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Function spaces and the Dirichlet problem for nonlocal operators

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ROZPRAWA DOKTORSKA

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Przestrzenie funkcyjne i zagadnienie Dirichleta dla operatorów nielokalnych

Promotor rozprawy: dr hab. inż. Bartłomiej Dyda

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Streszczenie

Punktem wyjścia naszych rozważań jest przestrzeń $\mathbb{R}^d$ dla $d = 1, 2, \ldots$ z metryką euklidesową indukowaną przez normę $|\cdot|$ i miarą Lebesgue'a $\mathrm{d}x$. Niech ν będzie niezerową, symetryczną miarą Lévy'ego, czyli miarą borelowską, która spełnia warunki

$$\nu(\mathbb{R}^d) \neq 0, \qquad \int_{\mathbb{R}^d} (1 \wedge |x|^2)\, \mathrm{d}\nu(x) < \infty \quad \text{oraz} \quad \nu(A) = \nu(-A)$$

dla dowolnego zbioru borelowskiego $A \subseteq \mathbb{R}^d$. Przyjmujemy również, że $\nu(\{0\}) = 0$. Tytułowy *operator nielokalny* działa na funkcjach $u\colon \mathbb{R}^d \to \mathbb{R}$ i wyraża się następującym wzorem:

$$Lu(x) = \lim_{\epsilon \to 0^+} \int_{|y|>\epsilon} (u(x) - u(x+y))\, \mathrm{d}\nu(y). \tag{L}$$

Powyższe wyrażenie jest dobrze określone dla wszystkich $x \in \mathbb{R}^d$ na przykład, gdy funkcja u jest dwukrotnie różniczkowalna w sposób ciągły i ma zwarty nośnik. Operator L nazywamy nielokalnym, gdyż jego wartość w punkcie x zawsze zależy od wartości funkcji u poza pewnym otoczeniem x. Dla porównania, operatory *lokalne* obliczamy na podstawie wartości funkcji na dowolnie małym otoczeniu punktu x, tak jak ma to miejsce dla klasycznych operatorów różniczkowania, na przykład dla laplasjanu Δ.

Definiowanie operatora bez precyzyjnego określenia jego dziedziny może być uznane za nonszalancję, jednak w tej rozprawie nie korzystamy z wyników teorii operatorów i wzór punktowy okazuje się dla nas w zupełności wystarczający. Sam operator służy nam przede wszystkim jako wygodny sposób przedstawienia rozważanych zagadnień.

Operatory nielokalne pojawiają się w wielu działach matematyki. Szczególnie ważny jest dla nas ich związek z teorią procesów Lévy'ego, z której czerpiemy bardzo wiele metod i inspiracji. Połączenie to najlepiej widać w znanym twierdzeniu o reprezentacji (zob. Sato [139, twierdzenie 31.5]), które mówi, że dla dostatecznie regularnych funkcji u, operator $-L$ pokrywa się z generatorem pewnego czysto skokowego procesu Lévy'ego (X_t), a intensywność skoków tego procesu jest opisana miarą ν. Dla porównania przypomnijmy, że generatorem ruchu Browna jest operator lokalny $\frac{1}{2}\Delta$. Skokowe procesy Lévy'ego odgrywają istotną rolę w modelowaniu niektórych losowych zjawisk, które są zbyt nieregularne, by można było je opisać ruchem Browna. W szczególności znalazły one zastosowanie w finansach, zob. Cont i Tankov [46], w genetyce i teorii ewolucji, zob. [14, 110] oraz w różnych działach fizyki, zob. Barndorff-Nielsen i inni [11].

Wiele fizycznych modeli niezwiązanych z procesami losowymi również stosuje operatory nielokalne, zob. np. [48, 117, 144]. Jest to jeden z motorów napędowych bardzo dynamicznie rozwijającej się teorii *nielokalnych równań różniczkowych*. Ze względu na różnicowo-całkową postać operatora L, nazwa ta może budzić pewne kontrowersje i zapewne bezpieczniej jest mówić po prostu o *równaniach nielokalnych*.

Bodaj najbardziej znanym przykładem operatora nielokalnego jest *ułamkowy laplasjan*, dany

dla $\alpha \in (0, 2)$ następującym wzorem:

$$(-\Delta)^{\alpha/2}u(x) = C_{d,\alpha} \lim_{\epsilon \to 0^+} \int_{|y|>\epsilon} \frac{u(x) - u(x+y)}{|y|^{d+\alpha}} \, \mathrm{d}y, \tag{FL}$$

przy czym

$$C_{d,\alpha} = \frac{\pi^{d/2}|\Gamma(-\alpha/2)|}{2^\alpha \Gamma((d+\alpha)/2)}.$$

Po ustaleniu odpowiedniej klasy funkcji u, można pokazać, że operator zadany wzorem (FL) w istocie pokrywa się z potęgą operatora $-\Delta$. Szczegóły można znaleźć np. w artykule Kwaśnickiego [109] o równoważnych definicjach ułamkowego laplasjanu. Spotyka się również notację $\Delta^{\alpha/2}$, którą należy rozumieć jako $-(-\Delta)^{\alpha/2}$. Zauważmy tu, że operator $\Delta^{\alpha/2}$ jest generatorem *izotropowego procesu α-stabilnego* i jest on ujemnie określony, podobnie jak laplasjan Δ. Z kolei (L) zadaje operator dodatnio określony. Z racji przemilczenia tematu dziedziny, powyższe rozważania należy traktować wyłącznie jako wyjaśnienie konwencji. Zwracamy uwagę na to, że nazwa „ułamkowy laplasjan" w literaturze odnosi się zarówno do $\Delta^{\alpha/2}$ jak i do $(-\Delta)^{\alpha/2}$. Mówiąc jednak poniżej o wyniku otrzymanym dla $\Delta^{\alpha/2}$, będziemy mieć na myśli tylko to, że używamy miary Lévy'ego o gęstości $C_{d,\alpha}|y|^{-d-\alpha}$.

Niech D będzie niepustym, otwartym (właściwym) podzbiorem $\mathbb{R}^d$. Jedną z głównych inspiracji dla badań zawartych w niniejszej rozprawie jest następujące zagadnienie Dirichleta:

$$\begin{cases} Lu = f & \text{w } D, \\ u = g & \text{w } \mathbb{R}^d \setminus D. \end{cases} \tag{DP}$$

O ile w klasycznych równaniach różniczkowych wystarczyłoby zadać drugi warunek jedynie na brzegu zbioru D, w większości przypadków wyznaczenie wartości operatora nielokalnego L na D wymaga zadania funkcji u także na D^c — jest to wyraźnie widoczne np. gdy miara Lévy'ego ν ma dodatnią gęstość. Dlatego, dla operatorów nielokalnych zwykle określa się warunek *zewnętrzny* zamiast brzegowego tak jak w (DP), zob. również [67, 90, 134, 137, 141, 142]. Dodajmy tu, że nielokalne zagadnienia *brzegowe* również są badane, lecz wymagają one pewnej modyfikacji operatora L. Takie podejście jest stosowane np. w pracach Guana i Ma [84] oraz Warmy [159].

Zagadnieniu Dirichleta jest poświęcony rozdział 3 rozprawy, oparty na pracy [137] autora. Operujemy tam na dowolnych symetrycznych miarach Lévy'ego oraz ograniczonych zbiorach otwartych D. Skupiamy się na rozwiązaniach wariacyjnych (DP), w szczególności twierdzenie 3.1.1, będące głównym wynikiem tego rozdziału, mówi o ich istnieniu i jednoznaczności. Spośród rezultatów tam zawartych, warto jeszcze wspomnieć o nierówności Poincarégo w twierdzeniu 3.3.2, oraz o zasadzie maksimum dla L i jej zastosowaniu do oszacowań norm supremum rozwiązań w podrozdziale 3.4.

Żeby mówić o rozwiązaniach wariacyjnych, należy zdefiniować funkcjonał energii. Punktem wyjścia dla naszej dyskusji jest następująca *forma kwadratowa* związana z L:

$$\mathcal{E}[u] = \tfrac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(x+y))^2 \, \mathrm{d}\nu(y)\mathrm{d}x. \tag{E}$$

Powyższe wyrażenie jest skończone np. dla funkcji lipschitzowskich o zwartym nośniku, zob. lemat 2.3.5. To właśnie forma $\mathcal{E}$ i jej różne warianty stanowią główny przedmiot tej rozprawy.

W celu uniknięcia uciążliwej notacji, do końca niniejszego streszczenia zakładamy, że miara Lévy'ego ν jest absolutnie ciągła względem miary Lebesgue'a. Większe zawiłości związane z singularnymi miarami Lévy'ego ograniczamy do rozdziału 3, bo właściwie tylko w nim (poza drobnymi wyjątkami w innych miejscach rozprawy) rozważamy takie miary. Są tam w szczególności podane odpowiedniki form występujących poniżej w tym akapicie. Dla absolutnie ciągłych miar Lévy'ego będziemy stosować notację $\mathrm{d}\nu(y) = \nu(y)\,\mathrm{d}y$. Oznaczamy też $\nu(x,y) = \nu(x-y)$. Używając tej konwencji oraz prostej zamiany zmiennych dostajemy:

$$\mathcal{E}[u] = \frac{1}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} (u(x)-u(y))^2 \nu(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Definicja ν dopuszcza niecałkowalną osobliwość w zerze, przez co skończoność formy $\mathcal{E}$ może wymagać pewnej *gładkości* funkcji u. Z punktu widzenia zagadnienia Dirichleta (DP) zakładanie takiej gładkości poza zbiorem D wydaje się ograniczające i zbędne. Z tego powodu w analizie (DP) w rozdziale 3 korzystamy z następującej modyfikacji formy $\mathcal{E}$:

$$\mathcal{E}_D[u] = \frac{1}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} (u(x)-u(y))^2 \nu(x,y)\,\mathrm{d}x\mathrm{d}y. \tag{ED}$$

Przy takim zmniejszeniu obszaru całkowania odcinamy się od osobliwości miary Lévy'ego dla punktów spoza zbioru D, więc przyrost $u(x)-u(y)$ nie musi jej kompensować, gdy x i y są oddalone od D. Zauważmy ponadto, że wyeliminowana całka po $D^c \times D^c$ nie ma żadnego znaczenia dla zagadnienia minimalizacji formy $\mathcal{E}$ przy ustalonych wartościach funkcji na D^c. Pomysł wykorzystania $\mathcal{E}_D$ w równaniach nielokalnych jest stosunkowo nowy — pojawił się około dziesięciu lat temu w pracach Servadei i Valdinociego [141, 142] oraz Caffarellego, Roquejoffre'a i Savina [36, Section 7]. Warto wspomnieć w tym miejscu o artykułach Felsingera, Kassmanna i Voigta [67], oraz Ros-Otona [134], które traktują stricte o zagadnieniu Dirichleta i są bliskie naszym rozważaniom z rozdziału 3.

Przedstawimy teraz problematykę rozdziału 4. Teoria rozwiązań wariacyjnych (DP) wskazuje, że aby móc jawnie zadać założenia na warunek zewnętrzny g, potrzebujemy zrozumieć zachowanie *śladów* $u|_{D^c}$ tych funkcji u, dla których $\mathcal{E}_D[u] < \infty$. Problem śladu jest ściśle związany z zagadnieniem *rozszerzania*, które polega na znalezieniu możliwie szerokiej klasy funkcji określonych na D^c, które można rozszerzyć do funkcji u, takiej że $\mathcal{E}_D[u] < \infty$. Jest to wysoce nietrywialne, wymaga bowiem precyzyjnego wyrażenia tego, że ślad jest regularny tylko blisko brzegu D. Pierwsze rozwiązanie uzyskali Dyda i Kassmann [62] metodami analitycznymi dla $\Delta^{\alpha/2}$. Podają oni następującą, czytelną charakteryzację śladów $g = u|_{D^c}$ funkcji u spełniających warunek $\mathcal{E}_D[u] < \infty$:

$$\iint\limits_{D^c\times D^c} \frac{(g(z)-g(w))^2}{(|z-w| + \delta_D(z) + \delta_D(w))^{d+\alpha}}\,\mathrm{d}z\mathrm{d}w < \infty,$$

gdzie $\delta_D(z) = d(z, \partial D)$, a d oznacza metrykę euklidesową. W twierdzeniu 4.2.1, pochodzącym z pracy Bogdana, Grzywnego, Pietruskiej-Pałuby, oraz autora niniejszej rozprawy [21], podajemy zupełnie inne podejście do tego zagadnienia, wykorzystujące teorię potencjału procesu Lévy'ego (X_t) związanego z operatorem L. Taka metodyka pozwala na rozpatrywanie dużo szerszej klasy operatorów Lévy'ego, choć wymaga wprowadzenia wielu dodatkowych pojęć. Oto niektóre z nich.

Dla punktów $z, w \in D^c$ definiujemy *jądro komunikacji poprzez* D wzorem

$$\gamma_D(z,w) = \iint\limits_{D\times D} \nu(z,x) G_D(x,y) \nu(y,w)\,\mathrm{d}y\mathrm{d}x,$$

gdzie G_D jest funkcją Greena zbioru D dla operatora L. W twierdzeniu 4.2.1 rozprawy dowodzimy, że $\mathcal{E}_D[u] < \infty$ pociąga

$$\iint\limits_{D^c \times D^c} (g(z) - g(w))^2 \gamma_D(z, w)\,\mathrm{d}z\mathrm{d}w < \infty.$$

Ponadto, powyższy warunek *charakteryzuje* ślady, więc dowolna funkcja g, która go spełnia posiada rozszerzenie u na $\mathbb{R}^d$, dla którego $\mathcal{E}_D[u] < \infty$. Jest to zatem odpowiednik wyniku Dydy i Kassmanna. Rozszerzenie funkcji g zadajemy jawnym wzorem, jako całkę Poissona $P_D[g]$ i otrzymujemy następującą tożsamość typu Douglasa:

$$\mathcal{E}_D[P_D[u]] = \tfrac{1}{2} \iint\limits_{D^c \times D^c} (g(z) - g(w))^2 \gamma_D(z, w)\,\mathrm{d}z\mathrm{d}w. \tag{DI}$$

Dla lepszego opisu jądra γ_D odsyłamy do jego oszacowań w twierdzeniu 4.2.5, które dla $\Delta^{\alpha/2}$ przyjmują prostą jawną postać, zob. przykład 4.2.6.

Twierdzenie 4.2.1 współgra z teorią funkcji harmonicznych dla operatora L — przykładową funkcją z tej klasy jest $P_D[g]$. Podrozdział 4.4 poświęcony jest badaniu ich własności. Kulminacją zawartych w nim rozważań są twierdzenie 4.4.14, ustalające równoważność kilku definicji harmoniczności dla funkcji w przestrzeni Sobolewa związanej z $\mathcal{E}_D$, oraz wniosek 4.4.15, w którym stwierdzamy hipoeliptyczność pewnej klasy operatorów nielokalnych L. Hipoeliptyczność oznacza tu, że funkcje słabo harmoniczne dla operatora L są nieskończenie wiele razy różniczkowalne.

O wiele starsze i lepiej zbadane (choć mające zupełnie inne przeznaczenie) podejście do form kwadratowych na podzbiorach przestrzeni $\mathbb{R}^d$, polega na zupełnym odcięciu się od zewnętrza D i rozważaniu wyrażenia

$$\mathcal{E}_D^{\mathrm{cen}}[u] = \tfrac{1}{2} \iint\limits_{D \times D} (u(x) - u(y))^2\, \nu(x, y)\,\mathrm{d}x\mathrm{d}y.$$

Powyższa forma jest zgodna z podejściem stosowanym w pracach [84, 159] przy nielokalnych zagadnieniach *brzegowych.* Skrót „cen" w indeksie górnym pochodzi od cenzurowanych procesów stabilnych wprowadzonych przez Bogdana, Burdzego i Chena [18], dla których $\mathcal{E}_D^{\mathrm{cen}}$ jest formą Dirichleta. W rozdziale 4, dla porównania z $\mathcal{E}_D$, przedstawiamy twierdzenie 4.5.6 o rozszerzaniu związane z formami typu $\mathcal{E}_D^{\mathrm{cen}}$.

Dalszemu badaniu form typu $\mathcal{E}_D^{\mathrm{cen}}$ poświęcony jest rozdział 5. Rozważamy tam przestrzenie Triebela–Lizorkina zdefiniowane przy pomocy półnorm

$$\left(\int_D \left(\int_D \frac{|u(x) - u(y)|^q}{|x - y|^d \phi(|x - y|)^q}\,\mathrm{d}y \right)^{\frac{p}{q}} \mathrm{d}x \right)^{\frac{1}{p}}. \tag{TL}$$

Zakładamy zawsze, że $1 < q \le p < \infty$, oraz że jądro w powyższej całce spełnia warunek

$$\int_{\mathbb{R}^d} (1 \wedge |y|^q) |y|^{-d} \phi(|y|)^{-q}\,\mathrm{d}y < \infty,$$

który dla $q = 2$ pokrywa się z warunkiem całkowalności miary Lévy'ego. Dla otrzymywania nierówności dotyczących półnormy (TL) warto wiedzieć, że można ją kontrolować pozornie mniejszym wyrażeniem. W rozdziale 5, który powstał na podstawie artykułu autora tej rozprawy [138], badamy kiedy półnorma (TL) jest porównywalna z

$$\left(\int_D \left(\int_{B(x, \theta\delta_D(x))} \frac{|u(x) - u(y)|^q}{|x - y|^d \phi(|x - y|)^q}\,\mathrm{d}y \right)^{\frac{p}{q}} \mathrm{d}x \right)^{\frac{1}{p}}, \tag{RED}$$

gdzie θ jest pewną liczbą z przedziału $(0,1]$. Porównywalność lub jej brak ustalamy przy wielu różnych założeniach na ν i D, lecz warto podkreślić, że w sporej części wyników tego rozdziału D jest obszarem *jednostajnym* (*uniform domain*). Główną motywacją dla tych badań jest praca Pratsa i Saksmana [128], którzy rozważają $\phi(|y|) = |y|^{-\alpha/2}$. W twierdzeniu 5.1.1 wskazujemy szerszą klasę funkcji ϕ, dla których zachodzi porównywalność. W dowodzie korzystamy z metod użytych w pracy [128]. Dobieramy możliwie słabe założenia pozwalające zastosować te techniki, lecz jak dowodzimy w podrozdziale 5.4.2, sytuacja w dużym stopniu sprowadza się do wymagania silnego (potęgowego) zaniku ϕ w zerze. Nie jest zatem niespodzianką, że dla $\phi \equiv 1$ porównywalność nie zachodzi, zob. przykład 5.4.7, niemniej jednak dla takiej funkcji ϕ otrzymujemy pewne zanurzenie powiązanych przestrzeni funkcyjnych, zob. twierdzenie 5.5.1. W wynikach podrozdziału 5.6, w szczególności w twierdzeniu 5.1.2 oraz w przykładzie 5.6.1, jawi się subtelna zależność między geometrią zbioru oraz singularnością jądra całkowego, mająca znaczenie dla porównywalności (TL) z (RED). Warto wspomnieć, że w rozdziale 5 używamy wyłącznie metod analitycznych.

