Partial Differential Equations

Problem Set 1: First order equations and shock waves

1. Find all solutions $u = u(x,t)$ to the following PDEs

$$a)\ u_x = 0, \quad b)\ 3u_t + u_{xt} = 0, \quad c)\ \frac{\partial^n u}{\partial x^n} = 0,\ n \in \mathbb{N}, \quad d)\ u_{xx} + u = 0, \quad e)\ u_{xt} = 0.$$

2. Which of the following operators are linear?

$$a)\ Lu = u_x + xu_t, \quad b)\ Lu = u_x + uu_t, \quad c)\ Lu = u_t + u_x + 1, \quad d)\ Lu = u_x^2 + u_y^2 - 1.$$

3. (*Geometrical method of solution*) The quantity $au_x + bu_y$ is the directional derivative of u in the direction of a vector $\mathbf{v} = (a,b) \in \mathbb{R}^2$ with $a^2 + b^2 \neq 0$. Hence, the equation

$$au_x + bu_y = 0,$$

tells us that u does not change in the direction of $\mathbf{v}$. We will show how to solve the above by using only geometrical means.

   (a) Argue that if u is constant on lines generated by $\mathbf{v}$ then it should change on any line perpendicular to this vector. Draw everything on the $x-y$ plane and prove that the solution of the above equation is given by $u(x,y) = f(bx - ay)$ for some differentiable f.

   (b) We can find the same solution as before in a more systematic way. The idea is to rotate the coordinate plane. Let the new coordinates (x', y') be defined by $x' = ax + by$ and $y' = bx - ay$. By the chain rule calculate u_x and u_y in terms of $u_{x'}$ and $u_{y'}$. Show that if u is a solution of the above PDE then $u_{x'} = 0$ which is easy to solve.

4. Solve the following equations using the Problem 3

$$a)\ 4u_x - 3u_y = 0 \ \text{ with } \ u(0,y) = y^3, \quad b)\ 2u_t + 3u_x = 0 \ \text{ with } \ u(x,0) = \sin x.$$

5. Use the rotation of coordinates as in Problem 3 to solve the following equation

$$au_x + bu_y + cu = 0.$$

6. (*Conservation*) Let $u = u(x,t)$ denote the density per unit length of some substance, which satisfies the conservation law without any sources. Show that, provided the flux q vanishes at infinity and u is x-integrable, the total amount of u in space is constant in time. *Hint.* The total amount of u is defined by some integral. Which one?

7. (*Method of characteristics*) Solve the following problems. Draw their characteristics on the $x-t$ plane and sketch the solution $u(x,t)$ for several times.

$$a) \begin{cases} u_t + 10u_x = 0, & t > 0, \quad x \in \mathbb{R}; \\ u(x,0) = \frac{1}{1+x^2} & x \in \mathbb{R}, \end{cases} \qquad b) \begin{cases} u_t + 2tu_x = -u, & t > 0, \quad x \in \mathbb{R}; \\ u(x,0) = e^{-x^2}, & x \in \mathbb{R}. \end{cases}$$

$$c) \begin{cases} u_t - x^2 u_x = \sin u, & t > 0, \quad x > 0; \\ u(x,0) = 2\arctan x, & x \geq 0, \end{cases} \qquad d) \begin{cases} u_t + 2tu_x = xtu, & t > 0, \quad x \in \mathbb{R}; \\ u(x,0) = x, & x \in \mathbb{R}. \end{cases}$$

8. (*Signaling problem*) Suppose that a lighthouse is located at $x = 0$ and constantly sends its signals with the intensity $\psi(t)$.

(a) As you know, light travels through any medium with a constant speed c. Let $u = u(x,t)$ denote the light intensity at (x,t). Find $u(x,t)$ for $x > 0$ and $t \in \mathbb{R}$ (lighthouse has been sending its signals for ever).

(b) Solve the above problem but now assume that the light is absorbed by clouds and other aerosols with a rate proportional to its intensity.

9. (*Initial-boundary value problems*) Sometimes we have to solve a problem where both the initial and boundary data is given. Consider the following equations.

$$a)\begin{cases} u_t + c\,u_x = 0, & t>0, \quad x>0, \quad c>0; \\ u(x,0) = \phi(x), & x>0; \\ u(0,t) = \psi(t), & t>0, \quad \phi(0)=\psi(0)=0, \end{cases} \qquad b)\begin{cases} u_t + tu_x = -u^2, & t>0, \quad x>0; \\ u(x,0) = x, & x>0; \\ u(0,t) = \sin t, & t>0, \end{cases}$$

What conditions have to be satisfied for the $a)$ case to have a solution with $c < 0$?

