Partial Differential Equations

Problem Set 2: Fourier's method (separation of variables)

This is a collection of problems concerning separation of variables for constant coefficient PDEs. Many of the considered equations will be derived, explained, and investigated in the further part of the semester. Here, you will only practice solving them. Unless stated otherwise we will always assume that the initial condition is

$$u(x,0) = \varphi(x),$$

and, if needed for second order in time equation,

$$u_t(x,0) = \psi(x).$$

Depending on the time, we do not have to solve each of the following problems before moving to the next list. We can go back to them when working through the lists concerning a specific type of the PDE: heat, Poisson, and wave. Do not worry that you do not fully understand the interpretation of these PDEs yet. Problems with an exclamation mark (!) are compulsory to solve.

0. (*Orthogonal expansion*) Let $(\phi_n)_n$ be a set of orthogonal functions with respect to the inner product $\langle \cdot, \cdot \rangle$. Recall that for every sufficiently regular function we can associate the series expansion

   $$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x).$$

   We write " $\sim$ " instead of " $=$ " inasmuch we need some advanced methods to prove the convergence. We will return to this issue later.

   By using standard orthogonality argument known from algebra show that

   $$a_n = \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2},$$

   where the norm is defined as $\|u\| = \sqrt{\langle u, u \rangle}$. How to normalize the orthogonal set, that is how to redefine ϕ_n to have $\|\phi_n\| = 1$?

Heat equation

1. (*Heat equation*) (!) Solve the homogeneous heat equation $u_t = u_{xx}$ with the following boundary conditions. In each case compute the exact Fourier coefficients for $\phi(x) = U_0 = \text{const}$. This problems models heat conduction in an insulated bar of length L.

   a) $u(0,t) = 0; \quad u_x(L,t) = 0,$ b) $u_x(0,t) = 0; \quad u_x(L,t) = 0,$
   c) $u_x(0,t) = 0; \quad u(L,t) = 0,$ d) $u_x(0,t) = -hu(0,t); \quad u(L,t) = 0.$
   e) $u_x(0,t) = -hu(0,t); \quad u_x(L,t) = 0,$ f) $u_x(0,t) = -hu(0,t); \quad u_x(L,t) = -ku(L,t).$

2. (*Periodic boundary conditions*) Suppose that a homogeneous thin wire is bent into a circular ring of length $2L$. Assume that its surface is insulated and formulate a boundary value problem modelling temperature distribution in the ring by using the heat equation $u_t = u_{xx}$. Finally, solve the stated problem.

Laplace's equation

3. (*Laplace equation*) Let $u = u(x, y)$ be the electrostatic potential. During the lecture you will learn that in the absence of the sources it satisfies the following Laplace's equation

$$\Delta u = u_{xx} + u_{yy} = 0.$$

Solve the above PDF on a square $(x, y) \in (0, 1)\times(0, 1)$ with the following boundary conditions (the measurement of the potential or its flux at one side of the rectangle)

$$a) \begin{cases} u(x,0) = f(x), & x \in [0,1]; \\ u(x,1) = 0, & x \in [0,1]; \\ u(0,y) = 0, & y \in [0,1]; \\ u(1,y) = 0, & y \in [0,1], \end{cases} \qquad b) \begin{cases} u_y(x,0) = g(x), & x \in [0,1]; \\ u(x,1) = 0, & x \in [0,1]; \\ u(0,y) = 0, & y \in [0,1]; \\ u(1,y) = 0, & y \in [0,1]. \end{cases}$$

4. (*Potential of a disc*) (!) A unit disc has been charged with electricity. We will find its potential, that is a function $u = u(r, \theta)$ where r is the radius, and θ the angle (it is easier to work in polar coordinates). The potential satisfies the Laplace's equation written in polar coordinates

$$\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

Use separation of variables $u(r, \theta) = R(r)\Theta(\theta)$ to solve the above PDE with the periodicity conditions

$$u(r, 0) = u(r, 2\pi), \quad u_\theta(r, 0) = u_\theta(r, 2\pi),$$

and the boundedness condition $\lim_{r\to 0^+} |u(r, \theta)| < \infty$.

Hint. In the process you will meet Euler's ODE: $ax^2y'' + bxy' + cy = 0$. Consult lecture notes, your ODE course, or the literature to recall how it can be solved.

5. (*Potential of a quarter of a pizza*) Solve Laplace equation on a quarter of a circle, i.e.

$$\begin{cases} \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = 0, & 0 < r < 1,\ 0 \le \theta \le \frac{\pi}{2}, \\ u_r(1,\theta) = f(\theta), & 0 \le \theta \le \frac{\pi}{2}, \\ u(r,0) = 0, \quad u(r,\frac{\pi}{2}) = 0, & 0 \le r \le 1. \end{cases}$$

6. (*Potential of a ring*) Find a solution of the Laplace equation on an annulus

$$\begin{cases} \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = 0, & a < r < b,\ -\pi \le \theta \le \pi, \\ u(a,\theta) = 0, \quad u(b,\theta) = f(\theta), & -\pi \le \theta \le \pi. \end{cases}$$

7. (*Potential of a punctured plate*) Use a separation of variables to solve the Laplace equation on the *exterior* of an unit circle, i.e.

$$\begin{cases} \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta} = 0, & r > 1,\ -\pi \le \theta \le \pi, \\ u(1,\theta) = f(\theta), & -\pi \le \theta \le \pi. \end{cases}$$

Wave equation

8. (*Guitar string*) (!) Vibrations of the guitar string with length $L > 0$ are described by the wave equation
$$u_{tt} = c^2 u_{xx}, \quad x \in (0, L),$$
with boundary conditions
$$u(0,t) = u(L,t) = 0.$$
Use separation of variables to solve this problem with a typical initial condition for plucking the string at $x = x_0$ (draw a picture)
$$u(x,0) = \varphi(x) = A \begin{cases} \dfrac{x}{x_0}, & 0 \leq x < x_0, \\ \dfrac{L-x}{L-x_0}, & x_0 \leq x \leq L, \end{cases} \qquad u_t(x,0) = \psi(x) = 0,$$
where A is the amplitude. It would be great to use a computer to visualize these vibrations.

9. (*Vibrating loop*) Find the oscillations of the vibrating loop satisfying wave equation $u_{tt} = u_{xx}$ with periodic boundary conditions
$$u(0,t) = u(2\pi, t), \quad u_x(0,t) = u_x(2\pi, t).$$

10. (*Friction*) In real-world situation we always have to deal with the frictional loss of energy. A typical example is the damped wave equation
$$u_{tt} = c^2 u_{xx} - \beta u_t, \quad x \in (0, L),$$
where $\beta > 0$ is the damping coefficient. Solve the above with $u(x,0) = u(L,0) = 0$ and the initial condition $u(x,0) = A$. What happens with the solution after a long time?

11. (*String with a loaded endpoint*) When solving a boundary value problem of a string having a mass m attached to one of its end point one has to solve the wave equation
$$u_{tt} = c^2 u_{xx}, \quad x \in (0, L),$$
with the boundary conditions
$$u(0,t) = 0, \quad m u_{tt}(L,t) = -k u_x(L,t).$$
Use the separation of variables to solve this problem.

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