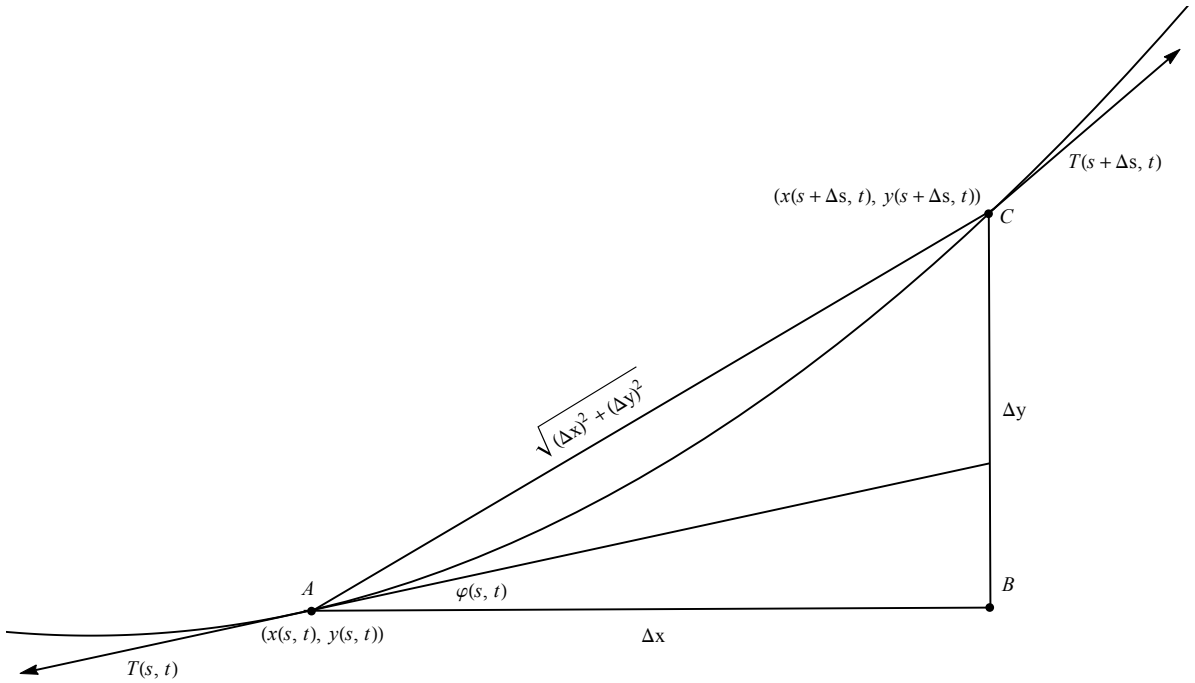

Figure 19: A schematic of the string fragment.

7 Wave equation

In this section we will meet second order hyperbolic PDEs that model many wavelike phenomena. They are essentially relevant for describing small-amplitude oscillations near equilibrium. We will start with several examples.

Example. (*String vibrations*) Consider a one-dimensional string [27] with linear density ρ, which in equilibrium position is stressed by the force $\mathbf{T}_0$ (tension) and is subjected to force per unit length $\mathbf{F}$ (e.g. gravity, load). Without lose of generality for our future considerations, we can assume that the force $\mathbf{F}$ works in the vertical direction, i.e. $\mathbf{F} = (0, F)$. We assume here that the string is very thin, that is its length dimension is much larger than its thickness [28]. For simplicity, let us imagine that our string is attached to the guitar at points $x = 0$ and $x = L$. We will only be interested in vibration of the string in a direction perpendicular to the Ox axis (transverse), but it is very informative to derive equations in which we allow vibrations along the Ox axis (longitudinal) as well. Let $(x(s,t), y(s,t))$ be the parametric equation of the string shape at time t with parameter s.

Fix any time instant t. Consider a small piece of string describes by an increase of s by Δs (Fig 19). It is obvious that the Δs increment produces the projections Δx and

[27] It can also be a cable, chain or anything we can assume for simplicity that it is essentially one-dimensional.

[28] The standard guitar string diameter E6 (i.e. the thickest one) is about 0.1 cm and its active length is 65 cm (for 4/4 measure). So we can see that the string is about 650 times longer than thicker. We can define the dimensionless thickness parameter as $\epsilon =$ thickness / length. The assumption of one-dimensional string is sensible if $\epsilon \ll 1$. In our case, $\epsilon = 0.1/65 \approx 0.0015 \ll 1$ for the string E6 and $\epsilon = 0.025/65 \approx 0.0004 \ll 1$ for the thinnest string E1. Piano strings are somewhat thicker however, this assumption is also valid for them.

Δy. Let us consider the forces (per unit length) acting on that small fragment. These are: tension $\mathbf{T} = \mathbf{T}(s,t)$ and a predetermined force per unit length $\mathbf{F} = \mathbf{F}(s,t)$. The resultant of these two forces must be equal to the mass of the segment multiplied by the acceleration it experiences.

Note that thanks to the one-dimensionality of the string the tension acts in tangent direction. We can write

$$\mathbf{T}(s,t) = T(s,t)(\cos\varphi(s,t), \sin\varphi(s,t)), \tag{7.1}$$

where $\varphi = \varphi(s,t)$ is the angle between the string and the axis Ox (see Fig 19) and $T(s,t)$ being the length of the vector $\mathbf{T}(s,t)$. Recall from the calculus that the mass of the arc of a parametric curve with the density $\rho(s)$ can be calculated from the formula [29]

$$M(s) = \int_{s_0}^{s} \rho(\nu)\sqrt{\left(\frac{\partial x}{\partial s}(\nu,t)\right)^2 + \left(\frac{\partial y}{\partial s}(\nu,t)\right)^2}\,d\nu. \tag{7.2}$$

In our case, the mass of the interval is

$$\begin{aligned} M(s+\Delta s) - M(s) &= \int_{s}^{s+\Delta s} \rho(\nu)\sqrt{\left(\frac{\partial x}{\partial s}(\nu,t)\right)^2 + \left(\frac{\partial y}{\partial s}(\nu,t)\right)^2}\,d\nu = \\ &= \Delta s\rho(\sigma)\sqrt{\left(\frac{\partial x}{\partial s}(\sigma,t)\right)^2 + \left(\frac{\partial y}{\partial s}(\sigma,t)\right)^2}, \end{aligned} \tag{7.3}$$

for some $\sigma \in (s, s+\Delta s)$ which follows from the mean-value theorem for integrals. Notice that $\sigma \to s$ when $\Delta s \to 0$. In a similar way we can compute a the result of a force $\mathbf{F}$ on the interval Δs. From the assumption, its x-coordinate is equal to zero, while the y-coordinate is

$$\int_{s}^{s+\Delta s} F(\nu,t)\sqrt{\left(\frac{\partial x}{\partial s}(\nu,t)\right)^2 + \left(\frac{\partial y}{\partial s}(\nu,t)\right)^2}\,d\nu = \Delta s F(\omega,t)\sqrt{\left(\frac{\partial x}{\partial s}(\omega,t)\right)^2 + \left(\frac{\partial y}{\partial s}(\omega,t)\right)^2}, \tag{7.4}$$

for some $\omega \in (s, s+\Delta s)$.

The net tension acting on Δs is equal to $\mathbf{T}(s+\Delta s,t) - \mathbf{T}(s,t)$. We can therefore write the x-coordinate as

$$T(s+\Delta s,t)\cos\varphi(s+\Delta s,t) - T(s,t)\cos\varphi(s,t) = \rho(\sigma)\Delta s\sqrt{\left(\frac{\partial x}{\partial s}(\sigma,t)\right)^2 + \left(\frac{\partial y}{\partial s}(\sigma,t)\right)^2}\frac{\partial^2 x}{\partial t^2}. \tag{7.5}$$

Similarly, the ycoordinate is

$$\begin{aligned} &T(s+\Delta s,t)\sin\varphi(s+\Delta s,t) - T(s,t)\sin\varphi(s,t) = \\ &= \rho(\sigma)\Delta s\sqrt{\left(\frac{\partial y}{\partial s}(\sigma,t)\right)^2 + \left(\frac{\partial y}{\partial s}(\sigma,t)\right)^2}\frac{\partial^2 x}{\partial t^2} + \Delta s F(\omega,t)\sqrt{\left(\frac{\partial y}{\partial s}(\omega,t)\right)^2 + \left(\frac{\partial y}{\partial s}(\omega,t)\right)^2}. \end{aligned} \tag{7.6}$$

[29] For a homogeneous curve, the mass is equal to the density multiplied by its length. For any mass distribution, ρ must be put under the integral. To better understand this formula, calculate the mass of the straight line. Then take any curve and zoom it in.