10. (*Quasi-linear equations*) Solve the following problems and sketch the characteristics.

$$a)\begin{cases} u_t + \ln u\, u_x = 0, & t>0, \quad x\in\mathbb{R}; \\ u(x,0) = e^x, & x\in\mathbb{R}, \end{cases} \qquad b)\begin{cases} u_t + u_x = u^2, & t\in\mathbb{R}, \quad x>0; \\ u(0,t) = \psi(t), & t\in\mathbb{R}. \end{cases}$$

$$c)\begin{cases} u_t + u^2u_x = 0, & t>0, \quad x\in\mathbb{R}; \\ u(x,0) = x, & x\geq 0, \end{cases} \qquad d)\begin{cases} u_t + u^{-1}u_x = 0, & t>0, \quad x\geq 0; \\ u(x,0) = \frac{1}{1+x}, & x\geq 0. \end{cases}$$

11. Solve the following full quasi-linear problem.

$$a)\begin{cases} u_t + uu_x = -u, & t>0, \quad x\in\mathbb{R}; \\ u(x,0) = x, & x\in\mathbb{R}. \end{cases} \qquad b)\begin{cases} u_t + uu_x = 2t, & t>0, \quad x\in\mathbb{R}; \\ u(x,0) = x, & x\in\mathbb{R}. \end{cases}$$

12. (*Shock waves*) Find the solutions of given problems. Draw the characteristics and a shock wave trajectory.

$$a)\begin{cases} u_t + u^2u_x = 0, \quad t>0, \quad x\in\mathbb{R}; \\ u(x,0) = \begin{cases} 2, & x<0; \\ 1, & 0>0, \end{cases} \end{cases} \qquad b)\begin{cases} u_t + 2uu_x = 0, \quad t>0, \quad x\in\mathbb{R}; \\ u(x,0) = \begin{cases} 3, & x<0; \\ 2, & x>0, \end{cases} \end{cases}$$

13. (*Shock fitting*) Analyse the following shock-fitting problems. Draw the solution $u(x,t)$ for several times.

$$a)\begin{cases} u_t + uu_x = 0, \quad t>0, \quad x\in\mathbb{R}; \\ u(x,0) = \begin{cases} 1, & x<0; \\ -1, & 0<x<1; \\ 0, & x>1, \end{cases} \end{cases} \qquad b)\begin{cases} u_t + uu_x = 0, \quad t>0, \quad x\in\mathbb{R}; \\ u(x,0) = \begin{cases} 1, & x<0; \\ 1-x, & 0<x<1; \\ 0, & x>1. \end{cases} \end{cases}$$

$$c)\begin{cases} u_t + u^2u_x = 0, \quad t>0, \quad x\in\mathbb{R}; \\ u(x,0) = \begin{cases} 0, & x<0; \\ 1-x, & 0<x<1; \\ 0, & x>1. \end{cases} \end{cases} \qquad d)\begin{cases} u_t + u^2u_x = 0, \quad t>0, \quad x\in\mathbb{R}; \\ u(x,0) = \begin{cases} 0, & x<0; \\ x, & 0<x<1; \\ 0, & x>1. \end{cases} \end{cases}$$

14. (*Paint*) The paint flows down the wall and has a thickness $u(x,t)$. As can be shown, the governing equation has the form

$$u_t + u^2u_x = 0, \quad t>0, \quad x\in\mathbb{R}.$$

Solve the paint flow problem with the given initial profile

$$u(x,0) = \begin{cases} 0, & x<0 \text{ or } x>1; \\ 1, & 0<x<1. \end{cases}$$

15. (*Car traffic*) Let $u = u(x,t)$ be a density of cars per unit length on a road at time t. Define the traffic flux $q(x,t)$ as the number of cars per unit time passing through a fixed x at time t.

   (a) Argue that the conservation of cars leads to the equation

$$u_t + q_x = 0.$$

   (b) Define the car velocity as $v = q/u$ (as in the example with convection). A simple model of velocity distribution assumes that there exists a critical density at which cars stop (traffic jam)

$$v(u) = v_m \left(1 - \frac{u}{u_c}\right).$$

   From this, find q and plug it in the conversation of cars to obtain the governing equation.