Rozdział 6 powstał na bazie niedawnego preprintu Bogdana, Grzywnego, Pietruskiej-Pałuby, oraz autora niniejszej rozprawy [22]. Powracamy tutaj do zakresu całkowania jak we wzorze (ED) i badamy nieliniowe tożsamości Douglasa. Jednym z głównych wyników jest twierdzenie 6.4.1 (zob. także wzory (6.2.9) oraz (6.2.11)), które dla $p > 1$ daje następującą równość:

$$\begin{aligned}&\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} (P_D[g](x)^{\langle p-1\rangle} - P_D[g](y)^{\langle p-1\rangle})(P_D[g](x) - P_D[g](y))\nu(x,y)\,\mathrm{d}x\mathrm{d}y\\ &= \iint\limits_{D^c\times D^c} (g(z)^{\langle p-1\rangle} - g(w)^{\langle p-1\rangle})(g(z) - g(w))\gamma_D(z,w)\,\mathrm{d}z\mathrm{d}w, \end{aligned} \tag{NDI}$$

gdzie $r^{\langle\kappa\rangle} = |r|^\kappa \operatorname{sgn}(r)$ dla $r \in \mathbb{R}$ oraz $\kappa > 0$. Zauważmy, że dla $p = 2$ otrzymujemy poprzednią tożsamość Douglasa (DI). Powyższa nieliniowa postać jest silnie zainspirowana wynikami Bogdana, Dydy i Luksa [20], w szczególności tożsamością Hardy'ego–Steina, którą rozszerzamy do naszego kontekstu w podrozdziale 6.3, a następnie wykorzystujemy w dowodzie równości (NDI).

Dla tożsamości Douglasa kluczowe jest to, że w formie po lewej stronie znajduje się funkcja harmoniczna $P_D[u]$. Bez tego zwykle nie możemy liczyć na równość, jednakże w twierdzeniu 6.5.4 otrzymujemy wersję tożsamości Douglasa z resztą dla dostatecznie regularnych funkcji innych niż harmoniczne. Wynik ten jest nowy nawet dla $p = 2$ i $\Delta^{\alpha/2}$.

Przestrzenie zdefiniowane przy pomocy formy po lewej stronie wzoru (NDI) mają istotnie różny charakter od *ułamkowych przestrzeni Sobolewa* $W^{s,p}$ znanych z literatury, zob. przykład 2.3.9. Z tego powodu, w pracy [22] zdecydowaliśmy się na nowe określenie — przestrzeń *Sobolewa–Bregmana*, zob. podrozdział 6.1. W podrozdziale 6.6 porównujemy różne typy przestrzeni funkcyjnych związanych z rozważanymi formami nieliniowymi (nieco trafniej, choć mniej elegancko, można by je zwać formami niekwadratowymi).

Omówimy teraz założenia, z których korzystamy w rozdziałach 4 i 6 (z pominięciem „analitycznego" podrozdziału 4.5) — w obu są one bardzo podobne. Przyjmujemy tam, że miara ν jest nieskończona oraz *unimodalna*, co znaczy że jej gęstość $\nu(x)$ zależy wyłącznie od promienia $|x|$ i maleje (słabo) wraz z jego wzrostem. Znaczna część wyników korzysta również z górnych oszacowań na ν w zerze, oraz ograniczonego tempa jej zaniku: $\nu(r) \approx \nu(r+1)$ dla $r > 1$, zob. warunek **A2** w podrozdziale 4.2. Gładkość funkcji harmonicznych, która odgrywa rolę w obu rozdziałach, jest związana z oszacowaniami pochodnych miary Lévy'ego w nieskończoności w warunku **A1**. Oszacowania jądra γ_D wymagają więcej założeń o skalowaniu ν, zob. warunek **A3**, oraz podrozdział A.1.2. O D zakładamy zwykle, że ma brzeg miary Lebesgue'a zero, oraz że D^c spełnia warunek VDC, zob. podrozdział 2.1.2. Oszacowania γ_D otrzymujemy natomiast dla półprzestrzeni oraz zbiorów klasy $C^{1,1}$.

Większość pojęć, które pojawiają się powyżej, jest szczegółowo przedyskutowana w rozdziale 2. Omawiamy tam geometrię różnych podzbiorów $\mathbb{R}^d$, ustalamy związek procesu Lévy'ego (X_t) z rozważanymi obiektami, wprowadzamy niektóre elementy jego teorii potencjału oraz przedstawiamy podstawowe fakty na temat operatora L i przestrzeni Sobolewa związanych z formami, o których jest mowa powyżej.

Chapter 1

Introduction

With the intent to contain this introduction within a few pages we do not go into too much details here. Most of the notions are discussed exhaustively in Chapter 2, according to the needs of this dissertation. The introductions to the respective chapters in turn elaborate on more specific issues studied in the dissertation. In particular they contain a detailed placement of our results in the context of the literature, as well as some historical remarks.

In the whole dissertation our reference space is the Euclidean space $\mathbb{R}^d$, $d = 1, 2, \dots$ with the Euclidean metric induced by the norm $|\cdot|$ and the Lebesgue measure $\mathrm{d}x$. Let ν be a nonzero symmetric Lévy measure, that is, a Borel measure such that

$$\nu(\mathbb{R}^d) \neq 0, \qquad \int_{\mathbb{R}^d} (1 \wedge |x|^2)\, \mathrm{d}\nu(x) < \infty \quad \text{and} \quad \nu(A) = \nu(-A)$$

for all Borel sets $A \subseteq \mathbb{R}^d$. For conventional reasons we shall also assume that $\nu(\{0\}) = 0$. The titular *nonlocal operator* acts on functions $u\colon \mathbb{R}^d \to \mathbb{R}$ and is given by the following formula

$$Lu(x) = \lim_{\epsilon \to 0^+} \int_{|y|>\epsilon} (u(x) - u(x+y))\, \mathrm{d}\nu(y). \tag{L}$$

The above expression is well-defined and finite for all $x \in \mathbb{R}^d$, e.g., for the twice continuously differentiable functions with compact support, see Proposition 2.3.2 below. The name 'nonlocal' refers to the fact that the value of $Lu(x)$ depends on the values of u at the points far away from x. This is in contrast with the *local* operators, by which we usually mean various differential operators, e.g., the Laplacian Δ.

At this early stage, we note that the contents of this dissertation are independent of the operator theory, which is why we refrain from specifying the domain of L. Throughout the text the operator L is always understood pointwise, as in the definition (L). In fact, L turns out to be a mean rather than an end in our studies, as becomes clear later on. Nevertheless, it is a very convenient way to present the results and motivations for our research.

Nonlocal operators emerge in many branches of mathematics. An invaluable source of inspiration and methods for our work is the theory of stochastic processes. Namely, according to the well-known representation formula (see, e.g., Sato [139, Theorem 31.5]), for sufficiently regular u, $-L$ coincides with the generator of a pure-jump Lévy process (X_t). The intensity of jumps of this process is described by the Lévy measure ν. For reference, we recall that $\frac{1}{2}\Delta$ is the generator of the *Brownian motion.* Jump Lévy processes have numerous real-world applications for the phenomena which are too irregular to be described by the Brownian motion. These include: financial modeling, see Cont and Tankov [46] and the references therein, genetics, see, e.g., [14, 110] and various branches of physics, see Barndorff-Nielsen et al. [11].

Putting aside the context of Lévy processes, many other physical models use nonlocal operators, see, e.g., [48, 117, 144], giving rise to equations governed by L, e.g., $Lu = 0$. These have many names in the literature: integro-differential equations, pseudo-differential equations, or even nonlocal PDEs, which may be somewhat disturbing if we take 'PDE' in the usual sense. We will usually use the term *nonlocal equations*.

Arguably the most prominent example of a nonlocal operator is the *fractional Laplacian* $(-\Delta)^{\alpha/2}$ for $\alpha \in (0, 2)$:

$$(-\Delta)^{\alpha/2} u(x) = C_{d,\alpha} \lim_{\epsilon \to 0^+} \int_{|y|>\epsilon} \frac{u(x) - u(x+y)}{|y|^{d+\alpha}} \, \mathrm{d}y, \tag{FL}$$

where

$$C_{d,\alpha} = \frac{\pi^{d/2} |\Gamma(-\alpha/2)|}{2^\alpha \Gamma((d+\alpha)/2)}.$$

Note that $C_{d,\alpha}|y|^{-d-\alpha}\,\mathrm{d}y$ is in fact a Lévy measure, because $\alpha \in (0, 2)$. It requires some caution to describe for which u the definition (FL) agrees with the actual fractional power of the negative Laplacian defined, e.g., as a Fourier multiplier. This is however irrelevant to our development, we refer to Kwaśnicki [109] for a fuller discussion and a wider context. We note in passing that (L) yields a positive definite operator, cf. Subsection 2.3.2. Consequently, $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ is negative definite, as is the Laplacian Δ. This is, of course, only a conventional remark, because we do not specify the domains. We also note that the name 'fractional Laplacian' is also used for $\Delta^{\alpha/2}$ in the literature. However, below in this dissertation, whenever we say that a result was obtained for $\Delta^{\alpha/2}$ or the fractional Laplacian, we only mean that the underlying Lévy measure is that of $(-\Delta)^{\alpha/2}$.

Let D be a nonempty proper open subset of $\mathbb{R}^d$. A direct motivation for much of the studies included in this dissertation is the following nonlocal Dirichlet problem:

$$\begin{cases} Lu = f & \text{in } D, \\ u = g & \text{in } \mathbb{R}^d \setminus D. \end{cases} \tag{DP}$$

Unlike in the partial differential equations, here a *boundary* condition g would be insufficient, because in most cases (e.g., when ν has a strictly positive density), due to the nonlocality of L, we find at least some points $x \in D$ for which $Lu(x)$ depends on the values of u in $\mathbb{R}^d \setminus \overline{D}$. Therefore in problems driven by nonlocal operators it is customary to pose the condition g on the whole complement of D, as an *exterior* condition, see, e.g., [67, 90, 134, 137, 141, 142]. Nonlocal *boundary* value problems can also be considered, but they require a proper modification of the operator L, see, e.g., Guan and Ma [84] or Warma [159].

We study the Dirichlet problem in Chapter 3, which is based on the article by the author of this dissertation [137]. We work in the scope of all symmetric Lévy measures and bounded open sets D. We focus mainly on the variational solutions of (DP) and the main result of the chapter, which is Theorem 3.1.1, gives their existence and uniqueness. Another highlights of this part of the dissertation are the Poincaré inequality in Theorem 3.3.2, and a maximum principle, which is applied to obtain L^∞ bounds for solutions of (DP) in Section 3.4.

The study of the variational solutions of (DP) motivates our research of the *quadratic form* associated with L:

$$\mathcal{E}[u] = \tfrac{1}{2} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(x+y))^2 \, \mathrm{d}\nu(y) \mathrm{d}x. \tag{E}$$

The above expression is finite provided that, e.g., u is Lipschitz and compactly supported, see Lemma 2.3.5. Numerous modifications of the form $\mathcal{E}$ are, in fact, the central objects of this dissertation and we shall discuss them below.

For the sake of simplicity, in the remainder of this introduction we assume that ν is absolutely continuous and we remark that only the results of Chapter 3 allow singular Lévy measures (with rare exceptions elsewhere, which will be clearly visible). With a slight abuse of the notation we write $\mathrm{d}\nu(y) = \nu(y)\,\mathrm{d}y$. Also, most of the time we will denote $\nu(x,y) := \nu(x-y)$, but we stress that in spite of this notation, all the kernels ν (and K, see Chapter 5) considered in this dissertation are space-homogeneous. After a substitution, $\mathcal{E}$ takes on the convenient form

$$\mathcal{E}[u] = \tfrac{1}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} (u(x) - u(y))^2 \nu(x,y)\,\mathrm{d}x\mathrm{d}y.$$

The Lévy measure ν may be infinite on the neighborhood of the origin. Thus, requiring the finiteness of $\mathcal{E}[u]$ is overly restrictive for the Dirichlet problem, because it imposes the same regularity on the exterior condition g as on u in D. Furthermore, with the exterior condition g fixed, the integral over $D^c \times D^c$ is irrelevant for *minimizing* $\mathcal{E}$, which is the very essence of the variational solutions. Therefore, in Chapter 3 we use the following form:

$$\mathcal{E}_D[u] = \tfrac{1}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} (u(x) - u(y))^2 \nu(x,y)\,\mathrm{d}x\mathrm{d}y. \tag{ED}$$

Observe that now we avoid the singularity of the Lévy measure when two points $x, y \in D^c$ are close to each other and the increments $u(x) - u(y)$ do not need to compensate for it. The idea of considering $\mathcal{E}_D$ in the context of nonlocal equations is only about ten years old; early appearances include the works of Servadei and Valdinoci [141, 142] and Caffarelli, Roquejoffre and Savin [36, Section 7]. In this connection we mention the papers by Felsinger, Kassmann and Voigt [67] and Ros-Oton [134], which are strongly related to the development in [137], and we refer to Chapter 3 for further discussion on the Dirichlet problem in the context of $\mathcal{E}_D$.

Let us now outline the matters of Chapter 4. The approach to the Dirichlet problem using $\mathcal{E}_D$ (see the formulation of Theorem 3.1.1) indicates the importance of understanding precisely how regular the *trace* $u|_{D^c}$ is, given that $\mathcal{E}_D[u] < \infty$. This is directly connected with the *extension* problem, which is to determine a possibly wide class of functions $g\colon D^c \to \mathbb{R}$, that can be extended to a function u with finite $\mathcal{E}_D[u]$. Both problems are highly nontrivial, because they require a quantitative description of the fact that the trace $u|_{D^c}$ is regular only near the boundary of D. The solution to the extension and trace problem was given recently by Dyda and Kassmann [62] in the case of $\Delta^{\alpha/2}$ with the use of analytic methods, see Section 4.1. In Theorem 4.2.1 below, which is also the main result of the paper by Bogdan, Grzywny, Pietruska-Pałuba and the author [21], we provide a solution which uses the methods of the potential theory of the Lévy process (X_t) and allows for considering much more general Lévy measures than that of $\Delta^{\alpha/2}$. Before we proceed to the presentation of this result, we remark that the notions which appear in the following paragraph are properly defined and discussed in more detail in Subsection 2.2.2.

For points $z, w \in D^c$ we let

$$\gamma_D(z,w) = \iint\limits_{D\times D} \nu(z,x) G_D(x,y) \nu(y,w)\,\mathrm{d}y\mathrm{d}x,$$

where G_D is the Green function of D for L. In Theorem 4.2.1 we show that $\mathcal{E}_D[u] < \infty$ implies

$$\iint\limits_{D^c\times D^c} (g(z)-g(w))^2\gamma_D(z,w)\,\mathrm{d}z\mathrm{d}w < \infty,$$

where $g = u|_{D^c}$. Conversely, any function g satisfying the above condition may be extended to a function u with $\mathcal{E}_D[u] < \infty$. The extension is given by the Poisson integral $P_D[g]$ and we obtain the following Douglas-type identity:

$$\mathcal{E}_D[P_D[u]] = \tfrac{1}{2}\iint\limits_{D^c\times D^c} (g(z)-g(w))^2\gamma_D(z,w)\,\mathrm{d}z\mathrm{d}w. \tag{DI}$$

For a better comprehension of γ_D, we refer to the estimates in Theorem 4.2.5. An explicit analysis is given in Example 4.2.6 for $\Delta^{\alpha/2}$. There is a strong interplay between Theorem 4.2.1 and the theory of the harmonic functions of L. In fact, $P_D[g]$ is an important example of a harmonic function. Section 4.4 is devoted to studying this class of functions with the culmination in Theorem 4.4.14, which unifies several definitions of harmonicity in the Sobolev space associated with $\mathcal{E}_D$ (see also the discussion in Section 4.1), and Corollary 4.4.15, which asserts the hypoellipticity of a class of nonlocal operators L.

A more standard and established approach (but with completely different purposes) to the quadratic forms and Sobolev spaces on the subsets of $\mathbb{R}^d$ is via restricting the integrations to D only, that is,

$$\mathcal{E}_D^{\mathrm{cen}}[u] = \tfrac{1}{2}\iint\limits_{D\times D} (u(x)-u(y))^2\,\nu(x,y)\,\mathrm{d}x\mathrm{d}y.$$

This form is more in line with the nonlocal *boundary* value problems mentioned above. A probabilistic motivation for studying $\mathcal{E}_D^{\mathrm{cen}}$ comes from the theory of censored stable processes first developed by Bogdan, Burdzy and Chen [18], which explains 'cen' in the superscript. In Theorem 4.5.6 we establish an extension operator for this type of spaces. This result uses analytic and geometric methods in contrast to Theorem 4.2.1 discussed above.

Chapter 5 further investigates the forms of the type $\mathcal{E}_D^{\mathrm{cen}}$ and their analogues. We work in the setting of the Triebel–Lizorkin spaces, which are defined with the use of the following seminorm:

$$\left(\int_D\left(\int_D \frac{|u(x)-u(y)|^q}{|x-y|^d\phi(|x-y|)^q}\,\mathrm{d}y\right)^{\frac{p}{q}}\mathrm{d}x\right)^{\frac{1}{p}}. \tag{TL}$$

Here $1 < q \le p < \infty$ and we assume that the kernel satisfies $\int(1\wedge|y|^q)|y|^{-d}\phi(|y|)^{-q}\,\mathrm{d}y < \infty$, which is in agreement with the Lévy measure integrability condition for $q = 2$. For the sake of obtaining various inequalities concerning (TL) it is often useful to know that it is controlled by a seemingly smaller expression. In this vein, our task in Chapter 5, which is based on the author's article [138], is to investigate when the seminorm (TL) is comparable with

$$\left(\int_D\left(\int_{B(x,\theta\delta_D(x))} \frac{|u(x)-u(y)|^q}{|x-y|^d\phi(|x-y|)^q}\,\mathrm{d}y\right)^{\frac{p}{q}}\mathrm{d}x\right)^{\frac{1}{p}}, \tag{RED}$$

where $\theta \in (0,1]$ and $\delta_D(x) = d(x,\partial D)$ (here and below d is the Euclidean distance). We discuss the comparability and incomparability of (TL) and (RED) under various conditions for D and ϕ. These studies were inspired by the work of Prats and Saksman [128] who investigated the case of the uniform domains and $\phi(|y|) = |y|^{-\alpha/2}$. In Theorem 5.1.1 we give an affirmative result by using the methods of [128]. We propose possibly weak assumptions on ϕ, tailor-made for these methods, but as we find out in Subsection 5.4.2, the assumptions in fact reduce to certain scalings for ϕ, assuring strong singularity of the kernel. As we would expect, $\phi \equiv 1$ gives a negative result, see Example 5.4.7, but we obtain a certain embedding result for the associated function spaces in Theorem 5.5.1. Perhaps the most intriguing results of this chapter are obtained in Section 5.6, where the interplay between the kernel and the geometry of D becomes clearly visible, see in particular Theorem 5.1.2 and Example 5.6.1. We note that the methods of this chapter are purely analytic.

In Chapter 6, which mostly consists of the content of the recent preprint by Bogdan, Grzywny, Pietruska-Pałuba and the author [22], we return to the forms with integration domain as in (ED) and we develop the Douglas identity (DI) in a nonlinear setting. Namely, in Theorem 6.4.1 (see also (6.2.9) and (6.2.11)) we give the following formula (here $1 < p < \infty$):

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} &(P_D[g](x)^{\langle p-1\rangle} - P_D[g](y)^{\langle p-1\rangle})(P_D[g](x) - P_D[g](y))\nu(x,y)\,\mathrm{d}x\mathrm{d}y \\ &= \iint_{D^c\times D^c} (g(z)^{\langle p-1\rangle} - g(w)^{\langle p-1\rangle})(g(z) - g(w))\gamma_D(z,w)\,\mathrm{d}z\mathrm{d}w, \qquad \text{(NDI)}\end{aligned}$$

where $r^{\langle\kappa\rangle} = |r|^\kappa \operatorname{sgn}(r)$ for $r \in \mathbb{R}$ and $\kappa > 0$. For clarity we note that P_D and γ_D have exactly the same meaning as above, that is, they do not depend on p. Obviously, the situation reduces to (DI) for $p = 2$. The above nonlinear setting was strongly influenced by the Hardy–Stein identity of Bogdan, Dyda and Luks [20, (14)], whose extended version, given in Section 6.3, serves as a tool in the proof of (NDI).