We can now divide the two above equations by Δs and take the limit $\Delta s \to 0$ to obtain

$$\frac{\partial}{\partial s}\left(T(s,t)\cos\varphi(s,t)\right) = \rho(s)\sqrt{\left(\frac{\partial x}{\partial s}(s,t)\right)^2 + \left(\frac{\partial y}{\partial s}(s,t)\right)^2}\frac{\partial^2 x}{\partial t^2}, \tag{7.7}$$

and

$$\begin{aligned}&\frac{\partial}{\partial s}\left(T(s,t)\sin\varphi(s,t)\right) = \\ &= \rho(s)\sqrt{\left(\frac{\partial x}{\partial s}(s,t)\right)^2 + \left(\frac{\partial y}{\partial s}(s,t)\right)^2}\frac{\partial^2 y}{\partial t^2} + \sqrt{\left(\frac{\partial x}{\partial s}(s,t)\right)^2 + \left(\frac{\partial y}{\partial s}(s,t)\right)^2}F(s,t).\end{aligned} \tag{7.8}$$

The above equations can be simplified when we utilize the dependence of φ on the derivatives $\partial x/\partial s$ and $\partial y/\partial s$. Notice first that the angle $\varphi_{\Delta s}(s,t) := \angle CAB$ approaches $\varphi(s,t)$ as $\Delta s \to 0$. From the figure we immediately know that

$$\cos\varphi_{\Delta s}(s,t) = \frac{\Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}, \quad \sin\varphi_{\Delta s}(s,t) = \frac{\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}, \tag{7.9}$$

which is equivalent to

$$\cos\varphi_{\Delta s}(s,t) = \frac{\frac{\Delta x}{\Delta s}}{\sqrt{\left(\frac{\Delta x}{\Delta s}\right)^2 + \left(\frac{\Delta y}{\Delta s}\right)^2}}, \quad \sin\varphi_{\Delta s}(s,t) = \frac{\frac{\Delta y}{\Delta s}}{\sqrt{\left(\frac{\Delta x}{\Delta s}\right)^2 + \left(\frac{\Delta y}{\Delta s}\right)^2}}. \tag{7.10}$$

Taking the limit $\Delta s \to 0$ finally implies

$$\cos\varphi(s,t) = \frac{\frac{\partial x}{\partial s}}{\sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2}}, \quad \sin\varphi(s,t) = \frac{\frac{\partial y}{\partial s}}{\sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2}}, \tag{7.11}$$

where the derivatives are evaluated at (s,t). Finally, equations describing the motion of a point $(x(s,t), y(s,t))$ on the string have the form

$$\frac{\partial}{\partial s}\left(T(s,t)\frac{\frac{\partial x}{\partial s}}{\sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2}}\right) = \rho(s)\sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2}\frac{\partial^2 x}{\partial t^2}, \tag{7.12}$$

and

$$\frac{\partial}{\partial s}\left(T(s,t)\frac{\frac{\partial y}{\partial s}}{\sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2}}\right) = \rho(s)\sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2}\frac{\partial^2 y}{\partial t^2} + \sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2}F(s,t), \tag{7.13}$$

where for the sake of clarity we have abandoned explicitly writing the arguments of the derivative. This will not cause any misunderstandings. At first glance, the equations (7.12) and (**??**) look *scary*! They form a system of two second order nonlinear partial differential equations. Fortunately, in almost all applications we can make an assumption that simplifies them in a significant way. The result that we will get

then will be only an approximation of the real phenomenon [30]. This approximation, however, very often is of the order higher than the error of the experiment. Even if it was not, the simplified equation is always the first step to understanding the solution of the full generality.

Before we make the approximation, let us note that we have two equations: (7.12) and (7.13) and *three* unknowns: x, y and T. Is our problem well posed in this case? As we have already pointed out, the equations we have derived can describe vibrations not only of the strings but also of all other objects that can be considered to be significantly elastic and one-dimensional. What differs them is, apart from the density ρ, the distribution of the tension force $\mathbf{T}$. To complete the mathematical description of our phenomenon, we must provide $\mathbf{T}$. We can easily imagine that the greater the pulling force, the greater the deformation of the string. This is also the case for very many materials [31]. The experiment shows that the stress force is a function of *relative* deformation of the material. This is an extremely important relationship in physics, mechanics and construction engineering. The relative deformation of the (infinitesimally small) string piece is $((\partial x/\partial s)^2 + (\partial y/\partial s)^2)^{\frac{1}{2}} - 1$, which can be obtained by considering the geometry of the problem[32]. We can therefore write that the length of the stress vector is

$$T(s,t) = \mathcal{T}\left(\sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2} - 1, t\right), \tag{7.14}$$

for some function $\mathcal{T}$. We recall this the above is called the constitutive relation. We can expand the unknown function $\mathcal{T}$ in Taylor series which gives us

$$\mathcal{T}\left(\sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2} - 1, t\right) = T_0 + E\left(\sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2} - 1\right) + ..., \tag{7.15}$$

where T_0 is the tension in the equilibrium position (therefore it does not depend on time), while E is the so-called *Young's modulus*. We see that for small deformations, the relationship between stress and strain is linear. It is the celebrated *Hooke's law*.

Thus, we see that in order to solve the problem of string vibration we need three equations (7.12), (7.13) and (7.14). This will allow to find all unknowns. Let us now deal with a special case that is extremely important in applications.

Let us assume that we are only interested in transversal vibrations. When the string vibrates only vertically, the coordinate of x of any point on it remains constant for all time. This means that

$$\frac{\partial x}{\partial t}(s,t) = 0, \quad \text{that is} \quad x = x(s), \tag{7.16}$$

[30] You can also argue that *any* mathematical model is a certain idealization, i.e. an approximation of what is happening around us.

[31] The same is true with the textbook example of a spring: the tension force is proportional to the elongation.

[32] The string segment in the equilibrium position has the length $\Delta x = \Delta s$. When the string is tilted, this small segment deforms and obtains the length $\sqrt{(\Delta x)^2 + (\Delta y)^2}$. Going to the limit $\Delta s \to 0$ shows that the *relative* deviation is actually the one we gave above.

for all t and fixed s. Then, equations (7.12) and (7.13) take the form

$$\frac{\partial}{\partial s}\left(T(s,t)\frac{\frac{dx}{ds}}{\sqrt{\left(\frac{dx}{ds}\right)^2+\left(\frac{\partial y}{\partial s}\right)^2}}\right)=0, \tag{7.17}$$

and

$$\frac{\partial}{\partial s}\left(T(s,t)\frac{\frac{\partial y}{\partial s}}{\sqrt{\left(\frac{dx}{ds}\right)^2+\left(\frac{\partial y}{\partial s}\right)^2}}\right)=\rho(s)\sqrt{\left(\frac{dx}{ds}\right)^2+\left(\frac{\partial y}{\partial s}\right)^2}\frac{\partial^2 y}{\partial t^2}+\sqrt{\left(\frac{dx}{ds}\right)^2+\left(\frac{\partial y}{\partial s}\right)^2}F(s,t). \tag{7.18}$$

From (7.18) we can infer that

$$T(s,t)\frac{\frac{dx}{ds}}{\sqrt{\left(\frac{dx}{ds}\right)^2+\left(\frac{\partial y}{\partial s}\right)^2}}=T_0(t), \tag{7.19}$$

where $T_0(t)$ is the tension at $(x(s,t),y(s,t))$ satisfying $dx/ds=1$ and $\partial y/\partial s=0$, i.e. the angle φ is zero (equilibrium position). We can now use our constitutive equation (7.15) and substitute $dx/ds=1$ and $\partial y/\partial s=0$ in it. We get $T_0(t)=T_0$ for all times t, i.e. the right side of the equation (7.19) is a constant equal to the string tension in the equilibrium position[33]. Using this result, we are able to simplify the equation (7.18)

$$T_0\frac{\partial}{\partial s}\left(\frac{\frac{\partial y}{\partial s}}{\frac{dx}{ds}}\right)=\rho(s)\sqrt{\left(\frac{dx}{ds}\right)^2+\left(\frac{\partial y}{\partial s}\right)^2}\frac{\partial^2 y}{\partial t^2}+\sqrt{\left(\frac{dx}{ds}\right)^2+\left(\frac{\partial y}{\partial s}\right)^2}F(s,t). \tag{7.20}$$

Using parametric curves is often useful in theoretical considerations, but in practice it is much convenient to choose x as the independent variable. Suppose the string is not vertical at any time, that is

$$\frac{dx}{ds}\neq 0. \tag{7.21}$$

Then we known that there is a well defined inverse function $s=s(x)$ and we can define

$$u(x,t):=y(s(x),t). \tag{7.22}$$

Whence, from the chain rule we obtain

$$\frac{\partial u}{\partial x}=\frac{\partial y}{\partial s}\frac{ds}{dx}=\frac{\partial y}{\partial s}\left(\frac{dx}{ds}\right)^{-1}\quad\text{czyli}\quad\frac{\partial y}{\partial s}=\frac{dx}{ds}\frac{\partial u}{\partial x}, \tag{7.23}$$

where we used the theorem of a derivative of an inverse function. Moreover, from the chain rule we get

$$\frac{\partial}{\partial s}=\frac{dx}{ds}\frac{\partial}{\partial x}. \tag{7.24}$$

[33]This is an important discovery because we can very easy to measure. If it changed over time, it would be much more difficult to perform the appropriate experiment.

