Fixed point theorems for topological contractions

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Theorem (Banach fixed-point theorem, 1920)

Every Lipschitz contraction on complete metric space has unique fixed point.

Here $f : X \to X$ is a Lipschitz contraction iff existst $c \in [0, 1)$ s.t. for every $x, y \in X$

$$d(f(x), f(y)) \leq c \cdot d(x, y).$$

Topological contraction

Definition

Let X be a T_1-topological space and $f : X \to X$.
We say that f is a topological contraction on X iff for every open cover $\mathcal{U}$ of X there are $U \in \mathcal{U}$ and $n \in \omega$ s.t. $f^n[X] \subseteq U$

Theorem (Lebesgue number)

For every compact metric space, X and any open cover $\mathcal{U}$ there exists $\epsilon > 0$ s.t.

$$\forall x \in X \exists U \in \mathcal{U} \; B(x, \epsilon) \subseteq U.$$

Fact

Every Lipschitz contraction on a compact metric space is a topological contraction.

Fixed point theorem for compact T_1 spaces

Theorem

Let X be T_1 compact topological space and $f : X \to X$ be a closed topological contraction on X. Then there exsists an unique $x \in X$ s.t. $x = f(x)$.

Corollary

Every Lipschitz contraction on compact metric space has unique fixed point.

Example

Let (ω, τ) be T_1 topological space where

$$\tau = \{\emptyset\} \cup \{A \in \mathscr{P}(\omega) : A^c \text{ is finite } \}.$$

Then $\omega \ni n \mapsto f(n) = n + 1 \in \omega$ is a continuous, topological contraction without any fixed point, (f is not closed map !!!).

Lipschitz contraction is continuous but topological not neccessary.

Example

Let $X = \{1/n : n \in \mathbb{N}\} \cup \{0, 2, 3\}$ be endowed with the usual Euclidean metric from the real line. Let for $x \in X$:

$$f(x) := \begin{cases} 2 & \text{if} \quad x = 1/n, \\ 3 & \text{if} \quad x = 0, 2, 3. \end{cases}$$

The mapping f is a closed topological contraction because $f^2[X] = \{3\}$; it is closed because $f[X] = \{2, 3\}$; and it is not continuous because

$$f\left(\lim_n \frac{1}{n}\right) = f(0) = 3 \neq 2 = \lim_n f\left(\frac{1}{n}\right).$$

Here fixed point here is 3. Moreover, $f \subseteq X \times X$ is not closed set.

Weak Čech completeness

Definition

Tychonoff topological space X is Čech complete if

- exists $\{\mathcal{U}_i : i \in \omega\}$, $\mathcal{U}_i$ - open cover of X for $i \in \omega$,
- for every centered $\{F_m \in Clo(X) : m \in \omega\}$ s.t. $\forall i \in \omega\, \exists m \in \omega\, \exists U \in \mathcal{U}_i\ F_n \subseteq U$

then $\bigcap\{F_m \;\; m \in \omega\} \neq \emptyset$.

If we drop assumption that X is Tychonoff space then X is weak Čech complete.

Theorem

If X is a T_1 weak Čech complete space and $f : X \to X$ is a topological contraction, then f has a unique fixed point.

Weak contraction (or feebly topologically contractive)

Definition (Kupka)

Let X - topological space, then $f : X \to X$ is weak topological contraction if for each open cover $\mathcal{U}$ we have

$$\forall x, y \in X \, \exists n \in \omega \, \exists U \in \mathcal{U} \;\; f[\{x, y\}] \subseteq U$$

Theorem (Kupka, 1998)

If X top. space $f : X \to X$ s.t.

- *f has closed graph,*
- *f is weak top. contraction*

then f has fixed point. Moreover, if X is T_1 then fixed point is unique.

Corollary

If X is a Hausdorff topological space and f is a continuous weak topological contraction on X, then f has a unique fixed point.

Theorem

If X is a Hausdorff first-countable topological space and f is a closed weak topological contraction on X, then f has a unique fixed point.

Theorem

If X is a Hausdorff topological space, $f : X \to X$ is a continuous mapping, there exists $x_0 \in X$ such that for each open cover $\mathcal{U}$ of X there is $n \in \mathbb{N}_0$ such that $\{f^n(x_0), f^{n+1}(x_0)\} \subseteq U$ for some $U \in \mathcal{U}$, then f has a fixed point.

Definition (Atsuji space)

Complete metric space is Atsuji if Lebesgue number Theorem is true.