In general, if we put on the left-hand side of (NDI) a function u other than the harmonic $P_D[g]$, then the formula ceases to hold. It is however possible to obtain an identity with a remainder term for sufficiently regular non-harmonic u, which is done in Theorem 6.5.4. This is novel even for $p = 2$ and $\Delta^{\alpha/2}$.

We note in passing that for $\Delta^{\alpha/2}$ and $p \neq 2$, the spaces induced by the expression on the left-hand side of (NDI) have a different nature than the fractional Sobolev spaces $W^{s,p}$, cf. Example 2.3.9. Thus, in [22] we introduce the name *Sobolev–Bregman* spaces, cf. Section 6.1 below. In Section 6.6 we discuss the differences between the spaces built on various types of nonlinear (put differently — nonquadratic) forms studied in this dissertation.

Let us comment on the assumptions of Chapters 4 and 6 (excluding the 'analytic' Section 4.5). For the most part, they are identical in both chapters. We stipulate that ν is infinite and *unimodal*, which means that $\nu(x)$ depends only on the radius $|x|$ and decreases (weakly) with the growth of $|x|$. Oftentimes the results depend on upper bounds for ν at 0, see **A2** in Subsection 4.2 and on the decay control: $\nu(r) \approx \nu(r+1)$ for $r > 1$. The second-order differentiability of the harmonic functions, crucial for both discussed chapters, relies on the estimates of the derivatives of ν at infinity in **A1**. Further scaling conditions are required for the estimates of γ_D, see **A3** and Subsection A.1.2. As for the set D, we usually assume the volume density condition for its exterior and $|\partial D| = 0$, but the estimates of γ_D are only given for the half-space and for $C^{1,1}$ sets.

Last but not least, Chapter 2 gives the background for the notions discussed above: various classes of subsets of $\mathbb{R}^d$, the definition of the associated Lévy process (X_t) and some of its potential theory, and the basic information about the operator L and the nonlocal Sobolev spaces.

Chapter 2

Notation and preliminaries

This chapter introduces the fundamental notions and conventions which will appear throughout the dissertation. In Section 2.1 we discuss the geometry of the subsets of the Euclidean space $\mathbb{R}^d$ and Section 2.2 introduces the symmetric pure-jump Lévy processes and their potential theory, which will serve as one of our main toolboxes. Lastly, in Section 2.3 we discuss the nonlocal operators, the quadratic forms and related Sobolev spaces. We start with some notation.

By $C(D)$ we mean the class of the continuous functions on D. We also consider its subclasses: $C_0(\mathbb{R}^d)$ vanish at infinity, $C_c(D)$ are compactly supported in D and $C_b(D)$ are bounded on D. As usual, $C^k(D)$, $k = 1, 2, \ldots, \infty$, are the functions with continuous derivatives up to the order k, the classes $C_0^k(\mathbb{R}^d)$, $C_c^k(D)$ and $C_b^k(D)$ are defined accordingly, with the respective properties also applying to the derivatives up to the order k. By default, the Lebesgue spaces L^p, $p \in [1, \infty]$, will be understood as the spaces of equivalence classes of functions, but on rare occasions we will need to choose a representative.

For most of the time we do not track the constants. Accordingly, the value of $c > 0$ may vary between two unrelated discussions. More important constants will be capitalized, e.g., $C_1, C_2, \ldots$, and their value will not change throughout the text. By $a \lesssim b$ we mean that there exists $c > 0$ such that $a \leq cb$, and $a \approx b$ means that $a \lesssim b$ and $a \gtrsim b$, that is, $b \lesssim a$.

In case the Lévy measure is absolutely continuous with respect to the Lebesgue measure, we slightly abuse the notation and write $\mathrm{d}\nu(x) = \nu(x)\,\mathrm{d}x$ and $\nu(x, y) = \nu(x - y)$ for $x, y \in \mathbb{R}^d$. If, additionally, ν is radially symmetric, then we write $\nu(|x|) = \nu(x)$. We also let $\nu(x, G) := \nu(G - x) = \int_G \nu(x, y)\,\mathrm{d}y$.

In the whole dissertation for $A, B \subseteq \mathbb{R}^d$, by $d(A, B)$ we mean the standard Euclidean distance between the sets, that is, $d(A, B) = \inf\{|x - y| : x \in A,\, y \in B\}$. For $x \in \mathbb{R}^d$ we write $d(x, A) = d(\{x\}, A)$. The diameter of A is $\operatorname{diam} A = \sup\{|x - y| : x, y \in A\}$. As usual, we let $B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}$ for $x \in \mathbb{R}^d$, $r > 0$. Quite often the last coordinate in $\mathbb{R}^d$ will be distinguished, thus we denote $\mathbb{R}^d \ni x = (x', x_d)$, where $x' \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$.

Each subsequent chapter has its individual standing assumptions about ν and D, which will be given at its beginning. The **only global assumptions** are that $D \subseteq \mathbb{R}^d$ is open, D is nonempty and ν is a nonzero symmetric Lévy measure. We also assume that D is a proper subset, but on rare occasions, when it is convenient and does not cause confusion, we may write 'for $D = \mathbb{R}^d$' and such.

2.1 Geometry

In this section we introduce various definitions concerning the regularity of open subsets $D \subseteq \mathbb{R}^d$ and we present some of the connections between these definitions. We remark that unless we mention it, we do not assume the connectedness of D, i.e., in general D need not be a domain. We discuss three types of sets: those whose boundary is locally a graph of a function, the ones whose complement is *fat* near the boundary, and the uniform domains. It is rather impossible to provide a single position in the literature which covers exhaustively all these families, thus we give individual references in each subsection. Neat graphs showing the relationships between various classes of sets are given by Aikawa [2, 3.7 and 3.9].

2.1.1 Boundary as a graph of a function

We first define the classes of sets whose boundaries locally resemble a graph of a function satisfying certain regularity properties. In order to include 1-dimensional sets in the definitions below, we let $\mathbb{R}^0 := \{0\}$.

Definition 2.1.1. Let $f\colon \mathbb{R}^{d-1} \to \mathbb{R}$. The set $\{x : x_d > f(x')\}$ is called the *epigraph* of f, and the set $\{x : x_d < f(x')\}$ is its *hypograph*.

The epi- and hypographs are well-defined for any, even nonmeasurable, function f, but in order to make them useful we usually impose some regularity on f. Note that the half-space $H = \{x \in \mathbb{R}^d : x_d > 0\}$ is the epigraph of $f \equiv 0$.

Definition 2.1.2. We say that $D \subseteq \mathbb{R}^d$ has *continuous boundary* if ∂D is compact and there is $r_0 > 0$ such that for every $x \in \partial D$ there exist a rigid motion R_x and a continuous function $f_x\colon \mathbb{R}^{d-1} \to \mathbb{R}$ such that $R_x(B(x, r_0) \cap D) = B(0, r) \cap \{x \in \mathbb{R}^d : f(x') < x_d\}$.

The set D is Lipschitz (or, has *Lipschitz boundary*) with the Lipschitz constant $\lambda > 0$, if R_x and f_x can be chosen in a way that every f_x is Lipschitz with this constant, that is, for every $x \in \partial D$,

$$|f_x(x') - f_x(y')| \le \lambda |x' - y'|, \quad x', y' \in \mathbb{R}^{d-1}.$$

If we also require that all f_x have continuous, uniformly bounded gradients ∇f_x, then we say that D is of class C^1, and if all the gradients are Lipschitz with the same constant, then we call D a $C^{1,1}$ set. Note that for C^1 and $C^{1,1}$ sets the tangent plane exists at every point of the boundary. For convenience, we assume that R_x maps the tangent plane at x to the set $\{x \in \mathbb{R}^d : x_d = 0\}$, which implies that $f_x(0) = |\nabla f_x(0)| = 0$.

Example 2.1.3. The so-called crossed books, or crossed bricks domain in Figure 2.1 is a standard example of a domain which seems regular, but does not have even continuous boundary. As a digression, we remark that it is weakly Lipschitz. This means that for every point on the boundary its small neighborhood can be transformed via a bi-Lipschitz mapping into the unit cube, in the way that the points of D, and only the points of D, are mapped into the positive half-space H, see, e.g., Licht [115, Section 2] for a detailed treatment.

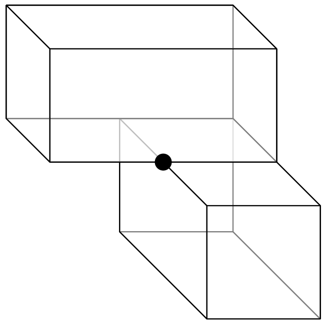

Figure 2.1: The crossed bricks domain, a black and white version of the illustration from [115].

It is well-known that $C^{1,1}$ sets can be characterized by the interior and exterior ball conditions. The proof of this fact is given, e.g., by Aikawa et al. [3, Lemma 2.2]. We give the statement below.

Lemma 2.1.4. *Assume that D is a $C^{1,1}$ open set. Then, D satisfies the interior and exterior ball conditions, that is, there exists $r > 0$ such that for every $x \in \partial D$ there are points x_I and x_E such that $B(x_I, r) \subseteq D$, $B(x_E, r) \subseteq D^c$ and $\overline{B(x_I, r)} \cap \overline{B(x_E, r)} = \{x\}$. Conversely, if an open D satisfies the interior and exterior ball conditions, then it is $C^{1,1}$.*

If the interior and exterior ball condition is satisfied with the radius $r > 0$, then we say that D is $C^{1,1}$ at scale r.

2.1.2 Volume density condition and d-sets

We proceed with another type of sets which will play a role in the extension theorems and Douglas identities in Chapters 4 and 6, through the crucial Lemma 2.2.2. In view of these results it may be instructive to think of $G = D^c$ below.

Definition 2.1.5. We say that a Borel set G satisfies the *volume density condition* VDC, if there exists $C_{\text{vdc}} > 0$ such that for every $x \in \partial G$ and $r > 0$ we have

$$|B(x, r) \cap G| \geq C_{\text{vdc}} r^d. \tag{2.1.1}$$

We say that VDC holds *locally* for G if VDC holds for $G \cup B^c$ for every ball B (the constant may depend on B).

If VDC holds, then it also holds locally because $(G^c \cap B)^c = G \cup B^c$ and $\partial(G \cup B^c) \subset \partial G^c \cup \partial B$ for every ball B. Furthermore, if G^c is bounded, then the local VDC for G is equivalent to VDC.

Example 2.1.6. Assume that G satisfies the following *interior cone condition*: there exists a fixed positive aperture such that for every $x \in \partial G$ there is an infinite cone with apex at x and this aperture, contained in G. Then, G satisfies VDC. In particular, the complement of the crossed bricks domain from Example 2.1.3 satisfies VDC. Furthermore, if G^c is bounded and Lipschitz, then G also satisfies VDC. Indeed, by using the Lipschitz property of f_x, we may easily construct a cone C_x with apex at $R_x(x)$, of fixed height and aperture depending on r_0 and λ, such that $R_x^{-1}(C_x) \subset G$, which gives (2.1.1) for small r. For large r the condition follows from the boundedness of G^c. Similarly, every Lipschitz epigraph satisfies the interior cone condition, hence also VDC. For $x \in \mathbb{R}$, let $f(x) = \sin(x^2)$, a locally Lipschitz function. The epigraph of f satisfies VDC only locally.

Example 2.1.7. The scope of the sets satisfying VDC goes far beyond the cone condition, e.g.,

$$G = \bigcup_{k\in\mathbb{Z}} \{x \in \mathbb{R}^d : 2^{2k} \leq |x| \leq 2^{2k+1}\}$$

satisfies VDC. More *wild* cases are given in Examples 2.1.10 and 2.1.14.

Example 2.1.8. Sets with merely continuous boundary need not satisfy VDC even locally. Indeed, it suffices to consider a domain with cusps as in Figure 2.2.

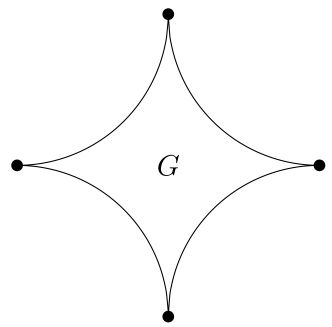

Figure 2.2: When x is one of the marked points, we have $|B(x,r)\cap G| = o(r^2)$ as $r \to 0^+$, so the local VDC fails to hold.

Lemma 2.1.9. *If G satisfies VDC, then* (2.1.1) *is satisfied also for every $x \in \operatorname{Int} G$.*

Proof. Assume that $\operatorname{Int} G \neq \emptyset$ and let $x \in \operatorname{Int} G$ and $r > 0$. If $d(x, \partial G) \geq r/2$, then $B(x, r/2) \subseteq G$, which gives (2.1.1). Thus, assume that $d(x, \partial G) \in (0, r/2)$ and let $x_0 \in \partial G$ satisfy $d(x, \partial G) = |x - x_0|$. Then we have $B(x_0, r/2) \subset B(x, r)$ and by VDC we get that

$$|B(x,r)\cap G| \geq |B(x_0, r/2)\cap G| \geq c(r/2)^d,$$

which ends the proof. □

In view of the above fact we see that there is very little difference between the sets satisfying VDC and the d-sets in $\mathbb{R}^d$, which in the literature would usually be called n-sets due to having $\mathbb{R}^n$ as the reference space instead of $\mathbb{R}^d$, see, e.g., Jonsson and Wallin [98, page 205]. Namely, d-sets in $\mathbb{R}^d$ only require $r \in (0,1)$ in (2.1.1). Note that the other inequality from the definition in [98] is trivially satisfied for $d = n$. In fact, being a d-set in $\mathbb{R}^d$ is equivalent to VDC when G^c is bounded and to local VDC when G is bounded. At least two other names for this type of sets function in the literature: measure density condition and (Ahlfors) regular sets, cf. Hajłasz, Koskela and Tuominen [86].

In our development in Chapters 4 and 6 we rely on the fact that $|\partial D| = 0$. In the sequel we discuss whether VDC implies that property. Using Lemma 2.1.9 we point out an inconsistence in the literature: Wu in [161, page 284] states that VDC (called (VDC_b) therein) for D^c yields $|\partial D| = 0$, while Shvartsman [143, page 1213] mentions that, e.g., fat Cantor sets are d-sets in $\mathbb{R}^d$, which means that a set with the boundary of positive Lebesgue measure can admit VDC. Neither of the papers provide a proof or a reference to one and the author was unable to find a definite answer, therefore below we give an argument which confirms the option of Shvartsman.

Example 2.1.10. We will show that there exists $G \subset [0,1]$ with $|\partial G| \neq 0$, which satisfies (2.1.1) for small r. Then, by considering $(-\infty, 0)\cup G\cup(1,\infty)$ we obtain a closed set with VDC and the boundary of positive Lebesgue measure. We let G be the fat Cantor set constructed as follows: we start with the interval $[0,1]$, then, in the first step, we remove an open interval of the length $\frac{1}{4}$ from the middle of $[0,1]$ so that we are left with two identical intervals. For $n \geq 2$, in the nth step we remove from the middle of every remaining interval an interval of the length $\frac{1}{4^n}$. In the end we will have removed the intervals of the total length

$$\sum_{n=0}^{\infty} \frac{2^n}{4^{n+1}} = \frac{1}{4}\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{2}.$$

By construction, G has empty interior, hence $|\partial G| = |G| = \frac{1}{2}$. Now it suffices to show that (2.1.1) holds for $x \in G$ and $r \in (0,1)$. We call the intervals which remain after n steps G_n, the nth generation. The length of the interval from the nth generation is given by the recurrence formula

$$l_{n+1} = \frac{1}{2}(l_n - 4^{-n-1}), \quad l_0 = 1,$$

which is satisfied by $l_n = 2^{-n-1} + 2^{-2n-1}$.

In order to prove (2.1.1), we first reduce the problem to considering as x the endpoints of the intervals from $G_1, G_2, \ldots$ and then we prove that it suffices to consider $x = 0$.

Assume that (2.1.1) holds for all the endpoints of the intervals from $G_1, G_2, \ldots$. Note that the set of all these endpoints is dense in G. Thus, if $y \in G$ and $r \in (0,1)$, then there is an endpoint x such that $|x-y| < r/2$ and consequently,

$$|B(y,r) \cap G| \geq |B(x,r/2) \cap G| \tag{2.1.2}$$

To see that it suffices to consider $x = 0$, assume that $x \in (0,1/2)$ is an endpoint of an interval from the nth generation. We claim that for every $r > 0$ we have

$$|B(x,r) \cap G| \geq |B(0,r) \cap G|. \tag{2.1.3}$$

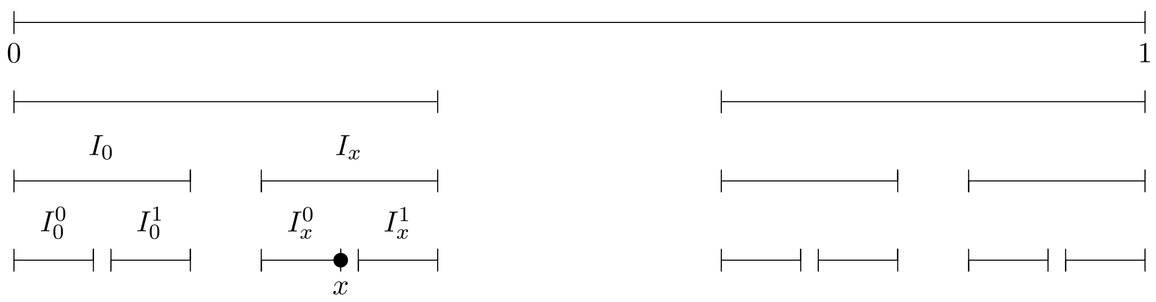

Figure 2.3: Illustration of three steps of the construction of G and an exemplary x which is an endpoint of the interval from the third generation, but not from the second.

In order to obtain (2.1.3) we think of the balls as of clouds which grow over time with the same speed from $r = 0$ to $r = 1$, and cover G on their way. Thus (2.1.3) is equivalent to the fact that at each moment in time, the cloud based in x (x-cloud) covers a larger part of G than the cloud starting from 0 (0-cloud). The latter property is true indeed, as we argue below. Since $G \cap I$ is identical and symmetric with respect to the midpoint of I for each $I \in G_n$, and the covering of the intervals always begins at the endpoint, it suffices to think of covering the intervals from $G_1, \ldots, G_n$ instead of G. We group the intervals of each generation into pairs (twins) having the same parent, as I_0^0, I_0^1 and I_x^0, I_x^1 in Figure 2.3 (but the following argument is, of course, independent of the picture). The covering of I_0^0 and I_x^0 (the intervals adjacent to 0 and x respectively) by the 0-cloud and the x-cloud respectively, will start and end at the same time. Furthermore, the x-cloud will start (and thus, end) covering I_x^1 at the same or earlier time than the 0-cloud starts covering I_0^1. Obviously, the 0-cloud cannot cover G at any other place at this moment. Therefore the x-cloud covers a larger or equal portion of G up to the time when I_0 is covered by the 0-cloud. The parents I_0, I_x also have exactly one twin interval each and the distance between the twins is identical. The twin of I_x will begin being covered by the x-cloud not later than the twin of I_0 by the 0-cloud. By repeating this argument we obtain (2.1.3).