Thanks to the two aobve equations we can write (7.18) as a one-dimensional *nonliner wave equation*

$$T_0 \frac{\partial^2 y}{\partial x^2} = \rho(x)\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\frac{\partial^2 y}{\partial t^2} + \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} F(x,t), \tag{7.25}$$

where for clarity of notation we have expressed ρ and F as functions of x.

The equation (**??**) has a very interesting stationary solution. Suppose the string (although now it is better to think about a cable or hanging chain[34]) hangs under the gravity. This gives us $F(x,t) = \rho(x)g$. In addition, assume that the string has already reached its steady state, i.e. $\partial y/\partial t = 0$. We then get the *hanging cable equation*

$$T_0 \frac{d^2 y}{dx^2} = \rho g\sqrt{1+\left(\frac{dy}{dx}\right)^2}. \tag{7.26}$$

The solution of (7.26) is a hyperbolic cosine (exercise).

Note that nowhere before have we made any simplifying assumptions, but have only considered specific cases. Let us return to the equation (**??**) and make a very important assumption that the deflections of our string are *small*[35]. This means that

$$\left|\frac{\partial y}{\partial x}\right| \ll 1. \tag{7.27}$$

Due to this assumption, all (greater than the first) powers of $\partial y/\partial x$ are at least one order smaller than the derivative itself. Therefore, it is justified to neglect them. We then get the one-dimensional *wave equation*

$$T_0 \frac{\partial^2 y}{\partial x^2} = \rho(x)\frac{\partial^2 y}{\partial t^2} + F(x,t). \tag{7.28}$$

Note also that, according to the assumption of small deflections, the constitutive equation (7.15) tells us that the stress force is constant at every point of the string and at all times. In many situations, we are interested in a case where the string is homogeneous (i.e. $\rho(x) = \rho = \text{const.}$) And no external forces (i.e. $F \equiv 0$) act on it. We then get the very well known *wave equation*

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \tag{7.29}$$

where $c = \sqrt{T_0/\rho}$ is the wave velocity. □

Example. (*Membrane and acoustics*) It turns out that the wave equation also appears when describing the vibration of the membrane (2D) and air at the propagation of the acoustic wave (3D). However, it is no longer just one-dimensional and has the form

$$u_{tt} = c^2 \Delta u, \tag{7.30}$$

[34] And also about the arches supporting the ceiling in cathedrals.

[35] For the guitar string this is a very appropriate assumption. We can see this by making a very simple *back-of-the-envelope* estimate. As we mentioned above, the length of a typical guitar string is 65 cm. It seems reasonable to assume that its typical deflection when playing is $1-2$ mm. Then $|\partial y/\partial x| \approx 2/650 \approx 0.003 \ll 1$.

where Δ is the Laplacian. $\square$

Example. (*Maxwell's equations*) One of the most profound and fundamental achievements of physics was a unification of the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields done by Scottish mathematician and physicist James Clerk Maxwell. He devised that

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{B} - 4\pi\mathbf{J}, \quad \frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}, \tag{7.31}$$

where c is the speed of light in vacuum, ρ charge density, and $\mathbf{J}$ current density. The interpretation of the equations is as follows. Starting from the left: Coulomb's law, source-free magnetic field, Ampere's law (for $\mathbf{E}$ independent of time) and Faraday's law. To comply with Ampere's law, we must have $\nabla \cdot \mathbf{J} = 0$. In case this is not met, for the mathematical closure of equations, a term with the time derivative of $\mathbf{E}$ is needed. Adding it allowed Maxwell to obtain a unified theory of electric and magnetic fields.

We will show that the fields $\mathbf{E}$ and $\mathbf{B}$ satisfy wave equation. If we compute the second derivative then

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c\nabla \times \frac{\partial B}{\partial t} = -c^2 \nabla \times (\nabla \times \mathbf{E}) \,. \tag{7.32}$$

Poprawi? strumie? pr?du! A standard identity in vector calculus tells us that $\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \Delta \mathbf{E}$. From the other equation we then have

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \Delta \mathbf{E} - 4\pi c^2 \nabla \rho. \tag{7.33}$$

For a given charge this is a nonhomogeneous wave equation. Similarly we can show that

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \Delta \mathbf{B} + 4\pi c \nabla \times \mathbf{J}. \tag{7.34}$$

Notice that the mathematics of E and B is essentially the same. This is a trace of a unification of seemingly different fields. Moreover, we see that each wave travels with the speed of light. $\square$

Example. (*Shallow water waves*) It can be shown that for a water flow in a shallow channel we have the so-called *shallow water equations*

$$\begin{cases} h_t + (uh)_x = 0, \\ u_t + uu_x = -gh_x, \end{cases} \tag{7.35}$$

where $u = u(x,t)$ is the horizontal velocity of the flow, while $h = h(x,t)$ the depth. If a wave has a small amplitude then

$$h = H + \zeta, \quad u = U, \tag{7.36}$$

where $\left|\frac{\zeta}{H}\right| \ll 1$, $\left|\frac{U}{u}\right| \ll 1$ and H is the mean depth. Dropping quadratic terms leaves us with

$$\zeta_t + HU_x = 0, \quad U_t + g\zeta_x = 0. \tag{7.37}$$

If we cross-differentiate the above we obtain

$$0 = \zeta_{tt} + HU_{xt} = \zeta_{tt} - gh\zeta_{xx}, \tag{7.38}$$

which is a linear wave equation $\zeta_{tt} = c^2 \zeta_{xx}$ with $c = \sqrt{gH}$. $\square$

7.1 Boundary conditions

We will start from one-dimensional wave equation, i.e.

$$u_{tt} = c^2 u_{xx} + f, \quad 0 < x < L. \tag{7.39}$$

Since this is a second order equation with respect to time we have to impose two initial conditions: one for the deflection u and one for the velocity u_t, therefore

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x). \tag{7.40}$$

Similarly as with the previous cases we have several options for prescribing boundary conditions

1. (*Dirichlet condition*) The string endpoints move according to a given rule

$$u(0,t) = \mu(t), \quad u(L,t) = \nu(t). \tag{7.41}$$

   If $\mu = \nu = 0$ then the endpoints are fixed, that is the string is attached.

2. (*Neumann condition*) Suppose that there is a force F acting on an endpoint. Since for small deflections string tension is equal to $T_0 u_x$ the boundary condition takes the form

$$u_x(0,t) = \mu(t) \quad \text{lub} \quad u(L,t) = \nu(t). \tag{7.42}$$

   A homogeneous case where μ or ν vanishes, states that the corresponding endpoint is free.

3. (*Robin condition*) If one of the string endpoints is elastically attached to some support than there arises a force proportional to the deflection, that is $T_0 u_x = -ku$ at that endpoint. If additionally, the support moves according to a known rule we have

$$u_x(0,t) = -\lambda_1(u(0,t) - \theta_1), \quad \text{lub} \quad u_x(L,t) = -\lambda_2(u(L,t) - \theta_2). \tag{7.43}$$

4. (*Load*) An interesting boundary condition arises when we assume that one endpoint is loaded with a mass m that is attached to a spring. Then, from Newton's second law we have

$$mu_{tt} = -ku_x + mg \quad \text{at one endpoint.} \tag{7.44}$$

5. (*Periodic condition*) If both ends of the string are joined, we have to impose continuity conditions

$$u(0,t) = u(L,t), \quad u_x(0,t) = u_x(L,t). \tag{7.45}$$

We will only deal with Dirichlet's condition because it is sufficient to illustrate many issues related to the wave equation. Of course, due to linearity, we can separate the problem of nonhomogeneous load f, initial conditions, and boundary conditions into three separate problems in which there is only one nonhomogeneity.

7.2 Separation of variables

Similarly as for the heat and Poisson's equation we can use the method of separation of variables. Remember that is works only for simple domains. Much have been said about this method and to illustrate the theory we will consider two musical examples.