   (c) (*Red light turns green*) Solve the car flow problem for a case where the red light at $x = 0$ turns green at $t = 0$, i.e.

$$u(x,0) = \begin{cases} u_c, & x < 0; \\ 0, & x > 0. \end{cases}$$

   (d) (*Green light turns red*) Now, a red light suddenly turns on and the traffic with a density u_0 suddenly have to stop at $x = 0$.

$$u(x,0) = u_0 < u_c, \quad x \leq 0, \quad u(0,t) = u_c, \quad t > 0.$$

   Consider only the domain $x \leq 0$.

16. (*Continuation of the discontinuity*) Solve the following equation

$$u_t + uu_x = 0, \quad t > 0, \quad x \in \mathbb{R},$$

for each of the two initial conditions

$$a)\ u(x,0) = \Phi_1(x) := \begin{cases} 1, & x \leq 0; \\ 1 - \frac{x}{\epsilon}, & 0 < x < \epsilon; \\ 0, & x \geq \epsilon, \end{cases} \qquad b)\ u(x,0) = \Phi_2(x) := \begin{cases} 0, & x \leq 0; \\ \frac{x}{\epsilon}, & 0 < x < \epsilon; \\ 1, & x \geq \epsilon. \end{cases}$$

In both cases examine the limit $\epsilon \to 0$.

17. (*Flood hydrograph*) Imagine a long and narrow river. We can introduce a curvilinear variable x which denotes the position measured along its bed. Moreover, let $A = A(x)$ be the wetted cross-section at x. Conservation of mass along with Manning's Law states that

$$A_t + A^m A_x = 0, \quad m > 0.$$

Assume also that $A(x,0) = A_0(x)$.

   (a) Find a solution of the above equation given in an implicit form.

   (b) A flood can arise as a result of a sudden rainfall at some point x. To model this situation assume that $A_0(x) = \delta(x)$, where δ is the Dirac delta[1]. Draw the characteristics.

   (c) Find the shock wave. You do not have to solve the equation in order to do that.

   (d) A flood hydrograph is a graph of the flux at some fixed point x_0. Hydrologists use this tool a lot. Draw the hydrograph by yourself.

[1] If you do not feel comfortable with distributions, think about some narrow and high spike of unit mass such as $\frac{1}{\epsilon}\chi_{(-\epsilon/2,\epsilon/2)}(x)$ or $\frac{1}{\epsilon}\varphi(x/\epsilon)$, where φ is any integrable positive function with $\int |\varphi| dx = 1$.

18. (*Blow-up time*) For a general initial condition it is not usually straightforward to determine the shock wave development. It is, however, possible to find the first time of a blow-up. Consider

$$\begin{cases} u_t + c(u)u_x = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(x,0) = \phi(x) \end{cases},$$

where $c > 0$ and $\phi > 0$ have an opposite monotonicity (say, $c' > 0$ and $\phi' < 0$).

(a) Pick any characteristic $X = X(t)$ and define $P(t) := u_x(X(t), t)$. Compute P' in terms of u and its derivatives.

(b) Use the differential equation for u in order to get rid of the second derivatives in the expression for P'.

(c) Solve the obtained equation for P and whence, for u_x.

(d) Define $F(\xi) := c(\phi(\xi))$ and show that the first time t_b for which u_x becomes infinite is

$$t_b = \frac{1}{\max_\xi |F'(\xi)|}.$$

19. Find the first time of a blow-up for the following problems.

$$\text{a)} \begin{cases} u_t + uu_x = 0, & t > 0, \quad x \in \mathbb{R}; \\ u(x,0) = e^{-x^2} \end{cases} \qquad \text{b)} \begin{cases} u_t + uu_x = 0, & t > 0, \quad x \in \mathbb{R}; \\ u(x,0) = \frac{1}{\cosh x^2}. \end{cases}$$

$$\text{c)} \begin{cases} u_t + uu_x = 0, & t > 0, \quad x \in \mathbb{R}; \\ u(x,0) = \begin{cases} 2 - x^2, & x < 1; \\ 1, & x > 1. \end{cases} \end{cases} \qquad \text{d)} \begin{cases} u_t + uu_x = 0, & t > 0, \quad x \in \mathbb{R}; \\ u(x,0) = \begin{cases} 1, & x \le 0; \\ \cos x, & 0 < x \le \frac{\pi}{2}; \\ 0, & x > \frac{\pi}{2}. \end{cases} \end{cases}$$

Łukasz Płociniczak