Now, since

$$1 \geq \frac{l_{n+1}}{l_n} = \frac{2^{-n-2} + 2^{-2n-3}}{2^{-n-1} + 2^{-2n-1}} \geq \frac{1}{4},$$

it suffices to consider $r = l_n$. Then we get

$$|B(0, l_n) \cap G| = 2^{-n}|G| = 2^{-n-1} \geq \frac{l_n}{2}. \tag{2.1.4}$$

By (2.1.4), (2.1.3) and (2.1.2) we get that G satisfies (2.1.1) for $r < 1$.

Example 2.1.11. Any set G with *local* exterior or interior cone condition (meaning that for every ball B the cones of fixed aperture and finite height exist for all $x \in \partial G \cap B$), has the boundary of zero Lebesgue measure. In particular, if G is either a Lipschitz set or a locally Lipschitz epigraph, then we have $|\partial G| = 0$. This follows from the *porosity* or *thinness* of $\partial G \cap B$ for every B, that is: there exist $\theta \in (0,1)$ and $r_0 > 0$ such that for every $x \in \partial G \cap B$ and $0 < r < r_0$ we have $B(x', \theta r) \subset B(x, r) \setminus (\partial G \cap B)$ for some $x' \in B(x, r)$, cf. Grandlund, Lindqvist and Martio [80, 4.15 and 4.16].

2.1.3 Whitney decomposition and uniform domains

Some properties of subsets of $\mathbb{R}^d$ are easy to express and to exploit with the use of the Whitney decomposition. This is the case, e.g., for the uniform domains. This subsection is devoted to introducing these two notions, which are crucial for the results of Chapter 5.

A *cube* in $\mathbb{R}^d$ is, as usual, the set $[0, r]^d + x_0$, where $r > 0$ and $x_0 \in \mathbb{R}^d$. Thus, by default, we consider closed cubes. The family of *dyadic* cubes consists of those with $r = 2^k$ and $x_0 = 2^k(c_1, \ldots, c_n)$, where $k, c_1, \ldots, c_n \in \mathbb{Z}$.

For cubes Q, R in $\mathbb{R}^d$ we consider $l(R)$ — the length of the side of R and $D(Q, R) = l(Q) + d(Q, R) + l(R)$, the long distance between Q and R. We ask the reader to bear with the letters D and d serving two purposes each, as it will always be clear what the present meaning is. **The scaling of the cube**, denoted as αQ, $\alpha > 0$, **is done from its center** x_Q. In particular, unless $x_Q = 0$, we **do not** have $\alpha Q = \{\alpha x : x \in Q\}$.

Definition 2.1.12. We say that a family of dyadic cubes $\mathcal{W}$ is a *Whitney decomposition* of D if $\bigcup \mathcal{W} = D$ and for every $Q, S \in \mathcal{W}$,

1. If $Q \neq S$, then $\operatorname{Int} Q \cap \operatorname{Int} S = \emptyset$.
2. There is a constant $C_{\mathcal{W}}$ such that $C_{\mathcal{W}} l(Q) \leq d(Q, \partial D) \leq 4 C_{\mathcal{W}} l(Q)$.
3. If $Q \cap S \neq \emptyset$, then $l(Q) \leq 2 l(S)$.
4. If $Q \subseteq 5S$, then $l(S) \leq 2 l(Q)$.

Here we refer to the seminal paper of Whitney [160, page 67], the monograph of Stein [148, Theorem VI.1], and to a version for *triadic* cubes of Krantz and Parks [105, Theorem 5.3.1].

We note that Definition 2.1.12 is slightly more restrictive than the standard Whitney decomposition because of the two latter conditions, cf. Proposition 1 in [148, Chapter VI]. These conditions were proposed by Prats and Tolsa [129, Definition 3.1] and Prats and Saksman [128, Definition 2.1], with the remark that they are granted if the constant $C_{\mathcal{W}}$ is large enough. They are also used in the paper of the author [138]. For completeness, we present a construction of the decomposition from Definition 2.1.12, by slightly modifying the one given by Stein in [148].

Proposition 2.1.13. *Every open $D \subsetneq \mathbb{R}^d$ admits a Whitney decomposition of Definition 2.1.12.*

Proof. For $k \in \mathbb{Z}$ we let $\mathcal{M}_k$ be the collection of all dyadic cubes in $\mathbb{R}^d$ with side length 2^{-k}. We also define the layers

$$D_k = \{x \in \mathbb{R}^d : c2^{-k} \le d(x, D^c) \le c2^{-k+1}\},$$

where $c > 0$ will be specified later. Consider the family of cubes

$$\mathcal{W}_0 = \bigcup_{k\in\mathbb{Z}} \{Q \in \mathcal{M}_k : Q \cap D_k \neq \emptyset\}.$$

We have $D \subseteq \bigcup \mathcal{W}_0$, because each family $\mathcal{M}_k$ covers $\mathbb{R}^d$, hence D_k as well. Furthermore, if $Q \in \mathcal{W}_0$ and $l(Q) = 2^{-k}$, then there exists a point $x \in Q \cap D_k$, and we have

$$d(Q, D^c) \le d(x, D^c) \le c2^{-k+1} = 2c \cdot l(Q),$$

and

$$d(Q, D^c) \ge d(x, D^c) - \operatorname{diam} Q \ge c2^{-k} - \sqrt{d} l(Q) = l(Q)(c - \sqrt{d}).$$

Together, if we let $c = c'\sqrt{d}$, then we obtain

$$(c' - 1)\sqrt{d} \cdot l(Q) \le d(Q, D^c) \le 2c'\sqrt{d} \cdot l(Q). \tag{2.1.5}$$

Since $(1, \infty) \ni c' \mapsto 2c'/(c' - 1)$ decreases, we see that any $c' \ge 2$ is fit for the second condition of Definition 2.1.12.

Assume that $Q \cap S \neq \emptyset$ and, say, $l(Q) \le l(S)$. Let $x \in Q \cap S$. Then, by (2.1.5) we have $d(x, D^c) \ge d(S, D^c) \ge (c' - 1)\sqrt{d} \cdot l(S)$. On the other hand, $d(x, D^c) \le d(Q, D^c) + \operatorname{diam} Q \le (2c' + 1)\sqrt{d} \cdot l(Q)$. Therefore we obtain

$$l(S) \le \frac{2c' + 1}{c' - 1} l(Q).$$

If we take, e.g., $c' \ge 4$, then we get that $l(Q) \le l(S) \le 3l(Q)$, but since the cubes are dyadic, this in fact yields $l(S) \le 2l(Q)$, as the third condition asserts.

For the fourth property, assume that $Q \subseteq 5S$. Then we have $d(Q, D^c) \ge d(S, D^c) - 2 \operatorname{diam} S$ and by using (2.1.5) we obtain (for $c' > 3$)

$$l(S) \le \frac{2c'}{c' - 3} l(Q).$$

By taking any $c' > 6$ we obtain the last postulate of Definition 2.1.12.

Finally, we refine the family $\mathcal{W}_0$ so that the interiors of the cubes are disjoint: we first note that if the interiors of two dyadic cubes intersect, then one of them is a subset of the other. In order to obtain $\mathcal{W}$, we therefore remove from $\mathcal{W}_0$ all of the cubes which are proper subsets of some other cube in $\mathcal{W}_0$. Obviously all the previously shown properties stay true for $\mathcal{W}$. □

Example 2.1.14. By using the Whitney decomposition we may give another example of a set satisfying VDC but not the cone conditions. Let $D = \mathbb{R}^d \setminus \{0\}$ and let $\mathcal{W}$ be any Whitney decomposition of D. Then $D' = \bigcup_{Q \in \mathcal{W}} \operatorname{Int} Q$ satisfies VDC, because its complement has Lebesgue measure zero, but we fail to find *any* cone with apex at 0, contained in D'.

A sequence of cubes $(Q, R_1, \ldots, R_n, S)$ is a chain connecting Q and S, if every cube is a neighbor of its successor and predecessor (if it has one), by which we mean that their boundaries have nonempty intersection. We will denote the chain as $[Q, S]$ and the sum of the lengths of its cubes as $l([Q, S])$. We let $[Q, S) = [Q, S] \setminus \{S\}$.

Definition 2.1.15. Assume that D is a domain. The Whitney decomposition $\mathcal{W}$ of D is *admissible*, if there exists $\varepsilon > 0$ such that for every pair of cubes Q, S, there exists an ε-admissible chain $[Q, S] = (Q_1, Q_2, \ldots Q_n)$, i.e.,

- $l([Q, S]) \leq \frac{1}{\varepsilon} D(Q, S)$,
- there exists $j_0 \in \{1, \ldots, n\}$ for which $l(Q_j) \geq \varepsilon D(Q, Q_j)$ for every $1 \leq j \leq j_0$, and $l(Q_j) \geq \varepsilon D(Q_j, S)$ for every $j_0 \leq j \leq n$. Q_{j_0} will be denoted as Q_S — the central cube of the chain $[Q, S]$.

A domain which has an admissible Whitney decomposition is called a *uniform domain*.

Unless we state otherwise, for Q, S in an admissible $\mathcal{W}$ by $[Q, S]$ we mean an arbitrary admissible chain connecting Q and S, and Q_S is an arbitrary fixed central cube. Furthermore, $[Q, Q_S]$ is the subchain of cubes from Q to Q_S in $[Q, S]$.

The shadow of a cube is $\mathbf{Sh}_\rho(Q) = \{S \in \mathcal{W} : S \subseteq B(x_Q, \rho l(Q))\}$, $\rho > 0$. We also denote $\mathbf{SH}_\rho(Q) = \bigcup \mathbf{Sh}_\rho(Q)$. Note that we can take a sufficiently large ρ_ε so that

- for every ε-admissible chain $[Q, S]$, and every $P \in [Q, Q_S]$, we have $Q \in \mathbf{Sh}_{\rho_\varepsilon}(P)$,
- if $[Q, S]$ is ε-admissible, then every cube from it belongs to $\mathbf{Sh}_{\rho_\varepsilon}(Q_S)$,
- for every $Q \in \mathcal{W}$, $5Q \subseteq \mathbf{SH}_{\rho_\varepsilon}(Q)$.

From now on we fix ρ_ε and write $\mathbf{Sh}(Q) = \mathbf{Sh}_{\rho_\varepsilon}(Q)$ and $\mathbf{SH}(Q) = \mathbf{SH}_{\rho_\varepsilon}(Q)$. We remark that the shadow is not intended to be separated from the boundary of D, cf. [128, Figure 2].

Uniform domains were introduced by Martio and Sarvas [120] and were defined without the use of Whitney cubes. For further reading on uniform domains we refer to Herron and Koskela [87] and Väisälä [154]. Our use of the Whitney cubes, inspired by [128], facilitates in large the arguments which require the expression of communication from point to point in a domain, and often allows to change integration into summation. The uniformity excludes the domains with too narrow corridors and enforces that for every two points close to each other there is a short path connecting them through D.

Let us present some examples. The domain $\mathbb{R} \times [0, 1] \subset \mathbb{R}^2$ is not uniform, because an admissible chain between two distant points would require a proportionally large central cube. The half-space, on the other hand, is a uniform domain. Any Lipschitz domain (recall the assumption of compact boundary) and Lipschitz epigraph is a uniform domain, cf. [2, 3.7]. The crossed bricks domain from Example 2.1.3 is uniform as well.

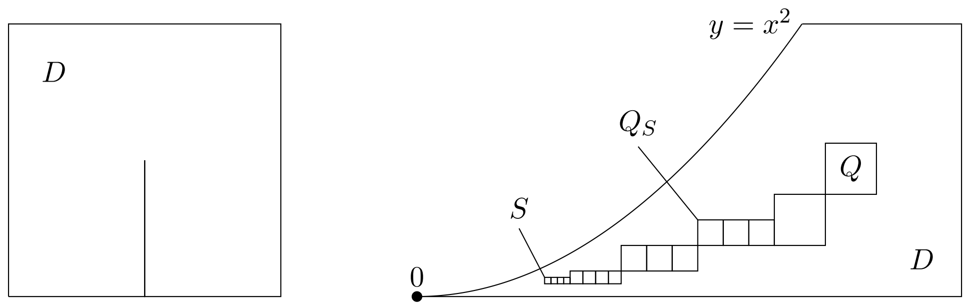

Figure 2.4: The domain on the left is not uniform because two points close on the plane can be far apart in D. The set on the right is not uniform because of the narrowness near 0.

In Figure 2.4, we give two more negative cases. For the one on the right we give a brief explanation: assume that there exists an ε-admissible Whitney decomposition $\mathcal{W}$ and fix Q. We will take S closer and closer to the origin. Note that the size of Q_S is controlled from below by $l(Q)$ because of $l(Q) \le D(Q, Q_S) \le \frac{1}{\varepsilon} l(Q_S)$. Thus, for sufficiently large $n \in \mathbb{N}$, if S intersects the line $x = \frac{1}{n^2}$, then there exists $P \in [S, Q_S]$ which intersects $x = \frac{1}{n}$. Then, $l(P) < \frac{1}{n^2}$ and we have $D(S, P) \ge d(S, P) \approx \frac{1}{n}$, which contradicts the admissibility of $\mathcal{W}$. By an easy modification of this argument we can replace x^2 with any function $xf(x)$ where $f > 0$ is continuous and $f(x) \to 0$ as $x \to 0^+$. Indeed, the cubes intersecting $x = \frac{1}{n}$ must have sides smaller than $\frac{1}{n} f(\frac{1}{n})$, so for every large n it suffices to choose S sufficiently close to 0. We note that $\operatorname{Int} D^c$ is also not uniform which is seen by taking the pairs of points $(\frac{1}{n}, \pm\frac{2}{n} f(\frac{1}{n}))$. The distance between them on the plane is equal to $\frac{4}{n} f(\frac{1}{n})$, but in D^c it exceeds $\frac{2}{n}$.

2.2 Lévy processes: construction and potential theory

As we argued in the introduction, the Lévy processes are one of the main motivations for studying the operator L. Furthermore, they provide powerful intuitions and tools which we apply further in this dissertation to analyze, among other topics, the harmonic functions and trace spaces associated with L. In the first subsection we discuss the construction of the related Lévy process, not only for the completeness of the presentation, but also in order to ensure that the process corresponds to the potential-theoretic notions which are crucial for our development in Chapters 4 and 6. These notions are introduced in the second subsection. In this section we tacitly assume that every considered set and function is Borel.

2.2.1 Construction and properties of Lévy processes

A Lévy process in $\mathbb{R}^d$ is a stochastic process which starts from 0, has independent and stationary increments and is stochastically continuous. There are a few approaches to obtaining a Lévy process corresponding to the symmetric Lévy measure ν, we will focus on the one using the transition densities. Many aspects of the underlying theory are quite technical, therefore we will avoid going into too much details, and we refer to the literature whenever possible.

For a more detailed reading about the following construction of the Lévy process see the first two chapters of the book by Sato [139]. First, we define the *Lévy–Khinchine exponent*:

$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos \xi x) \, \mathrm{d}\nu(x), \quad \xi \in \mathbb{R}^d. \tag{2.2.1}$$

Here and below, ξx is the scalar product in $\mathbb{R}^d$. We note that the above form of ψ is owed to the symmetry of ν. The Lévy–Khinchine representation [139, Theorem 8.1] implies that for every $t > 0$ the function $\xi \mapsto e^{-t\psi(\xi)}$ is a characteristic function of an $\mathbb{R}^d$-valued infinitely divisible random variable. Put differently, for every $t > 0$ there exists a probability measure $p_t(\mathrm{d}x)$ on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} e^{i\xi x}\, p_t(\mathrm{d}x) = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d.$$

For $t > 0$, $x \in \mathbb{R}^d$ and $A \subseteq \mathbb{R}^d$, denote $p_t(x, A) = \int_{A-x} p_t(\mathrm{d}y)$, the *transition kernel*. For the Brownian motion, the transition kernel is equal to the classical heat kernel $(4\pi t)^{-d/2} \exp(x^2/4t)$, so per analogy p_t is also often called the *heat kernel* of the process (X_t). The latter will be defined shortly.

For times $0 \leq t_1 < t_2 < \ldots t_n$ and sets $A_1, A_2, \ldots A_n \subseteq \mathbb{R}^d$ we define the finite-dimensional distributions:

$$\mathbb{P}(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) = \int_{A_1}\int_{A_2} \ldots \int_{A_n} p_{t_1}(\mathrm{d}x_1) p_{t_2-t_1}(x_1, \mathrm{d}x_2) \cdots p_{t_n-t_{n-1}}(x_{n-1}, \mathrm{d}x_n).$$

By the Kolmogorov extension theorem there exists a symmetric Lévy process (X_t) with the distribution $\mathbb{P}$, for details see [139, Theorem 7.10]. In the wording of [139, Section 11], (X_t) is the symmetric Lévy process in $\mathbb{R}^d$ with $(0, \nu, 0)$ as the Lévy triplet. By [139, Theorem 11.5] we may assume without loss of generality, that the trajectories of the process have left limits and are continuous from the right, in short, *càdlàg*. We let, as usual, $X_{t-} = \lim_{s\to t^-} X_s$ for $t > 0$ and $X_{0^-} = X_0$. We will take (X_t) as the canonical projection $X_t(\omega) = \omega(t)$ on the space of càdlàg functions $\omega\colon [0, \infty) \to \mathbb{R}^d$. We will also use the standard complete right-continuous filtration $(\mathcal{F}_t, t \geq 0)$ to analyze (X_t), see Protter [130, Theorem I.31]. The technical assumptions in the two preceding sentences still do not cost us any generality, but they are important for the martingale arguments in the proof of Proposition 6.3.2.

We will often consider the process (X_t) starting from $x \in \mathbb{R}^d$ with the distribution $\mathbb{P}^x$ defined by the relation $\mathbb{P}^x(X_t \in A) = \mathbb{P}(X_t + x \in A)$ for every $t > 0$ and Borel $A \subseteq \mathbb{R}^d$. Let Y be an $\mathbb{R}^d$-valued random variable, measurable with respect to $\mathcal{F} = \bigcup_{t\geq 0} \mathcal{F}_t$. We can also consider the process started at the random point Y with the distribution $\mathbb{P}^Y$. The corresponding expectations are denoted by $\mathbb{E}^x$ and $\mathbb{E}^Y$ respectively. For the detailed treatment of these symbols we refer to [139, Section 40]. For $t > 0$, let θ_t be the shift operator on the trajectories, that is $\theta_t(\omega(\cdot)) = \omega(\cdot + t)$. Accordingly, $Y \circ \theta_t(\omega) = Y(\theta_t(\omega))$. Every Lévy process enjoys the *strong Markov property*, see [139, Corollary 40.11], or Blumenthal and Getoor [15, I.8]. We are mostly interested in its following consequence given in [15, I.8.4]: for every stopping time τ and nonnegative $\mathcal{F}$-measurable random variable Z we have

$$\mathbb{E}^x[Z \circ \theta_\tau | \mathcal{F}_\tau] = \mathbb{E}^{X_\tau} Z, \quad x \in \mathbb{R}^d. \tag{2.2.2}$$

The kernels p_t give rise to a semigroup of operators:

$$P_t f(x) = \int_{\mathbb{R}^d} f(y) p_t(x, \mathrm{d}y), \quad t > 0,\ x \in \mathbb{R}^d. \tag{2.2.3}$$

It is well-known that P_t is a strongly continuous semigroup of contractions on $C_0(\mathbb{R}^d)$, see, e.g., Applebaum [4, Theorem 3.1.9]. This can be rephrased as: *Every Lévy process is a Feller process.*

Example 2.2.1. If $\alpha \in (0, 2)$ and $\mathrm{d}\nu(x) = C_{d,\alpha}|x|^{-d-\alpha}\,\mathrm{d}x$ for $x \in \mathbb{R}^d$, then the process (X_t) is called the *isotropic α-stable process*. In the literature the adjective *isotropic* is often replaced by *symmetric*, but this may be quite confusing when compared to our notion of symmetry of the Lévy measure.