Example. (*Guitar string*) Consider a string attached to the neck and the bridge of a guitar. We have

$$\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < L, \quad t > 0, \\ u(x,0) = \Phi(x), \\ u_t(x,0) = \Psi(x), \\ u(0,t) = u(L,t) = 0. \end{cases} \tag{7.46}$$

Separating variables starts with writing

$$u(x,t) = X(x)T(t), \tag{7.47}$$

which lead to familiar ODEs (compare the heat equation)

$$X'' + \lambda X = 0, \quad T'' + c^2 \lambda T = 0, \tag{7.48}$$

where the separation constant $\lambda > 0$. The boundary conditions are $X(0) = X(L) = 0$ and hence

$$X(x) = A \sin\left(\frac{n\pi}{L}x\right), \quad T(t) = C\cos\left(\frac{n\pi c}{L}t\right) + D\sin\left(\frac{n\pi c}{L}t\right), \tag{7.49}$$

and it follows that

$$u(x,t) = \sum_{n=1} \left(a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right)\right) \sin\left(\frac{n\pi}{L}x\right). \tag{7.50}$$

Further, we can apply initial conditions to determine a_n and b_n. The calculations are standard and we omit the details. It is interesting to give a musical interpretation of that we have found.

The vibrations of the string produce a sound that is then amplified by the resonant guitar body. Therefore, all the information of the quality of sound is encoded in Fourier coefficients a_n and b_n. The overall shape of the wave u is composed of pure sine waves called *harmonics*. Their frequency is

$$\text{frequency of } n\text{-th harmonic} = \frac{nc}{2L}, \tag{7.51}$$

and we see that higher pitches have a frequency which is a multiple of lower ones - hence the name. In general every sound, not necessarily produced by a guitar, can be decomposed in its Fourier series, i.e. we can find its expansion into harmonics. What distinguishes the sound of piano and the guitar, for instance, are the coefficients a_n and b_n that tell us how much of a given harmonic is included. This is called the *timbre* - the same sound has a completely different flavour played on different instruments.

Classical instruments cannot produce a pure sinusoidal wave and what we hear is a superposition of fundamental harmonic and its overtones. If a note has a frequency

ν then, from our solution, we know that we hear a linear superposition of waves with frequency $n\nu$ with various weights a_n and b_n. For the Fourier series to be convergent we must have $a_n, b_n \to 0$ with $n \to \infty$ so that higher harmonics have decaying amplitude. Therefore, the most important is the fundamental and its several overtones. For example the A sound consists of pure A (and its transpositions over octaves), E, $C^\sharp$, D, B, while E note expands into E, B, $G^\sharp$, A, $F^\sharp$, etc. Notice that the E and A are common in these spectra. This is what makes the *consonance*, i.e. a phenomenon occurring when different sounds played simultaneously are pleasant for the ear (this is of course somewhat subjective). Pythagoras found that consonance occurs for sounds produced by strings differing in length with a ratio of small integers[36]. For example, if two sounds with a frequency ratio of 3 : 2 are played, then the harmonics are

$$\begin{array}{cccc} \nu, & 2\nu, & 3\nu, & 4\nu, \quad ... \\ \frac{3}{2}\nu, & 3\nu, & \frac{9}{2}\nu, & 6\nu, \quad ... \end{array} \tag{7.52}$$

and we see that the third harmonic of the first sound is the same as the second harmonic of the other note. In this example, these sounds are in a *perfect fifth interval*. From here it is straightforward to stack different intervals over each other producing *chords*. We can carry this much further in developing music theory. It is remarkable however, that almost all of it can be explained with a Fourier solution of a simple PDE. Music, at least when it comes to composition, is mathematics. □

Example. (*Drum - a vibrating membrane*) Now, we consider a drum, that is a membrane spanned over a circular rim

$$\begin{cases} u_{tt} = c^2 \left(\frac{1}{r} \left(r u_r \right)_r + \frac{1}{r^2} u_{\theta\theta} \right), \quad 0 < r < a, \quad 0 < \theta < 2\pi, \\ u(r, \theta, 0) = \Phi(r, \theta), \\ u_t(r, \theta, 0) = \Psi(r, \theta), \end{cases} \tag{7.53}$$

where the Laplacian is written in polar coordinates. We separate variables as follows

$$u(r, \theta, t) = R(r)\Theta(\theta)T(t), \tag{7.54}$$

and it follows that

$$\frac{T''}{T} = c^2 \left(\frac{1}{rR} \left(rR' \right)' + \frac{1}{r^2 \Theta} \Theta'' \right). \tag{7.55}$$

After the first separation we obtain

$$T'' + \lambda T = 0, \quad \frac{r}{R} \left(rR' \right)' - \frac{\lambda}{c^2} r^2 = \frac{\Theta''}{\Theta}, \tag{7.56}$$

where we have anticipated the sign of the constant, i.e. $\lambda > 0$. The first ODE has the periodic solution with frequency dependent on λ

$$T(t) = C_1 \cos\left(\sqrt{\lambda} t \right) + C_2 \sin\left(\sqrt{\lambda} t \right). \tag{7.57}$$

[36] Note that we have shown that the frequency is inversely proportional to the string length.

A second separation gives us

$$\Theta'' + n^2\Theta = 0, \quad r^2R'' + rR' + \left(n^2 - \frac{\lambda}{c^2}r^2\right)R = 0. \tag{7.58}$$

It follows from the first ODE that we must have $n\mathbb{N}$ in order to have θ-periodic solutions. Moreover, the second equation is already known to us from our studies of Laplace's equation in cylindrical coordinates - Bessel equation (5.70). Since the membrane is fixed over its rim we have $R(a) = 0$, and for the well-posedness we require $R(0)$- bounded. Similarly to (5.76) the bounded solution is

$$R(r) = DJ_n\left(\frac{\sqrt{\lambda}}{c}r\right), \tag{7.59}$$

where J_n is the n-th order Bessel function of the first kind. In order to satisfy the boundary condition we must impose

$$\lambda = \left(\frac{c\, z_{nm}}{a}\right)^2, \tag{7.60}$$

where z_{nm} is the m-th zero of n-th Bessel function. We have thus arrived at a very important conclusion. The frequency of the elementary sine wave is

$$\text{frequency of } n\text{-th sine } = \frac{c\, z_{nm}}{2\pi a}, \tag{7.61}$$

which is *not* harmonic! Since z_{nm} are not integers the elementary notes produced by a drum are not multiples of each other (see Tab. 5.1.3). Note the completely different character of a percussion instrument such as drum from the pitched guitar or piano. □

7.3 One dimension - d'Alembert's solution

Historically, one of the first analytical solutions of partial equations was the solution of the wave equation derived by d'Alembert. We will trace his reasoning for the equation defined on the whole $\mathbb{R}$ line. We can think of an infinite string here. Let us solve then

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R} \\ u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x). \end{cases} \tag{7.62}$$

This is the initial value problem for the wave equation. d'Alembert introduced an appropriate transformation of variables in order to bring the wave equation into so-called canonical form. Of course, at that time the general theory of differential equations was unknown, which is why his reasoning is very innovative and brilliant. To follow d'Alemebert's intuition, let us write the wave equation using standard notation

$$0 = \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)u. \tag{7.63}$$

The form of the right-hand side encourages us to use factorization similar to the elementary school algebra. However, we have to keep in mind that we are dealing with operators and not numbers

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) = \frac{\partial^2}{\partial t^2} - c\frac{\partial^2}{\partial x \partial t} + c\frac{\partial^2}{\partial t \partial x} - c^2\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}, \quad (7.64)$$

since the mixed derivatives commute. Similarly

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) = \frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}, \quad (7.65)$$

The wave equation transforms thus into

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0 \quad \text{lub} \quad \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = 0. \quad (7.66)$$

Then, if u is a function satisfying one of the following first order equations

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u \quad \text{lub} \quad \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u, \quad (7.67)$$

then it is also a solution of the wave equation. We know that these solutions are $F(x-ct)$ and $G(x+ct)$, therefore from linearity we have

$$u(x,t) = F(x-ct) + G(x+ct). \quad (7.68)$$

Therefore, a general solution of the wave equation on $\mathbb{R}$ is composed of two waves: one going right, and one going left. Functions F and G are associated with initial conditions. We will find them. First of all,

$$\phi(x) = u(x,0) = F(x) + G(x), \quad (7.69)$$

and

$$\psi(x) = u_t(x,0) = -cF'(x) + cG'(x). \quad (7.70)$$

It follows that

$$-F(x) + G(x) = \frac{1}{c}\int_0^x \psi(s)ds + C, \quad (7.71)$$

where C is the integration constant. Immediately we have

$$F(x) = \frac{1}{2}\phi(x) - \frac{1}{2c}\int_0^x \psi(s)ds - \frac{C}{2}, \quad G(x) = \frac{1}{2}\phi(x) + \frac{1}{2c}\int_0^x \psi(s)ds + \frac{C}{2}. \quad (7.72)$$

Whence, we have proved the following famous d'Alembert's solution.

