Applications of fractional derivatives

Problem List

1. Derive the formula for a n-th integral of a function

$$\int_a^x \int_a^{x_1} \dots \int_a^{x_{n-1}} f(x_n) dx_1 dx_2 \dots dx_n = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt.$$

2. Show that $I_a^\alpha f(t) \to f(t)$ pointwise as $\alpha \to 0$ for $f \in C^1$ (can you do that if $f \in C$ only?).

3. Prove that the fractional integral possesses the semi-group property

$$I_a^\alpha I_a^\beta f(x) = I_a^{\alpha+\beta} f(x).$$

   Hint: You will need the Beta function.

4. Argue that the RL derivative is a right-inverse of the fractional integral

$$D_a^\alpha I_a^\alpha f(x) = f(x).$$

   Show by example that it is not a left-inverse.

5. Obtain the formulas for $I_0^\alpha x^\mu$, $D_0^\alpha x^\mu$ and ${}^C D_0^\alpha x^\mu$. What are the assumptions on μ?

6. Devise an example showing that the formula for the derivative of a product does not hold for RL derivative, i.e.

$$D_a^\alpha (f \cdot g)(x) \neq f(x) D_a^\alpha g(x) + g(x) D_a^\alpha f(x),$$

   in general for $\alpha \notin \mathbb{N}$. Do the same for the derivative of a composite function.

7. (*Computer*) Implement the Mittag-Leffler function (for our purposes the series representation is sufficient) and investigate how it behaves for positive and negative arguments. Check whether it reduces to the known elementary functions noted in the lecture.

8. (*Computer*) Plot the Mittag-Leffler function $E_\alpha(-t^\alpha)$ for different $0 < \alpha \leq 2$. How many zeros does it posses? What if $\alpha \to 2$? How does it compare to the classical case?

9. Obtain a solution to the following problem by the Laplace transform method

$${}^C D_0^\alpha y = \lambda y + f, \quad y(0) = y_0, \quad 0 < \alpha \leq 1.$$

10. (*Computer*) Implement the finite-difference method for computing the fractional integral I_0^α. Check it on the well-known formulas for integrals of the power functions.

11. (*Computer*) Write your own solver for fractional differential equations with Caputo derivative

$${}^C D_0^\alpha y = f(y,t), \quad y(0) = y_0, \quad 0 < \alpha \leq 1.$$

    Check your solutions with the theoretical ones for $f(y,t) = \lambda y$.

12. (*Computer*) Solve numerically the time-fractional diffusion equation

$${}^C \partial_t^\alpha u = u_{xx},$$

    with $u(x,0) = 0$, $u(0,t) = 1$ (you can use any scheme you want, for ex. explicit, implicit, Crank-Nicolson, ...). Compare it with the exact solution and the classical case.

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