In order to gain some intuition it is useful to think of ν as the jumping intensity of (X_t). This connection is subtle, but it is rather clear for symmetric Lévy processes. It can be seen, e.g., through the Lévy–Itô decomposition [139, Section 19].

2.2.2 Elements of potential theory

Below we introduce the notions which play an essential role in almost all of the results of Chapters 4 and 6. In order to better handle these objects we need to introduce some assumptions in this subsection. Namely, we assume that the Lévy measure has a radially symmetric density with a nonincreasing profile. In short we say that ν is *unimodal*. We also stipulate that ν is strictly positive in $\mathbb{R}^d$ and $\nu(\mathbb{R}^d) = \infty$. In this setting, $p_t(\mathrm{d}x)$ are absolutely continuous with respect to the Lebesgue measure and unimodal as well, see [139, Theorem 27.7] and Kulczycki and Ryznar [107, Lemma 2.5].

We let τ_D be the first exit time from the open set D, that is,

$$\tau_D = \inf\{t > 0 : X_t \notin D\}. \tag{2.2.4}$$

By [139, Theorem 40.13] τ_D is a stopping time.

The *Dirichlet heat kernel* $p_t^D(x, y)$ is

$$p_t^D(x, y) = p_t(x, y) - \mathbb{E}^x[p_{t-\tau_D}(X_{\tau_D}, y);\ \tau_D < t], \quad t > 0,\ x, y \in \mathbb{R}^d. \tag{2.2.5}$$

The above equation is commonly called the *Hunt's formula.* Note that it subtracts from the heat kernel the influence of all the trajectories which leave D before the time t. It is in fact the transition density of the killed process, see Chung and Zhao [44, Theorem 2.4], that is,

$$p_t^D(x, A) = \mathbb{P}^x(X_t \in A,\ \tau_D > t), \quad t > 0,\ x, y \in \mathbb{R}^d. \tag{2.2.6}$$

For a good glimpse at the behavior of p_t^D we refer to the works concerning the fractional Laplacian: the factorization formula for fat domains by Bogdan, Grzywny and Ryznar [23, Theorem 1] and the explicit estimates for $C^{1,1}$ domains by Chen, Kim and Song [42, Theorem 1.1].

The *Green function* of D is

$$G_D(x, y) = \int_0^\infty p_t^D(x, y)\,\mathrm{d}t, \quad x, y \in \mathbb{R}^d. \tag{2.2.7}$$

We discuss the finiteness of the Green function in Subsection A.1.3. This is not a great concern for us, because in the sequel G_D is finite whenever our arguments require that, in particular, for all bounded D and for all D satisfying (2.2.13), cf. (2.2.11).

For functions $f \geq 0$, by Tonelli's theorem and (2.2.6), we have

$$\int_{\mathbb{R}^d} G_D(x, y) f(y)\,\mathrm{d}y = \int_0^\infty \int_{\mathbb{R}^d} f(y) p_t^D(x, y)\,\mathrm{d}y\mathrm{d}t = \mathbb{E}^x \int_0^{\tau_D} f(X_t)\,\mathrm{d}t, \quad x \in \mathbb{R}^d. \tag{2.2.8}$$

Accordingly, $G_D(x, y)$ is interpreted as the occupation time density of (X_t) prior to the first exit from D. The expressions in (2.2.8) are called the Green operator, or the Green potential of f, and are denoted by $G_D[f](x)$. By taking $f \equiv 1$ we obtain that

$$G_D[1](x) = \int_{\mathbb{R}^d} G_D(x, y)\,\mathrm{d}y = \mathbb{E}^x \tau_D, \quad x \in \mathbb{R}^d. \tag{2.2.9}$$

We note that $G_D(x, y)$ and $p_t^D(x, y)$ are symmetric and they are equal to 0 whenever $x \in D^c$ or $y \in D^c$.

Understanding the process (X_t) upon exiting D is vital for our development. The following generalized Ikeda–Watanabe formula defines the joint distribution of $(\tau_D, X_{\tau_D-}, X_{\tau_D})$ restricted to the event $\{\tau_D < \infty,\, X_{\tau_D-} \neq X_{\tau_D}\}$: if $x \in D$, then

$$\mathbb{P}^x(\tau_D \in I,\, A \ni X_{\tau_D-} \neq X_{\tau_D} \in G) = \int\limits_I \int\limits_G \int\limits_A p_u^D(x,y)\nu(y,z)\,\mathrm{d}y\mathrm{d}z\mathrm{d}u, \tag{2.2.10}$$

see, e.g., Bogdan, Rosiński, Serafin and Wojciechowski [28, Section 4.2], or Ikeda and Watanabe [92] for the original contribution. By taking $I = (0,\infty)$, $A = D$ and $G \subset D^c$, we obtain a simpler form:

$$\mathbb{P}^x(\tau_D < \infty,\, X_{\tau_D-} \neq X_{\tau_D},\, X_{\tau_D} \in G) = \int_G \int_D G_D(x,y)\nu(y,z)\,\mathrm{d}y\mathrm{d}z, \quad x \in D. \tag{2.2.11}$$

This motivates the following definition of the *Poisson kernel*:

$$P_D(x,z) = \int_D G_D(x,y)\nu(y,z)\,\mathrm{d}y, \quad x \in D,\ z \in D^c. \tag{2.2.12}$$

For the rest of this work, the above equation, and none other, will be referred to as the *Ikeda–Watanabe formula.*

By (2.2.11), P_D is the density of the distribution of X_{τ_D} restricted to the event that (X_t) exits from D by a jump. Below we give conditions for ν and D under which this event almost surely holds.

Lemma 2.2.2. *Assume that for every $\lambda, r \in (0,1]$ we have $\nu(\lambda r) \le c\lambda^{-d-\alpha}\nu(r)$. If $|\partial D| = 0$ and VDC holds locally for D^c, then for every $x \in D$ we have $\mathbb{P}^x(X_{\tau_D} \in \partial D) = 0$.*

The proof is quite technical and it is given in the Appendix. For the narrower class of Lipschitz open sets and all isotropic pure-jump Lévy processes with infinite Lévy measure the result is stated by Sztonyk after Theorem 1 in [149]. By using the arguments from the proof of [16, Lemma 17] by Bogdan, we obtain the following consequence of Lemma 2.2.2.

Corollary 2.2.3. *Assume that $\mathbb{P}^x(\tau_D < \infty) = 1$ for $x \in \mathbb{R}^d$. Then, under the assumptions of Lemma 2.2.2,*

$$\mathbb{P}^x(\tau_D < \infty,\, X_{\tau_D-} \neq X_{\tau_D}) = 1. \tag{2.2.13}$$

As a consequence, we have

$$\mathbb{P}^x(X_{\tau_D} \in G) = \int_G P_D(x,z)\,\mathrm{d}z, \quad x \in D,\ G \subseteq D^c, \tag{2.2.14}$$

and

$$\int_{D^c} P_D(x,z)\,\mathrm{d}z = 1, \quad x \in \mathbb{R}^d. \tag{2.2.15}$$

Remark 2.2.4. Let us comment on the condition $\mathbb{P}^x(\tau_D < \infty) = 1$. It is satisfied, e.g., for all bounded sets D, cf. Pruitt [131] and for the half-space, because any one-dimensional projection of the unimodal process (X_t) is a nondegenerate symmetric Lévy process [139, Proposition 11.10] and thus oscillates by [139, Proposition 37.10]. The complement of a ball may however fail to satisfy this condition, see Grzywny and Kwaśnicki [82, Theorem 1.1].

The name 'Poisson kernel' also stems from the classical potential theory — it is well-known that the classical Poisson kernels for the disk and the half-plane are the distribution of the Brownian motion upon the first exit from the respective domain. The possibility of solving the classical Dirichlet problem by using the exit distribution of the Brownian motion was observed by Kakutani [99] and it serves as an important inspiration for probabilistic potential theory, as well as for our work. In rare instances, the Poisson kernel has an explicit form.

Example 2.2.5. If $\nu(x) = C_{d,\alpha}|x|^{-d-\alpha}$ and $\alpha \in (0,2)$, then we have the following formula for the ball $B = B(0,r)$,

$$P_B(x,z) = c\frac{(|z|^2 - r^2)^{\alpha/2}}{(r^2 - |x|^2)^{\alpha/2}}\frac{1}{|x-z|^d}, \quad x \in B,\ z \in B^c.$$

This result is due to M. Riesz [133], see also Landkof [111, (1.6.11)].

Remark 2.2.6. In Chapters 4 and 6 we will frequently stipulate that $\nu(r) \approx \nu(r+1)$ for $r \geq 1$. With this assumption, for bounded set D we easily see that for all $x, y \in D$ and $z \in D^c$ with $d(z, D) \geq \rho > 0$,

$$\nu(x,z) \approx \nu(y,z), \tag{2.2.16}$$

where comparability constants depend on ν, D and ρ. Consequently, (2.2.12) and (2.2.9) imply

$$P_D(x,z) \approx \nu(x,z)\mathbb{E}^x\tau_D, \quad x \in D,\ d(z,D) \geq \rho > 0, \tag{2.2.17}$$

with the same proviso on comparability constants. Note that if D is bounded and $x \in D$ is fixed, then $\mathbb{E}^x\tau_D$ is bounded by a positive constant, see [131].

For functions $g\colon D^c \to \mathbb{R}$ we define the *Poisson integral* (or *Poisson extension*):

$$P_D[g](x) = \begin{cases} \int_{D^c} g(z)P_D(x,z)\,\mathrm{d}z, & x \in D, \\ g(x), & x \in D^c. \end{cases} \tag{2.2.18}$$

In the above definition we assume that the integrals converge absolutely for every $x \in D$. Note that under the setting of Corollary 2.2.3 we have

$$P_D[g](x) = \mathbb{E}^x g(X_{\tau_D}), \quad x \in \mathbb{R}^d. \tag{2.2.19}$$

The following kernel of interaction via D, in short, *interaction kernel*, is essential for our expressions of trace spaces in Chapters 4 and 6:

$$\gamma_D(z,w) = \int_D\int_D \nu(w,x)G_D(x,y)\nu(y,z)\,\mathrm{d}x\mathrm{d}y. \tag{2.2.20}$$

The interaction kernel is a lesser known object than P_D and G_D, but it appears, e.g., in the book of Chen and Fukushima as a special case of a Feller measure [40, Theorem 5.7.6]. We immediately see that γ_D is symmetric and by the Ikeda–Watanabe formula we get

$$\gamma_D(w,z) = \int_D \nu(w,x)P_D(x,z)\,\mathrm{d}x = \int_D \nu(z,x)P_D(x,w)\,\mathrm{d}x = \gamma_D(z,w), \quad z,w \in D^c.$$

Example 2.2.7. Let $d = 1$, $D = (0,\infty) \subset \mathbb{R}$ and $\nu(w,x) = \pi^{-1}|x-w|^{-2}$ for $x, w \in \mathbb{R}$. The process corresponding to ν is called the Cauchy, or 1-stable process. Then

$$P_{(0,\infty)}(x,z) = \pi^{-1}x^{1/2}|z|^{-1/2}(x-z)^{-1}, \quad x > 0,\ z < 0,$$

see, e.g., Bogdan [17, (3.40)]. A direct calculation yields

$$\gamma_{(0,\infty)}(z,w) = \int_0^\infty \frac{1}{\pi^2}\frac{\sqrt{x}}{\sqrt{|z|}}\frac{\mathrm{d}x}{(x-z)(x-w)^2} = \frac{1}{2\pi\sqrt{zw}(\sqrt{|z|}+\sqrt{|w|})^2}, \quad z,w < 0.$$

We provide the estimates of γ_D for $C^{1,1}$ domains under certain assumptions on ν in Theorem 4.2.5.

We note that for $U \subset D$, the inequalities $p^U \leq p^D$ and $G_U \leq G_D$ hold true. Also, $P_U(x,z) \leq P_D(x,z)$ for $x \in U$, $z \in D^c$, and $\gamma_U(z,w) \leq \gamma_D(z,w)$ for $z,w \in D^c$. These inequalities are referred to as *domain monotonicity* and they all follow from Hunt's formula (2.2.5).

2.3 Nonlocal operators, quadratic forms and Sobolev spaces

In this section we discuss the nonlocal operator L, defined in (L), and the related quadratic forms and Sobolev spaces. Unless we say otherwise, we work in the context of general symmetric Lévy measures, mainly in order to avoid repeating too much material later on in Chapter 3, which contains the more tedious aspects of the discussion of singular ν.

2.3.1 The operator

We start with the basics concerning the well-definiteness of the operator L. For the convenience of the reader we once again display its definition:

$$Lu(x) = \lim_{\epsilon \to 0^+} \int_{|y|>\epsilon} (u(x) - u(x+y))\, \mathrm{d}\nu(y). \tag{2.3.1}$$

The operators of the above form constitute a prominent subclass of the operators appearing in the Courrège's theorem [47, 3.4]. The latter gives the representation of the operators satisfying the positive maximum principle. In this connection we mention that L may be approached as a Fourier multiplier, see, e.g., Hoh [89] or Bañuelos and Bogdan [6], but we note that the connections between various definitions of L usually are known to hold only for certain classes of functions, cf. Lemma 2.3.4 below and Kwaśnicki [109]. Explicit formulas for Lu are scarce, but they do exist for the fractional Laplacian and certain functions u, see Dyda [61] and the references therein.

As an initial step towards the well-definiteness of L, we note that for every $u \in C_b(\mathbb{R}^d)$ and $\epsilon > 0$, we have

$$\int\limits_{|y|>\epsilon} |u(x) - u(x+y)|\, \mathrm{d}\nu(y) \le 2\nu(B(0,\epsilon)^c)\|u\|_\infty < \infty. \tag{2.3.2}$$

Our formula for the operator L is pointwise and it may depend on the value of the function at a single point. This is because the measure ν is not necessarily absolutely continuous. Therefore, the formula (2.3.1) may yield different results for functions that are equal almost everywhere. Fortunately, the ambiguity is rather negligible, as we demonstrate below.

Proposition 2.3.1. *If the functions u, v are measurable, $u = v$ a.e. in $\mathbb{R}^d$ and Lu, Lv are well defined a.e. in $\mathbb{R}^d$, then $Lu = Lv$ a.e. in $\mathbb{R}^d$.*

Proof. First note that, being the limits of measurable functions, Lu and Lv are measurable. Furthermore, since they are well-defined and finite a.e., we have

$$\int_{\mathbb{R}^d} |Lu(x) - Lv(x)|\, \mathrm{d}x = \int_{\mathbb{R}^d} \lim_{\epsilon\to 0^+} \left| \int_{|y|>\epsilon} ((u-v)(x) - (u-v)(x+y))\, \mathrm{d}\nu(y) \right| \mathrm{d}x. \tag{2.3.3}$$

It suffices to show that the expression on the right-hand side is equal to 0. By using the triangle inequality, the monotone convergence theorem and Fubini's theorem, we can estimate (2.3.3) as follows

$$\begin{aligned}
&\int_{\mathbb{R}^d} \lim_{\epsilon\to 0^+} \left| \int_{|y|>\epsilon} ((u-v)(x) - (u-v)(x+y))\, \mathrm{d}\nu(y) \right| \mathrm{d}x \\
\le &\int_{\mathbb{R}^d} \lim_{\epsilon\to 0^+} \int_{|y|>\epsilon} |(u-v)(x) - (u-v)(x+y)|\, \mathrm{d}\nu(y)\mathrm{d}x \\
= &\lim_{\epsilon\to 0^+} \int_{\mathbb{R}^d} \int_{|y|>\epsilon} |(u-v)(x) - (u-v)(x+y)|\, \mathrm{d}\nu(y)\mathrm{d}x
\end{aligned}$$

$$= \lim_{\epsilon\to 0^+} \int_{|y|>\epsilon} \int_{\mathbb{R}^d} |(u-v)(x) - (u-v)(x+y)| \, \mathrm{d}x \mathrm{d}\nu(y).$$

Since the inner integral is equal to 0 for every $y \in \mathbb{R}^d$, the proposition is proved. □

Proposition 2.3.2. *If $u \in C_b^2(\mathbb{R}^d)$, then $Lu(x)$ is well defined for $x \in \mathbb{R}^d$ and Lu is bounded on $\mathbb{R}^d$. Consequently, for every bounded D we have $Lu \in L^2(D)$.*

Proof. Let $u \in C_b^2(\mathbb{R}^d)$. Substituting $-y$ for y in (2.3.1) and adding side by side gives

$$Lu(x) = \tfrac{1}{2} \lim_{\epsilon\to 0^+} \int_{|y|>\epsilon} (2u(x) - u(x+y) - u(x-y)) \, \mathrm{d}\nu(y). \tag{2.3.4}$$

By Taylor's expansion, for $x, y \in \mathbb{R}^d$:

$$\begin{aligned} 2u(x) - u(x+y) - u(x-y) &= 2u(x) - \left[u(x) + y \circ \nabla u(x) + \sum_{i,j=1}^{n} \frac{\partial^2 u(\xi)}{\partial x_i \partial x_j} y_i y_j \right] \\ &\quad - \left[u(x) - y \circ \nabla u(x) + \sum_{i,j=1}^{n} \frac{\partial^2 u(\xi)}{\partial x_i \partial x_j} y_i y_j \right] \\ &= -2 \sum_{i,j=1}^{n} \frac{\partial^2 u(\xi)}{\partial x_i \partial x_j} y_i y_j, \end{aligned}$$

where $\xi \in B(x, |y|)$. Since $u \in C_b^2(\mathbb{R}^d)$, we obtain

$$|2u(x) - u(x+y) - u(x-y)| \le c(1 \wedge |y|^2), \quad y \in \mathbb{R}^d, \tag{2.3.5}$$

for a constant c independent of x. As a consequence, $\int_{\mathbb{R}^d} (2u(x) - u(x+y) - u(x-y)) \, \mathrm{d}\nu(y)$ converges absolutely. By the dominated convergence theorem,

$$Lu(x) = \lim_{\epsilon\to 0^+} \int_{|y|>\epsilon} (u(x) - u(x+y)) \, \mathrm{d}\nu(y) = \tfrac{1}{2} \int_{\mathbb{R}^d} (2u(x) - u(x+y) - u(x-y)) \, \mathrm{d}\nu(y). \tag{2.3.6}$$

The boundedness of u follows from (2.3.5). □

Remark 2.3.3. As we have announced in the Introduction, we completely avoid the discussion of the domain of the operator. Since we are more focused on the quadratic forms, we are satisfied with the results for L which concern certain classes of functions. Kwaśnicki [109] precisely formulates the domains for $L = (-\Delta)^{\alpha/2}$; our definition (2.3.1) is called the singular integral therein.