Theorem 11 (d'Alembert). *The solution of (7.62), i.e. the wave equation on $\mathbb{R}$ with initial conditions is*

$$u(x,t) = \frac{\phi(x-ct) + \phi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \quad (7.73)$$

Let us return for a moment to the formula (7.68) for the general solution of the wave equation. As we have already mentioned, it is a superposition of two waves running in opposite directions. More specifically, in the $x - t$ diagram, we can draw two lines $x \pm ct$, on which each part of the solution is constant. These are our familiar characteristics. Because the equation is of the second order, it is equivalent to two first-order equations [37], Therefore there are two families of characteristics. Information about the initial conditions is therefore carried into spacetime, moving at a speed of c along straight lines. Note also that for a fixed time $t > 0$ and point $x_0 \in \mathbb{R}$ the characteristics bound a triangular area with sides $t = 0, x = \pm ct$. It is called the *area of influence* that is, the set that contain the information about initial conditions. The value of the function u at (x, t) is only affected by the values of the initial conditions in $|x - x_0| \leq ct$.

Example. We will consider several examples illustrating the wave-like behaviour of solutions. Consider a signal that has nonzero initial value u with vanishing velocity. Then (7.73) reduces into

$$u(x, t) = \frac{\phi(x - ct) + \phi(x + ct)}{2}, \tag{7.74}$$

which means that one part of the initial condition travels to the right while the other to the right. For instance consider a rectangular impulse (it is a very simple example of plucking the guitar string)

$$u(x, 0) = \phi(x) = \begin{cases} 1, & |x| \leq h; \\ 0, & |x| > h, \end{cases} \qquad u_t(x, 0) = \psi(x) = 0. \tag{7.75}$$

This solution can most easily be understood by seeing it on a spacetime diagram presented on Fig. 20. The most important fact is that a characteristic caries the value of u into the spacetime. □

Example. Suppose now that only the velocity is nonzero, that is

$$u(x, 0) = \psi(x) = 0, \quad u_t(x, 0) = \psi(x) = \begin{cases} 1, & |x| \leq h; \\ 0, & |x| > h, \end{cases} \tag{7.76}$$

This, in turn, is a simple model of hammer striking a piano string. From d'Alembert's solution it follows that

$$u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds, \tag{7.77}$$

and we can see that the solution depends on all values of ψ inside the area of influence. This is an important observation. □

Example. As a more complex application of d'Alembert's solution we will see how the solution of the wave equation for a half-line looks like. We can think of an infinite (very long) string hooked at one end. Everyone is familiar with the experiment that illustrates our considerations - a garden hose with one end tied to a stake driven into

[37] For example, $v_t = cu_x$ and $u_t = -cv_x$.

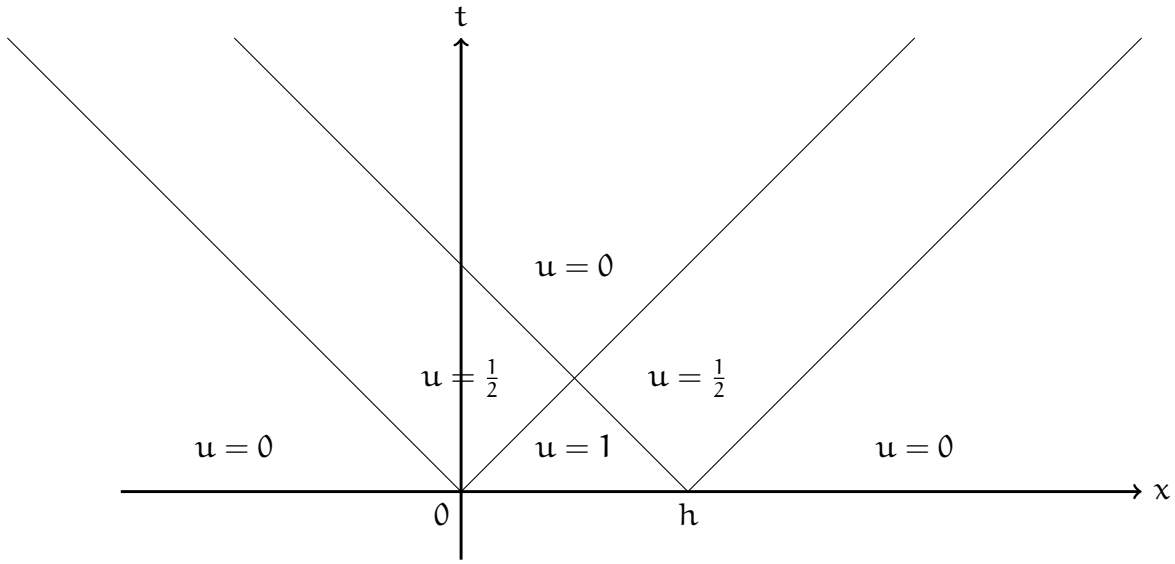

Figure 20: Spacetime diagram for wave equation on $\mathbb{R}$ with vanishing velocity and $\phi(x) = \chi_{[0,h]}(x)$ as in (7.75).

the ground, that is $u(0,t) = 0$. Suppose that the free end is forced by an impulse. We start with (7.68). Since the initial conditions are defined only for $x > 0$, by putting $t = 0$ we see that F and G can only take positive arguments. The argument of $G(x + ct)$ is always positive, this term remains unchanged. On the other hand, if $x > ct$, then of course $F(x - ct)$ is well defined by (7.72). However, the problem is the $x < ct$ area but we can use the boundary condition to obtain

$$0 = u(0,t) = F(-ct) + G(ct), \tag{7.78}$$

which gives $F(-z) = -G(z)$ for any $z > 0$. This means that F for negative arguments is an odd reflection of G. Therefore, for $x < ct$ we have

$$u(x,t) = F(x-ct)+G(x+ct) = -G(ct-x)+G(ct+x) = \frac{\phi(ct+x)-\phi(ct-x)}{2}+\frac{1}{2c}\int_{ct-x}^{ct+x}\psi(s)ds, \tag{7.79}$$

and for $x > ct$ the solution is given by the usual d'Alembert's formula. We can easily interpret this result. For $x > ct$ one part of the wave moves to the left until it is in the area $x < ct$ where it is reflected from the edge and begins to move to the right as $-G(ct - x)$. Thus, we see that just as with the heat equation the method of reflection works for the half-line - the initial condition is odd-reflected. A spacetime diagram for initial condition given by (7.75) is plotted on Fig. 21. Drawing it we have to remember that when a characteristic going to the left touches the $x = 0$ line it is reflected with a change sign of the value it carries. The value of the solution is then determined from the d'Alembert's formula (for our simple initial condition it is just the mean value). □

Example. Finally, let us see how d'Alembert's formula can be used to find a solution of the wave equation for a string attached on both endpoints. This problem can be solved without any difficulties by using the separation of variables, however d'Alembert's

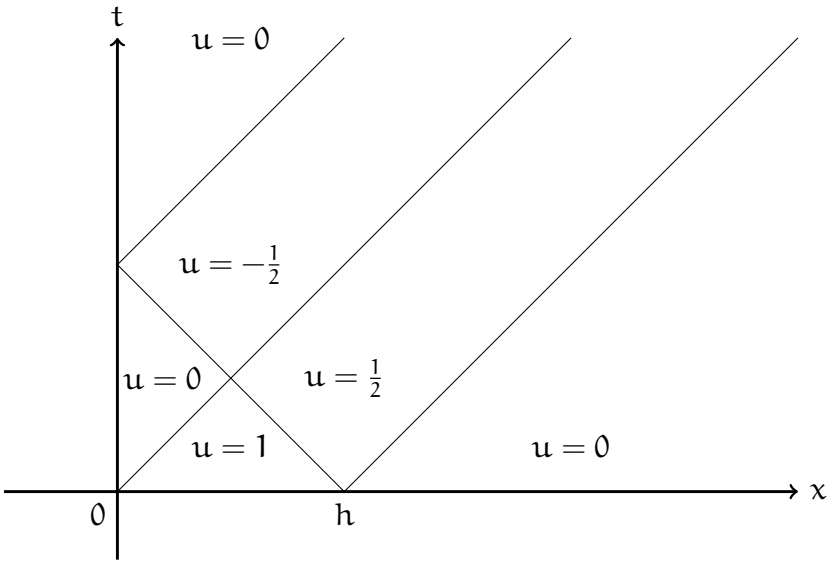

Figure 21: Spacetime diagram for wave equation on $\mathbb{R}_+$ with vanishing velocity and $\phi(x) = \chi_{[0,h]}(x)$ as in (7.75).

approach emphasizes the qualitative features of the wave much better. We therefore have

$$u(x,t) = F(x-ct) + G(x+ct). \tag{7.80}$$

If $x > ct$ and $x < L + ct$ the boundary conditions are not taken into account and hence the usual d'Alembert's solution is valid (7.73). However, when $x < ct$ or $x > L + ct$ we have to use the boundary conditions

$$0 = u(0,t) = F(-ct) + G(ct), \quad 0 = u(L,t) = F(L-ct) + G(L+ct). \tag{7.81}$$

The first one leads to $F(z) = -G(z)$, which implies $F(L+z) = F(L-z)$, therefore F is $2L$-periodic. A similar reasoning works for G, hence u is also $2L$-periodic. We see that in a way similar to the half-line we have a reflection at $x = 0$, but this time the wave is also reflected at $x = L$. The final solution is best seen in the space-time diagram. We see that if we only extend the initial conditions oddly with the period of $2L$, d'Alembert's solution for the whole straight line will be reduced to the solution on the interval $[0, L]$. This solution is much simpler and more revealing than solution given by the Fourier series[38]. The spacetime diagram for this problem is depicted on Fig. 22. Notice the $2L$-periodicity. □

7.4 Two and three dimensions

Very often, we want to find a solution to the wave equation for a signal propagating in each direction. This is an example of *spherical wave* that arises for example when sound is moving in a given medium. On the other hand, the light wave has a clearly defined direction of propagation and if it has a specific frequency it is an example of

[38]But they are the same - uniqueness!