Corollary (Beer)

Let (X, d) be an Atsuji space and $f : X \to X$ be a continuous (or closed) mapping. If there exists an $x_0 \in X$ such that $\liminf_{n\to\infty} d(f^n(x_0), f^{n+1}(x_0)) = 0$, *then f has a fixed point.*

Lacally Hausdorff space

Definition

A topological space X is *locally Hausdorff* if every point of the space has an open neighbourhood U such that the topology of X restricted to U is Hausdorff.

Theorem

If X is a locally Hausdorff T_1 topological space and f is a continuous weak topological contraction on X, then f has a unique fixed point.

Peripherally Hausdorff space

Definition

For every $\alpha \in On$ define a class $\mathcal{F}_\alpha$ as follows: for every T_1 topological space X, we say that $X \in \mathcal{F}_\alpha$ is α-Hausdorff space if

if $\alpha = 0$ then $X = \{x\}$ and,

if $\alpha > 0$ then $\forall x \in X \exists \beta < \alpha \ [x] \in \mathcal{F}_\beta$ where

$$[x] = \bigcap \{cl(U) : \ x \in U - \text{ is open in } X\}.$$

We say that X is peripherally Hausdorff iff $\exists \alpha \in On \, X \in \mathcal{F}_\alpha$,
We have

- If $\beta \leq \alpha$ then $\mathcal{F}_\beta \subseteq \mathcal{F}_\alpha$,
- $X \in \mathcal{F}_1$ iff X is a Hausdorff space.

Definition (Hausdorff rank)

Let X-peripherally Hausdorff space, define Hausdorff rank of X

$$rank_H(X) = \min\{\alpha \in On : \ X \in \mathcal{F}_\alpha\}.$$

Theorem

For every $\alpha \in On$ there is X-peripherally Hausdorff space s.t. $\alpha \leq rank_H(X)$.

Proposition

If $(X, \tau(X)) \in \mathcal{F}_\alpha$ and $Y \subseteq X$ is nonempty then $(Y, \tau(Y)) \in \mathcal{F}_\alpha$, where $\tau(Y) = \tau(X \upharpoonright Y) = \{U \cap Y : \ U \in \tau(X)\}$.

Here we used transfinite induction and $[x]_Y \subseteq [x]_X$ and $\tau([x]_X \upharpoonright [x]_Y)) = \tau([x]_Y))$ where $\tau([x]_X) = \{U \cap [x]_X : U \in \tau(X)\}$ and $\tau([x]_Y) = \{U \cap [x]_Y : U \in \tau(Y)\}$.

Theorem

If X, Y are peripherally Hausdorff spaces then

$$rank_H(X \times Y) = \max\{rank_H(X), rank_H(Y)\}.$$

Weak$^+$ topological contraction

Definition

Let X - topological space, then $f : X \to X$ is weak$^+$ topological contraction if for each open cover $\mathcal{U}$ we have

$$\forall x, y \in X \, \exists U \in \mathcal{U} \, \forall^\infty n \in \omega \ \ f^n[\{x, y\}] \subseteq U$$

Theorem

For every peripherally Hausdorff space X, every continuous weak$^+$ topological contraction on X has unique fixed-point.

Example

$$X = \{-1\} \cup [0,1].$$

Let the base of X consist of all sets of the form:

- $J \cap [0,1]$, where J is an open interval, and
- $((L \setminus \{0\}) \cap X) \cup \{-1\}$, where L is an open interval containing 0.

Let $f : X \to X$ be defined by

$$f(x) = \frac{1}{2} \cdot x \quad \text{where } x \in [0,1] \quad \text{and } f(-1) = 0.$$

Then X is a compact peripherally Hausdorff (in fact 2-Hausdorff) space and f is a continuous weak$^+$ contraction but $f \subseteq X \times X$ is not closed. Of course, the point 0 is a fixed point of f.

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Weak* topologies

Theorem

Let X be a linear topological space. Let V be a neighbourhood of the zero vector in X. We define Y as

$$Y := \{x^* \in X^* : |x(x^*)| \leq 1, \ \text{for each} \ x \in U\}$$

Let $f : Y \to Y$ be a weak-continuous mapping satisfying*

$$\lim_n |z(f^n(x^*) - f^n(y^*))| = 0$$

for every $z \in X$
Then f has a unique fixed point in Y.

We use the dual notation: $x(x^*) := x^*(x)$ for functionals x^* which are members of X^* and elements x of the space X.

Compact semigroups

Theorem

- *G is a Hausdorff compact topological monoid and*
- *$f : G \to G$ is a continuous mapping such that for each $x, y \in G$ and each neighbourhood V of the neutral element there exist $z \in G$ and*
- *$n \in \mathbb{N}$ such that $f^n(x), f^n(y) \in zV$*

then f has a unique fixed point.