Important motivations for studying L come from the theory of Lévy processes. For $x \in \mathbb{R}^d$ we let

$$\mathcal{L}u(x) = \lim_{t\to 0^+} \frac{P_t u(x) - u(x)}{t}, \tag{2.3.7}$$

whenever the limit exists. This is the *generator* of the semigroup P_t given by (2.2.3). We also consider a somewhat similar expression

$$\mathcal{U}u(x) = \lim_{r\to 0^+} \frac{\mathbb{E}^x u(X_{\tau_{B(x,r)}}) - u(x)}{\mathbb{E}^x \tau_{B(x,r)}}, \tag{2.3.8}$$

the *Dynkin characteristic operator*. We note that owing to the symmetry of ν, for every $u \in C_c^2(\mathbb{R}^d)$ we have

$$Lu(x) = -\int_{\mathbb{R}^d} (u(x+y) - u(x) - y \circ \nabla u(x) \mathbf{1}_{B(0,1)}(y)) \, \mathrm{d}\nu(y), \quad x \in \mathbb{R}^d.$$

The absolute convergence of the integral follows from the Taylor's formula. This, in fact, is the representation of the generator given in [139, Theorem 31.5]. Putting this together with the result of Dynkin [64, Chapter V.3] we obtain following fact.

Lemma 2.3.4. *Assume that $u \in C_c^2(\mathbb{R}^d)$. Then $\mathcal{L}u$ and $\mathcal{U}u$ are well-defined and*

$$-Lu(x) = \mathcal{L}u(x) = \mathcal{U}u(x), \quad x \in \mathbb{R}^d.$$

The above result justifies the name *Lévy operator* for L. The name Lévy-type operator usually concerns similar operators, but with the jumping kernels which need not be space-homogeneous.

Results of the type of Lemma 2.3.4 may be obtained for larger classes of functions under certain assumptions on ν. See, e.g., Kühn and Schilling [106, Theorem 3.2] for a detailed study for Lévy-type operators similar to the fractional Laplacian.

2.3.2 Quadratic forms and Sobolev spaces

In this subsection we establish basic definitions and facts concerning the quadratic forms associated with the Lévy measure ν. We show their explicit connection with the operator L and we discuss various approaches to the forms on the subsets of $\mathbb{R}^d$. In order to keep the presentation clear, we stay in the context of nonlocal Sobolev spaces (that is, $p = 2$). The more general classes announced in the Introduction are discussed further in the dissertation: Triebel–Lizorkin spaces in Chapter 5 and Sobolev–Bregman spaces in Chapter 6. In this subsection we focus on the differences stemming from considering different domains of integration and the importance of the underlying L^2 spaces.

We recall the definition of the form on the whole space:

$$\mathcal{E}[u] = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x+y))^2 \, \mathrm{d}\nu(x) dy.$$

For safety, we note that by Tonelli's theorem, the value of $\mathcal{E}$ stays the same if u is changed on the set of null Lebesgue measure. By Fukushima, Oshima and Takeda [72, Example 1.4.1] $\mathcal{E}$ is the *Dirichlet form* of the process (X_t), associated with ν, defined in Section 2.2, see also Section 5.7. It is an important representative of the class of nonlocal regular Dirichlet forms according to the Beurling–Deny formula [12], see [72, Theorem 3.2.1] for a more modern (and English) formulation. The connection between $\mathcal{E}$ and L is very straightforward. It is best seen on the following formal computation, which uses the substitution $y \to -y$ and $x \to x + y$, and the symmetry of ν:

$$\begin{aligned}\int_{\mathbb{R}^d} Lu(x)u(x) \, \mathrm{d}x &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(x+y))u(x) \, \mathrm{d}\nu(y)\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x+y) - u(x))u(x+y) \, \mathrm{d}\nu(y)\mathrm{d}x \\ &= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(x+y))^2 \, \mathrm{d}\nu(x)\mathrm{d}y = \mathcal{E}[u].\end{aligned}$$

This can be made strict, e.g., for every $u \in C_c^2(\mathbb{R}^d)$, see Proposition 3.2.4. Note that the above calculation yields the positive definiteness of L.

For the remainder of this subsection, in order to maintain clarity, we let ν be absolutely continuous, that is, $\mathrm{d}\nu(y) = \nu(y)\,\mathrm{d}y$, and we refer to Chapter 3 or [137] for the case of singular ν. Recall that

$$\mathcal{E}[u] = \tfrac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x)-u(y))^2\nu(x,y)\,\mathrm{d}y\mathrm{d}x.$$

Lemma 2.3.5. *If u is Lipschitz and compactly supported, then $\mathcal{E}[u] < \infty$.*

Proof. Let $K = \operatorname{supp} u$. Then $u(x) - u(y) = 0$ for $x, y \in K^c$ and $(u(x)-u(y))^2 \lesssim 1 \wedge |x-y|^2$ for $x, y \in \mathbb{R}^d$. Therefore

$$\begin{aligned}2\mathcal{E}[u] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x)-u(y))^2\nu(x,y)\,\mathrm{d}y\mathrm{d}x &= \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus K^c\times K^c}(u(x)-u(y))^2\nu(x,y)\,\mathrm{d}y\mathrm{d}x\\ &\lesssim \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus K^c\times K^c}(1\wedge|x-y|^2)\nu(x,y)\,\mathrm{d}y\mathrm{d}x.\end{aligned}$$

By the inclusion $\mathbb{R}^d\times\mathbb{R}^d\setminus K^c\times K^c \subseteq (K\times\mathbb{R}^d)\cup(\mathbb{R}^d\times K)$ and the symmetry of the integrand this is less than or equal to

$$2\int_K\int_{\mathbb{R}^d}(1\wedge|x-y|^2)\nu(x,y)\,\mathrm{d}y\mathrm{d}x,$$

which is finite by the definition of ν and the boundedness of K. □

For singular ν the symmetries are much less obvious, but the result follows by taking $D = \Omega \supset K$ in [137, Lemma 4.9 and Corollary 3.3].

We define the *nonlocal Sobolev space* as follows:

$$\widetilde{\mathcal{V}}_{\mathbb{R}^d} = \{u \in L^2(\mathbb{R}^d) : \mathcal{E}[u] < \infty\}. \tag{2.3.9}$$

The **norms** will be discussed shortly, see (2.3.15) and the discussion following it. We omit ν in the notation, which should be less intimidating and harmless, because ν will always be fixed within a discussion. The form $\mathcal{E}$ is a nonlocal analogue of the Dirichlet integral

$$\int_{\mathbb{R}^d}|\nabla u(x)|^2\,\mathrm{d}x,$$

and $\widetilde{\mathcal{V}}_{\mathbb{R}^d}$ is the counterpart of the classical Sobolev space $W^{1,2}(\mathbb{R}^d)$ (see, e.g., Evans [65, Chapter 5]). This analogy is best seen through the equality $\int u\Delta u = -\int|\nabla u|^2$ for $u \in C_c^2(\mathbb{R}^d)$. A more abstract similarity is that the Dirichlet integral with the domain $W^{1,2}(\mathbb{R}^d)$ is the Dirichlet form related to the Laplacian Δ, see Ma and Röckner [119, page 42]. We also note that for $\Delta^{\alpha/2}$, the form $\mathcal{E}$ converges to the Dirichlet integral as $\alpha \to 2^-$. This follows from the Fourier transform characterization of $\mathcal{E}$, see, e.g., [72, (1.4.28)].

Remark 2.3.6. If ν has a nonintegrable singularity at the origin, then in order for $\mathcal{E}[u]$ to be finite, u needs to compensate ν by having small increments $u(x) - u(y)$ for x close to y. This justifies the jargon that the finiteness of $\mathcal{E}$ is a way of measuring the smoothness of u.

In view of the application in the nonlocal Dirichlet problem (DP) it is unnecessarily restrictive to require the *smoothness* of u on D^c, but that is what $\mathcal{E}$ does. The remedy that we will use in Chapters 3, 4 and 6, is to consider the following quadratic form:

$$\mathcal{E}_D[u] = \frac{1}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} (u(x)-u(y))^2 \nu(x,y)\,\mathrm{d}x\mathrm{d}y. \tag{2.3.10}$$

The related Sobolev space is

$$\widetilde{\mathcal{V}}_D = \{u \in L^2(D) : \mathcal{E}_D[u] < \infty\}. \tag{2.3.11}$$

We note that we require the square integrability of u only in D rather than on the whole of $\mathbb{R}^d$. This suffices for our purposes in the nonlocal equations as we will see in the next chapter. In this vein we also consider an even less restrictive space

$$\mathcal{V}_D = \{u\colon \mathbb{R}^d \to \mathbb{R} \mid \mathcal{E}_D[u] < \infty\}. \tag{2.3.12}$$

We note the lack of the square integrability in D. We work with the spaces $\mathcal{V}_D$ in the context of the extension and trace operators in Chapter 4, and in its setting, in Lemma 4.3.1 we show that $\mathcal{V}_D \subseteq L^2_{loc}(D)$, which yields $\widetilde{\mathcal{V}}_D = \mathcal{V}_D$ for bounded D. Thanks to that, the results of Chapter 4 are applicable in Chapter 3, whose methods require the L^2 integrability on D. The underlying L^2 space is also very important when one wants to study $\mathcal{E}_D$ as a Dirichlet form and obtain a stochastic process related to that form. This topic recently became active, with the works of Gounoue, Kassmann and Voigt [70, Corollary 2.12] and Vondraček [156].

As customary, our formulation of weak solutions in Chapter 3 requires a class of test functions. Thus, we let

$$\mathcal{V}_D^0 = \{u \in \mathcal{V}_D : u = 0 \ a.e. \text{ on } D^c\} = \{u \in \mathcal{V}_{\mathbb{R}^d} : u = 0 \ a.e. \text{ on } D^c\}. \tag{2.3.13}$$

The above equality of spaces follows from the fact that if u vanishes on D^c, then $u(x)-u(y) = 0$ on $D^c \times D^c$, which yields $\mathcal{E}_D[u] = \mathcal{E}_{\mathbb{R}^d}[u]$. Accordingly, we define

$$\widetilde{\mathcal{V}}_D^0 = \mathcal{V}_D^0 \cap L^2(D). \tag{2.3.14}$$

Note that $\widetilde{\mathcal{V}}_D^0 = \widetilde{\mathcal{V}}_{\mathbb{R}^d}$. We endow $\widetilde{\mathcal{V}}_D^0$ with the following norm:

$$\|u\|_{\widetilde{\mathcal{V}}_D} = \Big(\|u\|^2_{L^2(D)} + \mathcal{E}[u]\Big)^{1/2}. \tag{2.3.15}$$

This is indeed a norm on $\widetilde{\mathcal{V}}_D^0$, because $\mathcal{E}[\cdot]^{1/2}$ is a seminorm and any nonzero function in $\widetilde{\mathcal{V}}_D^0$ has to have positive $L^2(D)$ norm. As we argue in Lemma 3.2.2, $(\widetilde{\mathcal{V}}_D^0, \|\cdot\|_{\widetilde{\mathcal{V}}_D})$ is a Hilbert space. Note that $\widetilde{\mathcal{V}}_{\mathbb{R}^d}^0 = \widetilde{\mathcal{V}}_{\mathbb{R}^d}$. We do not define any norms for the spaces $\mathcal{V}_D$ and $\widetilde{\mathcal{V}}_D$ here, because it is unnecessary for our development, but we note as a digression that for some ν and D, $\mathcal{V}_D$ may be normed in the way that it becomes a Hilbert space, see Lemma 4.3.3. However, for general ν and D it is unclear what the norm should be, as the following example demonstrates.

Example 2.3.7. Let $\nu(x) = \mathbf{1}_{B(0,1)}(x)$, $D = B(0,1)$ and $u(x) = \mathbf{1}_{B(0,2)^c}(x)$. Then clearly, $\int_D u(x)^2\,\mathrm{d}x = 0$. Furthermore, by inspecting the support of $u(x)-u(y)$ we see that

$$\mathcal{E}_D[u] = \int_{B(0,1)}\int_{B(0,2)^c} \nu(x,y)\,\mathrm{d}x\mathrm{d}y,$$

which is equal to 0 for the considered ν. Thus we have $u \in \widetilde{\mathcal{V}}_D$ and $\|u\|_{L^2(D)} = \mathcal{E}_D[u] = 0$.

From the above example we see that $\|\cdot\|_{\widetilde{\mathcal{V}}_D}$ need not be a norm on $\widetilde{\mathcal{V}}_D$, but we remark that despite the problems with norming, $\widetilde{\mathcal{V}}_D$ is still perfectly useful for studying weak solutions of (DP), see Theorem 3.1.1. To this end we will use the following bilinear version of $\mathcal{E}_D$, well-defined, e.g., for $u, v \in \mathcal{V}_D$:

$$\mathcal{E}_D(u,v) = \frac{1}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} (u(x)-u(y))(v(x)-v(y))\nu(x,y)\,\mathrm{d}x\mathrm{d}y. \tag{2.3.16}$$

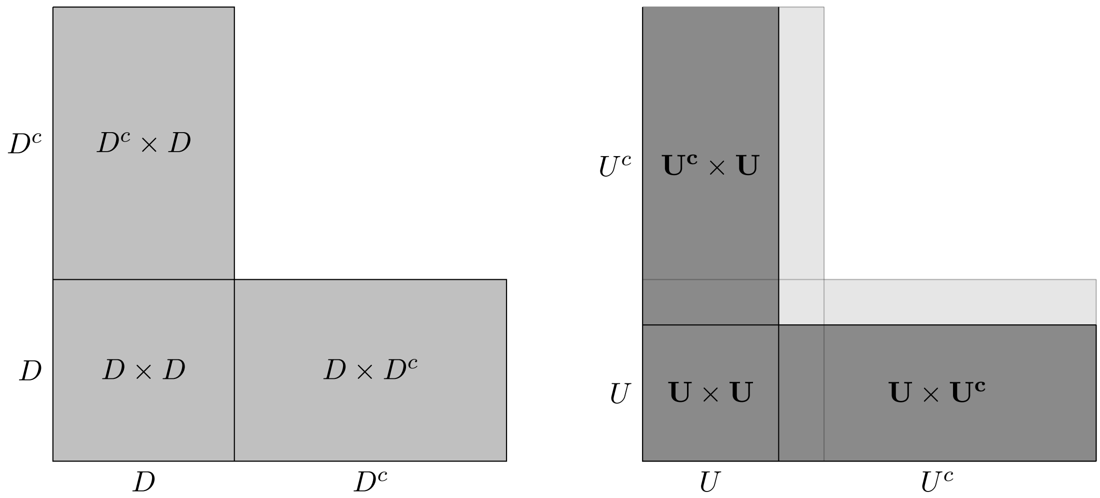

Figure 2.5: Visualization of the integration domain $\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c$ and the comparison with its counterpart for $U \subset D$. Note that $D^c \times D^c \subset U^c \times U^c$.

We also let $\mathcal{E}(u,v) = \mathcal{E}_{\mathbb{R}^d}(u,v)$. Observe that

$$\mathcal{E}_D(u,v) = \mathcal{E}_{\mathbb{R}^d}(u,v), \quad u \in \mathcal{V}_D,\ v \in \mathcal{V}_D^0. \tag{2.3.17}$$

The analogues of the above forms for singular Lévy measures are given in Section 3.2. In this connection we highlight the following formula, which was also suggested in the proof of Lemma 2.3.5 (see also Lemma 3.2.1):

$$\mathcal{E}_D[u] \approx \int_D \int_{\mathbb{R}^d} (u(x)-u(y))^2 \nu(x,y)\,\mathrm{d}x\mathrm{d}y.$$

We have the following monotonicity property, see also [137, Lemma 3.6].

Lemma 2.3.8. *If $U \subseteq D$, then $\mathcal{E}_U \le \mathcal{E}_D$ and $\mathcal{V}_D \subseteq \mathcal{V}_U$. In particular, for every $D \subseteq \mathbb{R}^d$ we have $\mathcal{V}_{\mathbb{R}^d} \subseteq \mathcal{V}_D$.*

Proof. The statements follow straight from the inclusions of the domains of integration, see Figure 2.5. □

There is another, just as popular approach to the forms and Sobolev spaces on the subsets of $\mathbb{R}^d$, which we will focus on in Chapter 5. We let

$$F_{2,2}(D) := \Big\{ u \in L^2(D) : \int_D \int_D (u(x)-u(y))^2 \nu(x,y)\, dydx < \infty \Big\}. \tag{2.3.18}$$

The norm is given by the formula

$$\|u\|_{F_{2,2}(D)} = \left(\|u\|^2_{L^2(D)} + \int_D \int_D (u(x) - u(y))^2 \nu(x, y)\, dydx \right)^{1/2}.$$

With that norm $F_{2,2}(D)$ is a Hilbert space, the verification is similar to the one given in Lemma 3.2.2. Obviously, we have $F_{2,2}(\mathbb{R}^d) = \widetilde{\mathcal{V}}_{\mathbb{R}^d}$. The notation $F_{2,2}$ is intended as a special case of Triebel–Lizorkin space $F_{p,q}$ which we introduce in Chapter 5.

The spaces $F_{2,2}(D)$ are much different than the previous ones in terms of the applications in the stochastic processes and the nonlocal equations, cf. [18, 84, 159]. This is partly due to the fact that there is no straight connection with the operator L, because in $F_{2,2}(D)$ we do not consider D^c at all. The spaces $F_{2,2}(D)$ are more established as the spaces on subsets of $\mathbb{R}^d$ than $\mathcal{V}_D$, especially from the point of view of analysis. This is seen, e.g., in the following example.

Example 2.3.9. The *fractional Sobolev spaces*, often associated with the names of Aronszajn [5], Gagliardo [74] and Slobodecki [145] are defined as follows:

$$W^{\alpha/2,p}(D) = \left\{ u \in L^p(D) : \int_D \int_D \frac{(u(x) - u(y))^p}{|x - y|^{d + \frac{p\alpha}{2}}}\, \mathrm{d}y\mathrm{d}x < \infty \right\}.$$

A neat introduction to the fractional Sobolev spaces is given by Di Nezza, Palatucci and Valdinoci in [55]. For $\Delta^{\alpha/2}$, the space $F_{2,2}(D)$ coincides with $W^{\alpha/2,2}(D)$, *notabene* the spaces $F_{p,p}(D)$ from Chapter 5 with an appropriate kernel coincide with $W^{\alpha/2,p}(D)$. With such explicit ν, the fractional Sobolev spaces display many relations with the L^q spaces, in particular the Sobolev embeddings, see [55, Sections 6 and 7] or Zhou [163]. For sufficiently regular D the spaces $W^{\alpha/2,p}(D)$ are the interpolation spaces between $L^p(D)$ and the classical Sobolev spaces $W^{1,p}(D)$, see, e.g., Tartar [151, p. 169].

Chapter 3

The Dirichlet problem and its weak solutions

3.1 Introduction

In this chapter we study the weak solutions of the Dirichlet problem (DP). The material below is taken from the article of the author [137]. Here we only assume that ν is a symmetric Lévy measure (possibly singular). Also, for the most part we will postulate the boundedness of D, but that will always be announced.

The definition of a weak solution dates back to the second quarter of the twentieth century with the works of Sobolev, Weyl, Friedrichs and Schwartz, among many others, we refer to the historical survey by Gårding [73]. A few years later this notion evolved into the context of abstract equations involving bilinear forms in Banach spaces and Lax and Milgram gave a method to find unique solutions for such problems. Their theorem still serves as one of the fundamental ways of proving the existence and uniqueness of weak solutions to various problems in PDEs. This abstract setting is also applicable for the nonlocal equations, which is the case for the main result of this chapter.