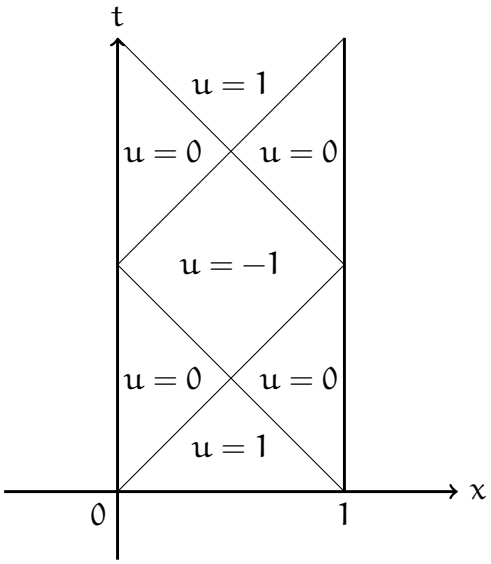

Figure 22: Spacetime diagram for wave equation on $[0, 1]$ with vanishing velocity and $\phi(x) = \chi_{[0,h]}(x)$ as in (7.75). Here, $L = 1$.

plane wave[39]. Let us consider both of these phenomena.

Example. (*Plane wave*) Consider the wave equation in $\mathbb{R}^3$

$$u_{tt} = c^2 \Delta u, \quad x \in \mathbb{R}^3. \tag{7.82}$$

Since the equation is linear with constant coefficients, we expect to find exponential solutions

$$u(x, t) = Ae^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \tag{7.83}$$

where A is a complex number (wave amplitude), $\mathbf{x} = (x, y, z)$, and $\mathbf{k} = (k_x, k_y, k_z)$. Looking for complex solutions of real equations is very practical since it simplifies calculations in a great extent. After them we always can take the real or imaginary part of the obtained solution and have a real function. Substituting (7.83) to (7.82) we get

$$\omega^2 = c^2 k^2, \tag{7.84}$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$. Therefore, we have obtained a generalization of the known result for one-dimensional wave - it is a dispersion relationship between wave frequency and wave number[40]. For one dimension we have $k = \frac{2\pi}{\lambda}$ which means that the wave number is the number of oscillations per unit space (times 2π). In addition, $\omega = \frac{2\,pi}{T} = \frac{2\pi c}{\lambda}$. Thus, $\omega = ck$. Our solution is therefore

$$u(x, t) = Ae^{i(\mathbf{k}\cdot\mathbf{x} \pm ckt)}. \tag{7.85}$$

Taking the real part we obtain

$$u(x, t) = A_r \cos(\mathbf{k}\cdot\mathbf{x} \pm ckt) - A_i \sin(\mathbf{k}\cdot\mathbf{x} \pm ckt), \tag{7.86}$$

[39]This is of course an idealization. In nature light consists of wave packets which, mathematically, are superpositions of waves of different frequency.

[40]Recall that from Physics 2

Where we have a decomposition $A = A_r + iA_i$. The wave vector $\mathbf{k}$ has a very important physical interpretation. From the solution we can see that the wave has exactly the same value on $(\mathbf{x}, t)$, for which $\mathbf{k} \cdot \mathbf{x} \pm ckt$ is constant and this implies

$$u = \text{const.} \quad \text{dla} \quad \mathbf{k} \cdot \mathbf{x} \pm ckt = C, \tag{7.87}$$

for some C. If we fix t, we get $\mathbf{k} \cdot \mathbf{x} = \mp ckt + C$, which is the equation of a plane with the normal vector $\mathbf{k}$. This is where the name of the plane wave comes from: its phase is constant on planes perpendicular to the wave vector. When the time t changes, the plane with a constant phase moves along the vector $\mathbf{k}$. Therefore, the wave moves into that direction. □

Example. (*Spherical wave*) Let us examine the spherical wave now. Assume that the solution of the equation (7.82) represents a wave that propagates in every direction (and not as it was previously - in a fixed one). This suggests introducing spherical coordinates

$$\frac{1}{c^2} u_{tt} = \frac{1}{r^2} \left(r^2 u_r \right)_r = u_{rr} + \frac{2}{r} u_r. \tag{7.88}$$

where we assumed that the solution is independent on both angles (since it is the same in every direction). Multiply the above equation by r

$$\frac{r}{c^2} u_{tt} = r u_{rr} + 2u_r = (ru)_{rr}, \tag{7.89}$$

since the left-hand side derivative is taken with respect to time we can write

$$(ru)_{tt} = c^2 (ru)_{rr}, \tag{7.90}$$

which has exactly the same form as one dimensional wave equation for a function ru. We now that its general solution is

$$u(r, t) = \frac{1}{r} \left[F(r - ct) + G(r + ct) \right]. \tag{7.91}$$

From here, it follows an important fact that the wave amplitude decays as $1/r$ while energy as $1/r^2$. □

The above result can be used to find a solution of the three dimensional wave equation with initial data[41]. Let u be a solution of

$$\begin{cases} u_{tt} = c^2 \Delta u, & t > 0, \quad \mathbf{x} \in \mathbb{R}^3, \\ u(\mathbf{x}, 0) = \phi(\mathbf{x}), \\ u_t(\mathbf{x}, 0) = \psi(\mathbf{x}). \end{cases} \tag{7.92}$$

We will use the *method of spherical means* and introduce another function $U = U(r)$ being the average value of u on a sphere with radius r and centre at a fixed point $\mathbf{x}_0$

$$U(r, t) := \frac{1}{4\pi r^2} \iint\limits_{\partial B(r, \mathbf{x}_0)} u \, dS. \tag{7.93}$$

[41] Another method is to use separation of variables in spherical coordinates.

We immediately have $U(0,t) = u(\mathbf{x}_0, t)$. Let us calculate the Laplacian of U. Since U does not depend on any angle we have

$$\Delta U = \frac{1}{r^2}\left(r^2 U_r\right)_r. \tag{7.94}$$

When we integrate (7.92) over a ball $B(r, \mathbf{x}_0)$ we obtain

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\iiint\limits_{B(r,\mathbf{x}_0)} u d\mathbf{x} = \iiint\limits_{B(r,\mathbf{x}_0)} \Delta u d\mathbf{x} = \iint\limits_{\partial B(r,\mathbf{x}_0)} \nabla u \cdot \mathbf{n} dS, \tag{7.95}$$

from Gauss-Ostrogradskii's (divergence) theorem. In the surface integral we can change the variable in order to deal with a sphere of unit radius

$$\begin{aligned}\iint\limits_{\partial B(r,\mathbf{x}_0)} \nabla u \cdot \mathbf{n} dS &= \iint\limits_{\partial B(r,\mathbf{x}_0)} \nabla u(\mathbf{x},t) \cdot \frac{\mathbf{x}-\mathbf{x}_0}{r} dS(\mathbf{x}) = r^2 \iint\limits_{\partial B(1,0)} \nabla u(\mathbf{x}_0 + r\mathbf{y}, t) \cdot \mathbf{y} dS(\mathbf{y}) \\ = r^2 \frac{\partial}{\partial r}\iint\limits_{\partial B(1,0)} u(\mathbf{x}_0 + r\mathbf{y}, t) dS(\mathbf{y}) &= 4\pi r^2 \frac{\partial}{\partial r}\left(\frac{1}{4\pi r^2}\iint\limits_{\partial B(r,\mathbf{x}_0)} u(\mathbf{x},t) dS(\mathbf{x})\right) = 4\pi r^2 U_r.\end{aligned} \tag{7.96}$$

where in the second to last equality we have returned to the original variable, hence the factor r^2. The left-hand side of (7.95) can be written as an integral over increasing family of spheres filling the whole ball $B(r, \mathbf{x}_0)$, that is we can use spherical coordinates to write

$$\iiint\limits_{B(r,\mathbf{x}_0)} u d\mathbf{x} = \int_0^r \rho^2 d\rho \iint\limits_{\partial B(\rho,\mathbf{x}_0)} u d\Omega = \int_0^r \rho^2 \left(\frac{1}{4\pi\rho^2}\iint\limits_{\partial B(\rho,\mathbf{x}_0)} u dS\right) d\rho = 4\pi \int_0^r \rho^2 U(\rho) d\rho, \tag{7.97}$$

since the surface and solid angle elements are related via $dS = \rho^2 d\Omega$. Combing the two above equations we arrive at