Theorem 3.1.1. *Assume that D is bounded and let ν be a symmetric Lévy measure. If $f \in L^2(D)$ and there exists $h \in \widetilde{\mathcal{V}}_D$, such that $g = h|_{D^c}$, then the Dirichlet problem* (DP) *has a unique weak solution in the sense of Definition 3.2.3.*

Noteworthy, our definition of weak solutions coincides with the variational solutions, see Lemma 3.2.5. The Lax–Milgram theorem requires coercivity of the bilinear form. In our case it is tantamount to the fact that a multiple of $\mathcal{E}$ dominates the squared L^2 norm on the class of the test functions $\widetilde{\mathcal{V}}_D^0$. This is referred to as the Poincaré inequality. We prove it in Theorem 3.3.2 below, it is certainly the most interesting and original aspect of the proof of Theorem 3.1.1. We note that we could strive for generality and consider all $f \in (\widetilde{\mathcal{V}}_D^0)^*$, but the situation with $f \in L^2(D)$ seems much more transparent.

The formulation of the above result provokes the following question: what conditions must a function $g\colon D^c \to \mathbb{R}$ satisfy in order to have an *extension* $h\colon \mathbb{R}^d \to \mathbb{R}$ which belongs to $\widetilde{\mathcal{V}}_D$? We address this problem in the next chapter and as a result we obtain more constructive formulations of the existence and uniqueness theorem in Corollaries 4.3.2 and 4.5.7.

Let us briefly present the related literature on the existence and uniqueness theory for nonlocal Dirichlet problem. Weak solutions of (DP) were studied by Hoh and Jacob [90, Sections 5 and 6] under the assumptions of certain growth at infinity for the homogeneous part of the symbol of the operator, see formula (1.1) and assumption P.3 therein. This excludes, e.g., any finite Lévy measure, cf. (2.2.1). More recently, Felsinger, Kassmann and Voigt [67] gave the existence and uniqueness results for operators with nonhomogeneous (depending on x and y, not only on $x - y$) and nonsymmetric *functions* k in place of ν. Thus, our context is original, but it does not *generalize* those in [67, 90].

In Section 3.4 we establish a weak maximum principle and L^∞ bounds for the weak and pointwise solutions of the Dirichlet problem. In [67, Theorem 4.1] a weak maximum principle is obtained under the assumption that the form (the counterpart of our $\mathcal{E}$) is bounded from below by the one corresponding to the fractional Laplacian with an arbitrary exponent $\alpha \in (0, 2)$, see also the survey article of Ros-Oton [134, Section 5]. For a slightly different problem involving the operator L, Jarohs and Weth [95] give a strong version of the maximum principle with one of the postulates being that ν does not vanish in any neighborhood of the origin. In Remark 3.4.7 we argue that for the solutions of (DP) this condition is in fact necessary for the strong maximum principle to hold in arbitrary open sets D.

For further reading on the above and other properties of solutions of the Dirichlet problem we refer to [134] and the references therein. We remark that our development, in particular, the choice of studied topics, is partly inspired by [67] and [134].

Nonlocal equations driven by operators with singular Lévy measures are a popular research topic in both probability and analysis. For example, there is vast literature concerning the anisotropic stable operators and the related cylindrical stable processes, see, e.g., [63, 108, 135, 136, 150]. More arbitrary Lévy measures were studied, e.g., by Endal, Jakobsen and del Teso [51, 52] in the context of nonlinear equations.

We note in passing that the Dirichlet problem may be approached via the Dynkin characteristic operator, cf. Lemma 2.3.4. With $\mathcal{U}$ in place of L the solutions of the Dirichlet problem may be represented as the sum of the Green potential and the Poisson integral defined in (2.2.8) and (2.2.18) respectively:

$$u = -G_D[f] + P_D[g]. \tag{3.1.1}$$

This is however delicate when we want to relate this representation to our version of the operator. For the general theory of such representations we refer to the book of Dynkin [64, Chapter V]. Results more directly related to our development are given by, e.g., Bogdan and Byczkowski [19, Section 3], or Grzywny, Kassmann and Leżaj [81, Theorem 1.1].

In this dissertation, apart from few very specific cases, we do not study Green potentials $G_D[f]$ — the Poisson integrals $P_D[g]$ are one of the central objects of interest in Chapters 4 and 6. We identify them as both pointwise and weak solutions of the Dirichlet problem with $f \equiv 0$ in Lemma 4.4.10 and Corollary 4.4.12, respectively, and we show their interior regularity, in particular we prove Weyl's lemma (hypoellipticity of L) for sufficiently regular operators L, see Corollary 4.4.15.

It goes without saying that apart from (3.1.1) and the weak formulation, there is a myriad of other approaches towards obtaining (and defining) the solutions of nonlocal problems, for only few of them see [10, 45, 58, 104, 155].

3.2 Weak and variational solutions

In this section we introduce the setup for the general symmetric Lévy measures ν and we prove several properties of the forms and Sobolev spaces related to ν, extending the discussion of Subsection 2.3.2. Finally, we provide the notions of weak and variational solutions of the

Dirichlet problem (DP) and we argue that the definitions are reasonable from the point of view of the pointwise solutions.

In order to generalize the form $\mathcal{E}_D$ to arbitrary symmetric Lévy measures we let

$$\nu_y(G) = \nu(G - y), \quad y \in \mathbb{R}^d,\ G \subseteq \mathbb{R}^d,$$

and we write

$$\mathcal{E}_D[u] = \frac{1}{2} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c} (u(x) - u(y))^2 \, \mathrm{d}\nu_y(x)\mathrm{d}y. \tag{3.2.1}$$

Note that this definition is in agreement with the one from Section 2.3, in particular we have $\mathcal{E}_{\mathbb{R}^d} = \mathcal{E}$. The spaces $\widetilde{\mathcal{V}}_D$ and $\widetilde{\mathcal{V}}_D^0$ and the norm $\|\cdot\|_{\widetilde{\mathcal{V}}_D}$ are defined respectively by (2.3.11), (2.3.14) and (2.3.15), and the bilinear version of $\mathcal{E}_D$ is

$$\mathcal{E}_D(u, v) = \frac{1}{2} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c} (u(x) - u(y))(v(x) - v(y)) \, \mathrm{d}\nu_y(x)\mathrm{d}y, \tag{3.2.2}$$

for $u, v \in \widetilde{\mathcal{V}}_D$. Crucially, (2.3.17) holds true, therefore $\|u\|_{\widetilde{\mathcal{V}}_D} = \|u\|_{\widetilde{\mathcal{V}}_{\mathbb{R}^d}}$ for $u \in \widetilde{\mathcal{V}}_D^0$. The following result should be somewhat reassuring. A slightly more general context is given by Bogdan and Sztonyk [29, (26)], or Endal, Jakobsen and del Teso [51, Lemma 6.4].

Lemma 3.2.1. *For every measurable function u we have*

$$\int_{\mathbb{R}^d} \int_D (u(x) - u(y))^2 \, \mathrm{d}\nu_y(x)\mathrm{d}y = \int_D \int_{\mathbb{R}^d} (u(x) - u(y))^2 \, \mathrm{d}\nu_y(x)\mathrm{d}y.$$

As a consequence,

$$\mathcal{E}_D[u] \le \int_D \int_{\mathbb{R}^d} (u(x) - u(y))^2 \, \mathrm{d}\nu_y(x)\mathrm{d}y \le 2\mathcal{E}_D[u].$$

Proof. By Tonelli's theorem, the translation invariance of Lebesgue measure, and the symmetry of ν, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_D (u(x) - u(y))^2 \, \mathrm{d}\nu_y(x)\mathrm{d}y = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x+y) - u(y))^2 \mathbf{1}_D(x+y) \, \mathrm{d}\nu(x)\mathrm{d}y \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x+y) - u(y))^2 \mathbf{1}_D(x+y) \, \mathrm{d}y\mathrm{d}\nu(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(y-x))^2 \mathbf{1}_D(y) \, \mathrm{d}y\mathrm{d}\nu(x) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(y) - u(y+x))^2 \mathbf{1}_D(y) \, \mathrm{d}y\mathrm{d}\nu(x) = \int_D \int_{\mathbb{R}^d} (u(x+y) - u(y))^2 \, \mathrm{d}\nu(x)\mathrm{d}y \\
&= \int_D \int_{\mathbb{R}^d} (u(x) - u(y))^2 \, \mathrm{d}\nu_y(x)\mathrm{d}y.
\end{aligned}$$

□

We can easily conclude that the corresponding integrals over $D \times D^c$ and $D^c \times D$ are also equal, provided that $\mathcal{E}_D[u] < \infty$.

The proof of the next result follows the analogue in [67, Lemma 2.3].

Lemma 3.2.2. *The spaces $\widetilde{\mathcal{V}}_{\mathbb{R}^d}$ and $\widetilde{\mathcal{V}}_D^0$ are Hilbert with the inner product $\mathcal{E}(u, v) + \int_{\mathbb{R}^d} uv$.*

Proof. It suffices to prove the proposition for $\widetilde{\mathcal{V}}_{\mathbb{R}^d}$, because $\widetilde{\mathcal{V}}_D^0$ is a closed subspace of $\widetilde{\mathcal{V}}_{\mathbb{R}^d}$. That $\widetilde{\mathcal{V}}_{\mathbb{R}^d}$ is an inner product space, is an easy consequence of Cauchy–Schwarz inequality. In order to prove the completeness, we let (u_n) be a Cauchy sequence in $\widetilde{\mathcal{V}}_{\mathbb{R}^d}$. This implies that (u_n) is a Cauchy sequence in $L^2(\mathbb{R}^d)$, so it converges in $L^2(\mathbb{R}^d)$ to a function u. Let us choose a subsequence (u_{n_k}) that converges to u a.e. From Fatou's lemma and the fact that (u_n) is Cauchy, hence bounded in $\widetilde{\mathcal{V}}_{\mathbb{R}^d}$, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x)-u(y))^2\,\mathrm{d}\nu_y(x)\mathrm{d}y &\le \liminf_{k\to\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u_{n_k}(x)-u_{n_k}(y))^2\,\mathrm{d}\nu_y(x)\mathrm{d}y\\ &\le \sup_{n\in\mathbb{N}}\|u_n\|^2_{\widetilde{\mathcal{V}}_{\mathbb{R}^d}}<\infty.\end{aligned}$$

Therefore, $u\in\widetilde{\mathcal{V}}_{\mathbb{R}^d}$. Now we will prove that $u_{n_k}\to u$ in $\widetilde{\mathcal{V}}_{\mathbb{R}^d}$ as $n\to\infty$. Again, by Fatou's lemma:

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u_{n_k}(x)-u_{n_k}(y)-(u(x)-u(y)))^2\,\mathrm{d}\nu_y(x)\mathrm{d}y\\ \le&\liminf_{l\to\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u_{n_k}(x)-u_{n_k}(y)-(u_{n_l}(x)-u_{n_l}(y))^2\,\mathrm{d}\nu_y(x)\mathrm{d}y.\end{aligned}$$

The latter expression gets arbitrarily small for k large enough, because (u_n) is Cauchy in $\widetilde{\mathcal{V}}_{\mathbb{R}^d}$. Thus, u_{n_k} converges to u in $\widetilde{\mathcal{V}}_{\mathbb{R}^d}$, and so $u_n\to u$ in $\widetilde{\mathcal{V}}_{\mathbb{R}^d}$ as $n\to\infty$. This finishes the proof of the completeness of $\widetilde{\mathcal{V}}_{\mathbb{R}^d}$. □

We define the *pointwise*, or *strong*, solutions of (DP) as the functions which satisfy its equations almost everywhere. However, our main target of consideration are the weak solutions.

Definition 3.2.3. Let $f\in L^2(D)$. We say that $u\in\widetilde{\mathcal{V}}_D$ is a weak solution of (DP), if $u=g$ a.e. on D^c, and for every $\phi\in\widetilde{\mathcal{V}}_D^0$

$$\tfrac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x)-u(y))(\phi(x)-\phi(y))\,\mathrm{d}\nu_y(x)\mathrm{d}y=\int_{\mathbb{R}^d}f(x)\phi(x)\,\mathrm{d}x. \tag{3.2.3}$$

The integral on the right-hand side above may be disturbing because we did not ascribe values of f on D^c Since the test functions ϕ are supported in D we may let, say, $f\equiv 0$ on D^c. We could have avoided this discussion by using (2.3.17), which yields the equivalent formulation

$$\mathcal{E}(u,\phi)=\mathcal{E}_D(u,\phi)=\int_D f(x)\phi(x)\,\mathrm{d}x,\quad \phi\in\widetilde{\mathcal{V}}_D^0,$$

but (3.2.3) immediately shows that a weak solution on D is also a weak solution on every $U\subseteq D$.

Below we show that our definition of weak solution is in accordance with the pointwise solutions. The result also confirms the formal calculation of $\int_{\mathbb{R}^d}uLu$ on page 23 for $u\in C_c^2(\mathbb{R}^d)$ if we take D large enough.

Proposition 3.2.4. *Assume that D is bounded. If $u\in C_b^2(\mathbb{R}^d)$ is a pointwise solution of* (DP), *then it is also a weak solution.*

Proof. Assume that $u\in C_b^2(\mathbb{R}^d)$ and let $\epsilon>0$. By Proposition 2.3.2, $Lu(x)=\frac{1}{2}\int_{\mathbb{R}^d}(2u(x)-u(x+y)-u(x-y))\,\mathrm{d}\nu(y)$ converges absolutely and $f:=Lu\in L^2(D)$. Furthermore, for every $\epsilon>0$ we have

$$\int_{|y|>\epsilon}|2u(x)-u(x+y)-u(x-y)|\,\mathrm{d}\nu(y)\le\int_{|y|>\epsilon}(1\wedge|y|^2)\,\mathrm{d}\nu(y)\le\int_{\mathbb{R}^d}(1\wedge|y|^2)\,\mathrm{d}\nu(y),$$

which is finite. Therefore, by the dominated convergence theorem, for every $\phi \in \widetilde{\mathcal{V}}_D^0$ we have

$$
\begin{aligned}
\int_{\mathbb{R}^d} f(x)\phi(x)\,\mathrm{d}x &= \int_D Lu(x)\phi(x)\,\mathrm{d}x \\
&= \int_D \int_{\mathbb{R}^d} \tfrac{1}{2}\phi(x)(2u(x) - u(x+y) - u(x-y))\,\mathrm{d}\nu(y)\mathrm{d}x \\
&= \lim_{\epsilon \to 0^+} \int_D \phi(x) \int_{|y|>\epsilon} \tfrac{1}{2}(2u(x) - u(x+y) - u(x-y))\,\mathrm{d}\nu(y)\mathrm{d}x \qquad (3.2.4) \\
&= \lim_{\epsilon \to 0^+} \int_D \phi(x) \int_{|y|>\epsilon} (u(x) - u(x+y))\,\mathrm{d}\nu(y)\mathrm{d}x \qquad (3.2.5) \\
&= \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^d} \phi(x) \int_{|y|>\epsilon} (u(x) - u(x+y))\,\mathrm{d}\nu(y)\mathrm{d}x.
\end{aligned}
$$

Splitting the integral in (3.2.4) is legitimate, because D is bounded and the integral over $\mathrm{d}\nu(y)$ in (3.2.5) is bounded as a function of x for every $\epsilon > 0$. That is because u is bounded and ν is finite away from the origin, cf (2.3.2). By the symmetry of ν and $\{y : |y| > \epsilon\}$, and the translation invariance of Lebesgue measure, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \phi(x) \int_{|y|>\epsilon} (u(x) - u(x+y))\,\mathrm{d}\nu(y)\mathrm{d}x \\
&= \int_{|y|>\epsilon} \int_{\mathbb{R}^d} \phi(x)(u(x) - u(x+y))\,\mathrm{d}x\mathrm{d}\nu(y) \\
&= \int_{|y|>\epsilon} \int_{\mathbb{R}^d} \phi(x-y)(u(x-y) - u(x))\,\mathrm{d}x\mathrm{d}\nu(y) \\
&= \int_{|y|>\epsilon} \int_{\mathbb{R}^d} \phi(x+y)(u(x+y) - u(x))\,\mathrm{d}x\mathrm{d}\nu(y) \\
&= -\int_{\mathbb{R}^d} \int_{|y|>\epsilon} \phi(x+y)(u(x) - u(x+y))\,\mathrm{d}\nu(y)\mathrm{d}x.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^d} \phi(x) \int_{|y|>\epsilon} (u(x) - u(x+y))\,\mathrm{d}\nu(y)\mathrm{d}x \\
&= \tfrac{1}{2} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^d} \int_{|y|>\epsilon} (u(x) - u(x+y))(\phi(x) - \phi(x+y))\,\mathrm{d}\nu(y)\mathrm{d}x \\
&= \tfrac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(x+y))(\phi(x) - \phi(x+y))\,\mathrm{d}\nu(y)\mathrm{d}x.
\end{aligned}
$$

The last equality follows from Lemma 2.3.5, which yields the absolute convergence of the last integral, and from the dominated convergence theorem.

□

For the classical Poisson equation the Dirichlet principle states that being a solution is equivalent to minimizing a certain energy functional. An analogous result holds for the weak solutions of nonlocal equations. Conversely, we may say that the weak solutions of (DP) solve the Euler–Lagrange equation for the energy form presented below. The following proof avoids the differentiation in Banach spaces and is rather standard, see, e.g., the lecture notes of Brokate [32, Proposition 1.2].

Lemma 3.2.5. *A function* $u \in \widetilde{\mathcal{V}}_D$ *is a solution of* (3.2.3) *if and only if* $u = g$ *a.e. in* D^c *and* u *minimizes the energy functional*

$$E[u] = \frac{1}{4} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c} (u(x) - u(y))^2 \, \mathrm{d}\nu_x(y) \mathrm{d}x - \int_D f(x) u(x) \, \mathrm{d}x \tag{3.2.6}$$

among the functions equal almost everywhere to g *on* D^c.

Proof. Let $u \in \widetilde{\mathcal{V}}_g = \{u \in \widetilde{\mathcal{V}}_D : u = g \text{ a.e. on } D^c\}$ minimize E among the functions from $\widetilde{\mathcal{V}}_g$. Then, for every $\phi \in \widetilde{\mathcal{V}}_D^0$ and every $\lambda \in \mathbb{R}$, we have $u + \lambda\phi \in \widetilde{\mathcal{V}}_g$, hence

$$0 \le E[u + \lambda\phi] - E[u] = \lambda\Big(\mathcal{E}_D(u, \phi) - \int_D f(x)\phi(x) \, \mathrm{d}x\Big) + \frac{\lambda^2}{2}\mathcal{E}_D[\phi].$$

For $\lambda > 0$, by dividing both sides by λ and taking the limit $\lambda \to 0^+$, we obtain

$$\mathcal{E}_D(u, \phi) - \int_D f(x)\phi(x) \, \mathrm{d}x \ge 0. \tag{3.2.7}$$

An analogous procedure for $\lambda < 0$ yields

$$\mathcal{E}_D(u, \phi) - \int_D f(x)\phi(x) \, \mathrm{d}x \le 0. \tag{3.2.8}$$

By (3.2.7) and (3.2.8), u is a weak solution.
Now, assume that u is a weak solution. Note that $\widetilde{\mathcal{V}}_g = u + \widetilde{\mathcal{V}}_D^0$, thus it suffices to verify that $E[u + \phi] - E[u] \ge 0$ for every $\phi \in \widetilde{\mathcal{V}}_D^0$. In fact, since u is a weak solution,

$$E[u + \phi] - E[u] = \mathcal{E}_D(u, \phi) - \int_D f(x)\phi(x) \, \mathrm{d}x + \tfrac{1}{2}\mathcal{E}_D[\phi] = \mathcal{E}_D[\phi] \ge 0.$$

□

3.3 Existence and uniqueness of solutions. Poincaré inequality

This section is devoted to proving Theorem 3.1.1. We will show that (3.2.3) may be transformed to a homogeneous problem, that is, with $g = 0$. Then in fact we will be looking for a solution in the space $\widetilde{\mathcal{V}}_D^0$, which is a Hilbert space. This will let us use the Lax–Milgram theorem. Its following formulation is taken from Theorem 6 in Chapter 6 of the book by Lax [113].