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_0^r \rho^2 U(\rho) d\rho = r^2 U_r. \tag{7.98}$$

To get rid of the integral we differentiate with respect to r obtaining

$$(rU)_{tt} = \frac{c^2}{r}\left(r^2 U_r\right)_r = \frac{c^2}{r}\left(2rU_r + r^2 U_{rr}\right) = c^2 (rU)_{rr}, \tag{7.99}$$

what shows that rU satisfies one-dimensional wave equation. It is exactly the same PDE as (7.88) however, set up on a half-line. For the boundary condition we have

$$\lim_{r\to 0^+} rU(r,t) = 0, \tag{7.100}$$

with initial conditions

$$rU(r,0) = \frac{1}{4\pi r}\iint_{\partial B(\mathbf{x},r)} \phi dS =: F(r), \quad rU_t(r,0) = \frac{1}{4\pi r}\iint_{\partial B(\mathbf{x},r)} \psi dS =: G(r). \tag{7.101}$$

Therefore, we can use d'Alembert's solution derived in (7.79) for a half-line to state

$$U(r,t) = \frac{F(ct+r) - F(ct-r)}{2r} + \frac{1}{2cr}\int_{ct-r}^{ct+r} G(s)ds \quad \text{for} \quad 0 < r \leq ct. \tag{7.102}$$

Now, we would like to return to the original solution by taking the limit

$$u(\mathbf{x}_0, t) = \lim_{r\to 0^+} U(r,t). \tag{7.103}$$

When we compute the limit in (7.102), suppress writing the subscript, and use

$$\lim_{r\to 0^+} \frac{F(ct+r) - F(ct-r)}{2r} = \frac{d}{dr}F(r)|_{r=ct} = \frac{1}{c}\frac{d}{dt}F(ct), \tag{7.104}$$

we obtain the following result first obtained by Poisson, however, known as Kirchhoff's solution.

Theorem 12 (Kirchhoff, Poisson)**.** *The solution of the three-dimensional wave equation with initial data (7.92) is given by*

$$u(\boldsymbol{x}, t) = \frac{1}{4\pi c}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\iint\limits_{B(\boldsymbol{x},ct)} \phi dS\right) + \frac{1}{t}\iint\limits_{B(\boldsymbol{x},ct)} \psi dS\right]. \tag{7.105}$$

Notice that the solution depends only on the value of initial conditions on the sphere and not on its interior. This is the so-called *Huygens principle*. This situation is completely different for two-dimensional case.

In order to derive the solution of a Cauchy problem for a two-dimensional wave equation we will use the Hadamard's *method of descent* which is based on utilizing three-dimensional solution for a initial data that does not depend on z. We thus solve

$$\begin{cases} u_{tt} = c^2\Delta u, & t > 0, \quad \mathbf{x} \in \mathbb{R}^2, \\ u(\mathbf{x}, 0) = \phi(\mathbf{x}), \\ u_t(\mathbf{x}, 0) = \psi(\mathbf{x}). \end{cases} \tag{7.106}$$

by introducing an auxiliary function with a dummy variable

$$\widetilde{u}(x, y, z, t) = u(x, y, t). \tag{7.107}$$

Then, $\widetilde{u}$ trivially satisfies three-dimensional wave equation and, hence, we can use (7.105). We only need to evaluate the surface integral keeping in mind that the initial data does not depend on z. A convenient parametrisation of the sphere is

$$\mathbf{r}(\rho, \theta) = \left(x + \rho\cos\theta, y + \rho\sin\theta, \pm\sqrt{(ct)^2 - \rho^2}\right), \quad \rho^2 = x^2 + y^2. \tag{7.108}$$

By computing the tangent vectors and taking their cross product we can obtain

$$dS = |\mathbf{r}_\rho \times \mathbf{r}_\theta| d\rho d\theta = \frac{ct\rho}{\sqrt{(ct)^2 - \rho^2}} d\rho d\theta. \tag{7.109}$$

Whence,

$$\iint\limits_{B(\boldsymbol{x},ct)} \phi dS = 2\int_0^{2\pi}\int_0^{ct} \phi(x+\rho\cos\theta, y+\rho\sin\theta)\frac{ct\rho}{\sqrt{(ct)^2-\rho^2}}d\rho d\theta. \tag{7.110}$$

We thus have proved the following corollary.

Corolary 2. *The solution of the two-dimensional wave equation with initial data (7.106) is given by*

$$\begin{aligned} u(\boldsymbol{x},t) = \frac{1}{2\pi}\Bigg[&\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{ct} \phi(x+\rho\cos\theta, y+\rho\sin\theta)\frac{\rho d\rho d\theta}{\sqrt{(ct)^2-\rho^2}} \\ &+\int_0^{2\pi}\int_0^{ct} \psi(x+\rho\cos\theta, y+\rho\sin\theta)\frac{\rho d\rho d\theta}{\sqrt{(ct)^2-\rho^2}}\Bigg]. \end{aligned} \tag{7.111}$$

We immediately notice a striking difference between three- and two-dimensional solutions of the wave equation. As we noticed, from (7.105) it follows that the solution depends only on values of initial condition on a *boundary* of a ball with radius ct. This means that On the other hand, (7.111) tells us that the wave uses all data *inside* a two-dimensional ball of radius ct. In other words, if the initial conditions are nonzero in some bounded region, then for all sufficiently long times the solution will also be nonzero. It can be shown that these phenomena remain true in higher even and odd dimensions, respectively. This two-dimensional behaviour can be observed in your bathtub in which a cork is floating in water. If you introduce a disturbance nearby, say you throw in a child's toy, after a while the cork starts to oscillate and does so for until the energy will be lost due to dissipation. Note also that if we lived in two dimensions we could not talk to each other since we would hear the sound for a (too) long time and one wave would interact with another. In three dimensions the acoustic wave has a sharp trailing edge noting its termination - we hear the sound and then the silence. It is a remarkable discovery and we can now see how it follows from the analysis of wave equation.

7.5 Green's functions (optional)

Similarly as with heat and Poisson's equations we can also write solutions to the wave equation as integrals over respective Green's function and its derivatives. We intuitively know that the Green's function is a solution to the respective linear PDE forces by point source in space and time. Added to that should be the respective condition of vanishing at the boundary. Note also that the Green's function should be causal, that is to vanish until the disturbance is introduces at some specific time.

Definition 15. *A* Green's function $G(\boldsymbol{x},t|\boldsymbol{x}_0,t_0)$ *for the wave equation is a solution of the following problem*

$$\begin{cases} G_{tt} = c^2\Delta_{\boldsymbol{x}}G + \delta(\boldsymbol{x}-\boldsymbol{x}_0)\delta(t-t_0), \quad t,t_0\in\mathbb{R}, \quad \boldsymbol{x},\boldsymbol{x}_0\in D\subseteq\mathbb{R}^n, \\ G\equiv 0, \quad \textit{for} \quad t>t_0, \\ G=0, \quad \textit{for} \quad \boldsymbol{x}\in\partial D. \end{cases} \tag{7.112}$$

Similarly as with Poisson's equation, by integrating the PDE defining Green's function and using Green's identities we can obtain an integral representation of the solution

$$\begin{cases} u_{tt} = c^2\Delta u + f(\mathbf{x},t), & t>0, \quad \mathbf{x}\in D\subseteq\mathbb{R}^n,\\ u(\mathbf{x},0)=\phi(\mathbf{x}),\\ u_t(\mathbf{x},0)=\psi(\mathbf{x}),\\ u(\mathbf{x},t)=g(\mathbf{x},t), & \text{for } \mathbf{x}\in\partial D. \end{cases} \tag{7.113}$$

in a form

$$\begin{aligned} u(\mathbf{x},t) = &\int_0^t\int_D G(\mathbf{x},t|\mathbf{x}_0,t_0)f(\mathbf{x}_0,t_0)\,d\mathbf{x}_0dt_0\\ &+\int_D\left(\psi(\mathbf{x}_0)G(\mathbf{x},t|\mathbf{x}_0,0)-\phi(\mathbf{x}_0,t)G_{t_0}(\mathbf{x},t|\mathbf{x}_0,0)\right)d\mathbf{x}_0\\ &-c^2\int_0^t\iint_{\partial D} g(\mathbf{x}_0,t_0)\nabla_{\mathbf{x}_0}G\cdot\mathbf{n}\,dS_0dt_0, \end{aligned} \tag{7.114}$$

where we can see that the various terms represent influences from different nonhomogeneities: source, initial conditions, and Dirichlet boundary data. One can spot many similarities between this and Green's function for Poisson's equation. For example, the boundary data enters the solution through a surface integral over the gradient of G.

As always, the Green's function for simple domains can be found by separation of variables. Here, we will consider unbounded domains $\mathbb{R}^n$ where $n=1,2,3$. Note that in that case a specific decay condition at infinity should be added.

It is beneficial to cast (7.112) into a different setting. Since it is easier to deal with initial value problems we will transform the above to such. We expect that a solution of a unforced wave equation with initial speed is equivalent to the above. Our motivation is a physical situation when a membrane is struck at time t_0. Initially there is no deflection but rather a sudden change in velocity. Therefore, we claim that

$$G(\mathbf{x},t|\mathbf{x}_0,t_0)=u(\mathbf{x}-\mathbf{x}_0,t)H(t-t_0), \tag{7.115}$$

where H is the Heaviside function, with

$$\begin{cases} u_{tt}=c^2\Delta_{\mathbf{x}}u, & t>t_0, \quad \mathbf{x},\mathbf{x}_0\in D\subseteq\mathbb{R}^n,\\ u(\mathbf{x},t_0|\mathbf{x}_0,t_0)=0,\\ u_t(\mathbf{x},0|\mathbf{x}_0,t_0)=\delta(\mathbf{x}-\mathbf{x}_0),\\ u=0, \quad \text{for} \quad \mathbf{x}\in\partial D. \end{cases} \tag{7.116}$$

To see this we just have to compute some derivatives. The spatial is straightforward

$$\Delta_{\mathbf{x}}G = H(t-t_0)\Delta_{\mathbf{x}}u, \tag{7.117}$$

while the temporal has to be computed in the language of distributions

$$\begin{aligned} G_t &= \delta(t-t_0)u + H(t-t_0)u_t\\ G_{tt} &= \delta'(t-t_0)u(\mathbf{x},t_0^+) + 2\delta(t-t_0)u_t(\mathbf{x},t_0^+) + H(t-t_0)u_{tt} = \Delta_{\mathbf{x}}G + \delta(\mathbf{x}-\mathbf{x}_0)\delta(t-t_0), \end{aligned} \tag{7.118}$$

where we have used the initial conditions. The fact that $H(t-t_0)u \equiv 0$ for $t < t_0$ is obvious. Therefore, in order to find the Green's function for a particular case we have to solve the initial value problem (7.116).

We start our revision of Green's functions for $D = \mathbb{R}^n$ with $n = 1, 2, 3$. For the one-dimensional wave equation we have the d'Alembert's solution (7.73) from which, after translating the initial time to t_0, we have

$$G(x,t|x,t_0) = \frac{1}{2c}\int_{x-c(t-t_0)}^{x+c(t-t_0)} \delta(\xi - x_0)\,d\xi = \frac{1}{2c}\begin{cases} 1, & x_0 \in (x-c(t-t_0), x+c(t-t_0)), \\ 0, & \text{otherwise.} \end{cases} \tag{7.119}$$

In short, we can write this as

$$G(x,t|x,t_0) = \frac{1}{2c}H(c(t-t_0) - |x-x_0|), \quad t > t_0. \tag{7.120}$$

We immediately see a striking difference between other evolution PDE that we know - heat equation. The Green's function for the latter - heat kernel - is infinitely differentiable. However, the above is not even continuous!

This situation becomes even more severe if we move into three dimensional space. In that case we have to use (7.105) with $\phi \equiv 0$ and $\psi = \delta(\mathbf{x} - \mathbf{x}_0)$. Without any lose of generality we take $t_0 = 0$ and obtain

$$G(\mathbf{x},t|\mathbf{x}_0,0) = \frac{1}{4\pi ct}\iint_{\partial B(\mathbf{x},ct)} \psi\,dS. \tag{7.121}$$

Now, we have to parametrize the surface integral and it is straightforward to introduce spherical coordinates centred at $\mathbf{x}$

$$G(\mathbf{x},t|\mathbf{x}_0,0) = \frac{1}{4\pi ct}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi(ct,\varphi,\theta)(ct)^2\cos\theta\,d\theta\,d\varphi. \tag{7.122}$$

At this point it is difficult to plug the Dirac delta and evaluate the double integral. However, we can artificially introduce an integral over the radius and use delta distribution to force us to stay on the sphere, i.e.

$$\begin{aligned} G(\mathbf{x},t|\mathbf{x}_0,0) &= \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\infty} \frac{\delta(r-ct)}{4\pi r}\psi(ct,\varphi,\theta)r^2\cos\theta\,dr\,d\theta\,d\varphi \\ &= \int_{\mathbb{R}^3} \frac{\delta(|\mathbf{x}-\mathbf{y}|-ct)}{4\pi|\mathbf{x}-\mathbf{y}|}\psi(\mathbf{y})\,d\mathbf{y}, \end{aligned} \tag{7.123}$$

where we have identified an integral over the whole three-dimensional space and used Cartesian coordinates. Finally, we can use the fact that $\psi(\mathbf{y}) = \delta(\mathbf{y} - \mathbf{x}_0)$ to write the Green's function

$$G(\mathbf{x},t|\mathbf{x}_0,t_0) = \frac{\delta(|\mathbf{x}-\mathbf{x}_0| - c(t-t_0))}{4\pi|\mathbf{x}-\mathbf{x}_0|}, \quad t > t_0. \tag{7.124}$$

Notice that the above is not even a function! Observe that the above formula describes a spherical wave with a very sharp front moving with a velocity c.

Lastly, we move to the two-dimensional case. Plugging $\psi \equiv 0$ and $\psi(x,y) = \delta(x-x_0)\delta(y-y_0)$ into (7.111) we get

$$\begin{aligned} G(x,y,t|x_0,y_0,t_0) &= \frac{1}{2\pi} \iiint\limits_{B((x,y),c(t-t_0))} \delta(\xi-x_0)\delta(\eta-y_0)\frac{d\xi d\eta}{\sqrt{(c(t-t_0))^2-((x-\xi)^2+(y-\eta)^2)}} \\ &= \frac{1}{2\pi\sqrt{(c(t-t_0))^2-((x-x_0)^2+(y-y_0)^2)}}, \quad t > t_0. \end{aligned} \tag{7.125}$$

This time, the Green's function is really a function however, with a singularity. Summing up, we see that (7.120), (7.125), and (7.124) are: a bounded (discontinuous) function, a singular (unbounded) function, and a distribution (not a function). The singularity of Green's function for wave equation becomes more severe if we move into higher dimensions. It can be shown that subsequent solutions will contain derivatives of Dirac delta which increase the lack of regularity. Once again, this has to be compared with heat equation when all fundamental solutions are infinitely smooth. Note that a difference of one temporal derivative order makes such a profound change in behaviour!

7.6 Uniqueness

We have seen how to solve wave equation in various situations. In some of them we were even able to write an explicit solution. However, in order to really know that we have obtained a physically meaningful solution we have to answer a question about uniqueness. For simplicity we will state the main result for one-dimensional equation with Dirichlet conditions and use the energy method. However, all of our calculations can be extended to higher dimensions.

Theorem 13. *A sufficiently smooth solution of*

$$\begin{cases} u_{tt} = c^2 u_{xx} + f(x,t), & t > 0, \quad x \in [0,L], \\ u(x,0) = \phi(x), & \\ u_t(x,0) = \psi(x), & \\ u(0,t) = \mu(t), & u(L,t) = \nu(t), \end{cases} \tag{7.126}$$

is unique.

Proof. By linearity it is sufficient to focus only on the case with homogeneous problem, i.e. with $f, \phi, \psi, \mu, \nu \equiv 0$ since we can always consider a difference of two solutions[42]. Let T be associated with kinetic energy

$$T(t) = \frac{1}{2}\int_0^L u_t^2 dx. \tag{7.127}$$

[42]Compare a similar argument for heat equation

Then, its change is

$$E'(t) = \int_0^L u_t u_{tt} dx = c^2 \int_0^L u_t u_{xx} dx = c^2 \left(u_t u_x|_{x=0}^{x=L} - \int_0^L u_{tx} u_x dx \right) = -c^2 \int_0^L u_{tx} u_x dx \tag{7.128}$$

where we have used the PDE, integrated by parts, and noticed that by boundary condition we have $u_t(x, 0, L) = 0$. Note that $(u_x^2)_t = u_x u_{tx}/2$ and hence

$$(T(t) + P(t))' = 0, \tag{7.129}$$

where the potential energy is

$$P(t) := \frac{1}{2} \int_0^L u_x^2 dx. \tag{7.130}$$

Therefore, the total energy is constant and equal to $T(t) + P(t) = T(0) + P(0) = 0$ by initial conditions. It follows that $u_t^2 + u_x^2 = 0$ which can take place only if $u \equiv 0$. □