Theorem 3.3.1. *Let* $\mathcal{H}$ *be a Hilbert space over* $\mathbb{R}$, *and let* $a \colon \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ *be bilinear. Assume that there exist* c_1, c_2 *such that*

$$|a(x, y)| \le c_1 \|x\| \cdot \|y\|, \quad x, y \in \mathcal{H},$$

and

$$|a(x, x)| \ge c_2 \|x\|^2, \quad x \in \mathcal{H}.$$

Then, for every $l \in \mathcal{H}^*$ *there is unique* $u \in \mathcal{H}$ *which satisfies*

$$a(u, v) = l(v), \quad v \in \mathcal{H}.$$

The first displayed requirement of the above result is the boundedness of the functional a, and the second is often referred to as coercivity. Obtaining the coercivity is the crux of this section, in order to get it we will prove the following *Poincaré inequality*.

Theorem 3.3.2. *Let D be bounded and let ν be a symmetric Lévy measure. Then, there exists c, which depends only on D and ν, such that for every $u \in \widetilde{\mathcal{V}}_D^0$ we have*

$$\|u\|_{L^2(D)}^2 \le c\mathcal{E}[u]. \tag{3.3.1}$$

Note that above we can use the pairs $L^2(D)$, $L^2(\mathbb{R}^d)$ and $\mathcal{E}_D$, $\mathcal{E} = \mathcal{E}_{\mathbb{R}^d}$ interchangeably because u vanishes on D^c.

Remark 3.3.3. It is possible that our Poincaré inequality may be deduced from the general versions from the potential theory, see, e.g., Fitzsimmons [69, (1.18)], or [72, Theorem 2.4.2], but in order to do this we would have to prove that the 0-order potential of $\mathbf{1}_D$ exists and is bounded. This may be quite tedious because the definitions are implicit and actually require solving a certain Dirichlet problem, cf. [72, (2.2.2)]. On the other hand the proof that we give below is straightforward and purely analytic.

In order to prove Theorem 3.3.2 we first establish it for atomic Lévy measures in Lemma 3.3.4 and then we represent $\mathcal{E}$ for general Lévy measures as an integral of the forms corresponding to the Dirac measures in Lemma 3.3.5.

For $x_0 \in \mathbb{R}^d \setminus \{0\}$ we let

$$\mathcal{E}_{x_0}[u] = \tfrac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x) - u(x+y))^2 \,\mathrm{d}\delta_{x_0}(y)\mathrm{d}x.$$

Note that the above expression significantly simplifies and in fact we have

$$\mathcal{E}_{x_0}[u] = \tfrac{1}{2}\int_{\mathbb{R}^d}(u(x) - u(x+x_0))^2 \,\mathrm{d}x. \tag{3.3.2}$$

We also observe the following fact which is a consequence of the formula $2a^2 + 2b^2 \ge (a+b)^2$. We have

$$\int_G (u(x) - u(x+x_0))^2 \,\mathrm{d}x \ge \tfrac{1}{2}\int_G u(x)^2 \,\mathrm{d}x - \int_{G+x_0} u(x)^2 \,\mathrm{d}x, \tag{3.3.3}$$

for every $x_0 \in \mathbb{R}^d$, Borel $G \subseteq \mathbb{R}^d$ and u for which the right-hand side is well-defined.

Lemma 3.3.4. *Assume that D is bounded and let $\nu = \delta_{x_0}$ for some $x_0 \in \mathbb{R}^d \setminus \{0\}$. Then there exists $C_{D,x_0} > 0$ such that for every $u \in \widetilde{\mathcal{V}}_D^0$ we have*

$$C_{D,x_0}\mathcal{E}_{x_0}[u] \ge \|u\|_{L^2(D)}^2. \tag{3.3.4}$$

Furthermore, if D is fixed, then for every $r > 0$ there exists $c(r) > 0$ such that $C_{D,x_0} < c(r)$ whenever $|x_0| > r$.

Proof. Let $x_0 \in \mathbb{R}^d$. By (3.3.2) we have

$$2\mathcal{E}_{x_0}[u] = \int_{D^c - x_0} u(x)^2 \,\mathrm{d}x + \int_{D - x_0}(u(x) - u(x+x_0))^2 \,\mathrm{d}x \tag{3.3.5}$$

$$= \int_{(D^c - x_0)\cap D} u(x)^2 \,\mathrm{d}x + \int_{D - x_0}(u(x) - u(x+x_0))^2 \,\mathrm{d}x. \tag{3.3.6}$$

By (3.3.5) and (3.3.6) we see that it is enough to show that $c\mathcal{E}_{x_0}[u] \ge \int_{(D-x_0)\cap D} u(x)^2 \,\mathrm{d}x$ with the constant c independent of u. By using (3.3.3), we get

$$2\mathcal{E}_{x_0}[u] \ge \int_{(D-x_0)\cap D}(u(x) - u(x+x_0))^2 \,\mathrm{d}x \ge \tfrac{1}{2}\int_{(D-x_0)\cap D} u(x)^2 \,\mathrm{d}x - \int_{D\cap(D+x_0)} u(x)^2 \,\mathrm{d}x. \tag{3.3.7}$$

By using (3.3.3), now with $G = D \cap (D + x_0)$, and by the fact that u is supported in D we further get

$$4\mathcal{E}_{x_0}[u] \geq 2\int_{D\cap(D+x_0)} (u(x)-u(x+x_0))^2\,\mathrm{d}x \geq \int_{D\cap(D+x_0)} u(x)^2\,\mathrm{d}x - 2\int_{D\cap(D+x_0)\cap(D+2x_0)} u(x)^2\,\mathrm{d}x. \tag{3.3.8}$$

By adding the sides of (3.3.7) and (3.3.8) we obtain

$$6\mathcal{E}_{x_0}[u] \geq \tfrac{1}{2}\int_{(D-x_0)\cap D} u(x)^2\,\mathrm{d}x - 2\int_{D\cap(D+x_0)\cap(D+2x_0)} u(x)^2\,\mathrm{d}x. \tag{3.3.9}$$

In the next step we use (3.3.3) with $8\mathcal{E}_{x_0}[u] \geq 4\int_{D\cap(D+x_0)\cap(D+2x_0)}(u(x) - u(x+x_0))^2\,\mathrm{d}x$ and we add the result to (3.3.9).

After k steps we obtain an inequality of the form:

$$c_k'\mathcal{E}_{x_0}[u] \geq \tfrac{1}{2}\int_{(D-x_0)\cap D} u(x)^2\,\mathrm{d}x - c_k\int_{D\cap(D+x_0)\cap\ldots\cap(D+kx_0)} u(x)^2\,\mathrm{d}x. \tag{3.3.10}$$

Since D is bounded, by taking k large enough we will get

$$D \cap (D + x_0) \cap (D + 2x_0) \cap \ldots \cap (D + kx_0) = \emptyset.$$

Then the subtracted integral in (3.3.10) disappears and we get the desired result.

The bound on the constant follows from the proof, because the required number of steps is controlled by the ratio of $\operatorname{diam}(D)$ to $|x_0|$. □

Lemma 3.3.5. *For every Lévy measure ν and $u, v \in \widetilde{\mathcal{V}}_D^0$ we have*

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \mathcal{E}_y(u, v)\,\mathrm{d}\nu(y). \tag{3.3.11}$$

Proof. Let $u, v \in \widetilde{\mathcal{V}}_D^0$. We have

$$\mathcal{E}(u, v) = \tfrac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x) - u(x + y))(v(x) - v(x + y))\,\mathrm{d}\nu(y)\mathrm{d}x$$

and the integral is absolutely convergent. Therefore, the statement follows by the bilinear counterpart of (3.3.2) and Fubini–Tonelli theorem. □

Proof of Theorem 3.3.2. Let $a > b > 0$ and let $R_a^b = \{x \in \mathbb{R}^d : a \leq |x| \leq b\}$. Note that for every nonzero Lévy measure ν there exist $\varepsilon_2 > \varepsilon_1 > 0$ such that $\nu(R_{\varepsilon_1}^{\varepsilon_2}) > 0$. By Lemma 3.3.4, there exists $c = c(\varepsilon_1) > 0$, such that for every $y \in R_{\varepsilon_1}^{\varepsilon_2}$ and $u \in \widetilde{\mathcal{V}}_D^0$ we have

$$\mathcal{E}_y[u] \geq c^{-1}\|u\|_{L^2(D)}^2.$$

Hence, by Lemma 3.3.5 we get

$$\mathcal{E}[u] \geq \int_{R_{\varepsilon_1}^{\varepsilon_2}} \mathcal{E}_y[u]\,\mathrm{d}\nu(y) \geq c^{-1}\nu(R_{\varepsilon_1}^{\varepsilon_2})\|u\|_{L^2(D)}^2. \tag{3.3.12}$$

□

Proof of Theorem 3.1.1. We first solve the homogeneous equation, i.e., $g = 0$ *a.e.* on D^c. We will use the Lax–Milgram theorem with $\mathcal{H} = \widetilde{\mathcal{V}}_D^0$, the norm $\|\cdot\|_{\widetilde{\mathcal{V}}_D}$, $a(u,v) = \mathcal{E}(u,v)$ and $l(v) = \int_{\mathbb{R}^d} fv$. Recall that on $\widetilde{\mathcal{V}}_D^0$, the norms $\|\cdot\|_{\widetilde{\mathcal{V}}_D}$ and $\|\cdot\|_{\widetilde{\mathcal{V}}_{\mathbb{R}^d}}$ are equal. Therefore, by Lemma 3.2.2 $\mathcal{H}$ is a Hilbert space. Let us verify the rest of the assumptions. For $u, v \in \widetilde{\mathcal{V}}_D^0$ the Cauchy–Schwarz inequality gives

$$\mathcal{E}(u,v)^2 \le \mathcal{E}[u]\mathcal{E}[v] \le \|u\|_{\widetilde{\mathcal{V}}_D}^2 \|v\|_{\widetilde{\mathcal{V}}_D}^2,$$

hence a is bounded. In our setting the coercivity is equivalent to

$$\mathcal{E}[u] \ge c_2(\|u\|_{L^2(D)}^2 + \mathcal{E}[u]), \quad u \in \widetilde{\mathcal{V}}_D^0.$$

For sufficiently small c_2 the above inequality is granted by Theorem 3.3.2. Finally, for every $\phi \in \widetilde{\mathcal{V}}_D^0$ we have

$$\left| \int_{\mathbb{R}^d} f(x)\phi(x)\,\mathrm{d}x \right| \le \|f\|_{L^2(D)} \|\phi\|_{L^2(D)} \le \|f\|_{L^2(D)} \|\phi\|_{\widetilde{\mathcal{V}}_D},$$

hence $l \in \mathcal{H}^*$. Thus we are in a position to use the Lax–Milgram theorem, from which we conclude that the equation

$$\mathcal{E}(u,\phi) = \int_{\mathbb{R}^d} f(x)\phi(x)\,\mathrm{d}x, \quad \phi \in \widetilde{\mathcal{V}}_D^0,$$

has a unique solution $u \in \widetilde{\mathcal{V}}_D^0$.

We now proceed with the case $g \ne 0$. The general idea follows [67, Theorem 3.5]. Consider an arbitrary (fixed) extension of g to a function in $\widetilde{\mathcal{V}}_D$ (which we also call g). Note that the statement '$u \in \widetilde{\mathcal{V}}_D$, $u = g$ *a.e.* in D^c' is now equivalent to '$u = \tilde{u} + g$ for some $\tilde{u} \in \widetilde{\mathcal{V}}_D^0$'. Let $u = \tilde{u} + g$ be such a function. Then,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(x) - u(y))(\phi(x) - \phi(y))\,\mathrm{d}\nu_y(x)\mathrm{d}y \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\tilde{u}(x) + g(x) - \tilde{u}(y) - g(y))(\phi(x) - \phi(y))\,\mathrm{d}\nu_y(x)\mathrm{d}y \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\tilde{u}(x) - \tilde{u}(y))(\phi(x) - \phi(y))\,\mathrm{d}\nu_y(x)\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (g(x) - g(y))(\phi(x) - \phi(y))\,\mathrm{d}\nu_y(x)\mathrm{d}y.
\end{aligned}$$

Thus, the solution of (3.2.3) exists, provided that there exists a solution $\tilde{u}$ of the following homogeneous equation:

$$\mathcal{E}(\tilde{u},\phi) = \int_{\mathbb{R}^d} f(x)\phi(x)\,\mathrm{d}x - \mathcal{E}(g,\phi), \quad \phi \in \widetilde{\mathcal{V}}_D^0. \tag{3.3.13}$$

It suffices to show that the right-hand side is a continuous linear functional on $\widetilde{\mathcal{V}}_D^0$. By the Cauchy–Schwarz inequality we have

$$\mathcal{E}(g,\phi)^2 \le \mathcal{E}[g]\mathcal{E}[\phi] \le \mathcal{E}[g]\|\phi\|_{\widetilde{\mathcal{V}}_D}^2, \quad \phi \in \widetilde{\mathcal{V}}_D^0.$$

Thus, since $g \in \widetilde{\mathcal{V}}_D$, we conclude that the equation (3.3.13) has a unique solution $\tilde{u}$. Therefore $u = \tilde{u} + g$ solves (3.2.3). We claim that u does not depend on the choice of the extension of

g. Let $g_1, g_2 \in \widetilde{\mathcal{V}}_D$ be extensions of g and let $\widetilde{u}, \overline{u}$ be solutions of (3.3.13) with $g = g_1$, $g = g_2$, respectively. For every $\phi \in \widetilde{\mathcal{V}}_D^0$ we have

$$\mathcal{E}(\widetilde{u}, \phi) = \int_{\mathbb{R}^d} f(x)\phi(x)\,\mathrm{d}x - \mathcal{E}(g_1, \phi),$$

and

$$\mathcal{E}(\overline{u}, \phi) = \int_{\mathbb{R}^d} f(x)\phi(x)\,\mathrm{d}x - \mathcal{E}(g_2, \phi).$$

Therefore

$$\mathcal{E}(\widetilde{u} + g_1 - (\overline{u} + g_2), \phi) = 0, \quad \phi \in \widetilde{\mathcal{V}}_D^0.$$

In particular,

$$\mathcal{E}[\widetilde{u} + g_1 - (\overline{u} + g_2)] = 0.$$

By the coercivity of $\mathcal{E}$ on $\widetilde{\mathcal{V}}_D^0$, we get $\widetilde{u} + g_1 = \overline{u} + g_2$ *a.e.* on $\mathbb{R}^d$, as claimed. This proves that the obtained solution is unique. □

3.4 Maximum principle and its applications

In this section we give the maximum and comparison principles for weak and pointwise solutions. We then construct the barriers and apply the comparison principle in order to obtain L^∞ bounds for the solutions.

3.4.1 Maximum and comparison principles

In the whole subsection we assume that D is bounded. First, we present the maximum principle for the weak solutions.

Theorem 3.4.1. *Let u be a weak solution of* (DP) *with $f \geq 0$, $g \geq 0$ a.e. Then $u \geq 0$ a.e.*

Proof. In order to obtain the result we will use $u_- = -(u \wedge 0)$ as the test function ϕ in (3.2.3). We claim that it belongs to $\widetilde{\mathcal{V}}_D^0$. Indeed, we have $g \geq 0$, hence $u_- = 0$ outside D. Obviously, $u_- \in L^2(\mathbb{R}^d)$. The finiteness of $\mathcal{E}[u_-]$ follows from the inequality $(u_-(x) - u_-(y))^2 \leq (u(x) - u(y))^2$, so the claim is proved. It follows that $u_+ = u \vee 0 \in \mathcal{V}_D^0$.

Now, since u is a weak solution, by the fact that for any function u we have $(u_+(x) - u_+(y))(u_-(x) - u_-(y)) \leq 0$, we obtain that

$$0 \leq \int_{\mathbb{R}^d} f(x)u_-(x)\,\mathrm{d}x = \mathcal{E}(u, u_-) = \mathcal{E}(u_+, u_-) - \mathcal{E}[u_-] \leq -\mathcal{E}[u_-].$$

Since we also have $\mathcal{E}[u_-] \geq 0$, we see that $\mathcal{E}[u_-] = 0$. By the Poincaré inequality (3.3.1) (which we can use, because $u_- \in \widetilde{\mathcal{V}}_D^0$) we conclude that $u_- = 0$ a.e. in D. □

The comparison principle follows.

Corollary 3.4.2. *Assume that u and v solve* (3.2.3) *with $f = f_u, g = g_u$ and $f = f_v$, $g = g_v$ respectively. If $f_u \geq f_v$ and $g_u \geq g_v$, then $u \geq v$.*

From the formulation Theorem 3.4.1 it may be unclear why we call it the maximum principle. This is better seen in the following versions for the pointwise solutions.

Theorem 3.4.3. *Assume that $u \in C(\mathbb{R}^d)$ and that Lu is well-defined in D. If $Lu \geq 0$ in D and $u \geq 0$ outside D, then $u \geq 0$ a.e. in D.*

Proof. With the purpose of obtaining a contradiction, we assume that $u(y) < 0$ for some $y \in D$. Then, from the continuity we conclude that u has a global minimum at some $x \in D$. Since $u(x)$ is the global minimum of u, we have $u(x) - u(x+y) \leq 0$ for every $y \in \mathbb{R}^d$. Therefore, by the monotone convergence theorem, (2.3.1) becomes

$$Lu(x) = \int_{\mathbb{R}^d} (u(x) - u(x+y))\, \mathrm{d}\nu(y) \leq 0.$$

If the inequality is strict, then we get the desired contradiction. Otherwise, let $A \subset \mathbb{R}^d$ be such that $\nu(A) > 0$ and $d(0, A) > 0$. In addition, we want $A + x$ to dominate $x = (x_1, \ldots, x_n)$ on at least one coordinate, i.e., that for some $k \in \{1, \ldots, n\}$ and every $y \in A + x$ we have $y_k - x_k \geq c > 0$. Since

$$\int_{\mathbb{R}^d} (u(x) - u(y))\, \mathrm{d}\nu_x(y) \leq \int_{A+x} (u(x) - u(y))\, \mathrm{d}\nu_x(y) = 0,$$

we obtain that $u(y) = u(x) < 0$ ν_x *a.e.* on $A + x$. In particular, there exists $x_{(1)} \in A + x$ such that $u(x_{(1)}) = u(x)$. We have $(x_{(1)})_k \geq x_k + d$. Once again, if $Lu(x_{(1)}) < 0$, then we have a contradiction, and if $Lu(x_{(1)}) = 0$ we repeat the procedure, obtaining $x_{(2)}$ and so on. Since A dominates 0 and D is bounded, we will eventually get that for some m either $Lu(x_{(m)}) < 0$, or $x_{(m)} \in D^c$ and $u(x_{(m)}) = u(x) < 0$ which contradicts $u(y) > 0$ for $y \in D^c$. □

The first iteration of the argument above gives the proof of the negative minimum (equivalently — positive maximum) principle.

Proposition 3.4.4. *Assume that $u \in C(\mathbb{R}^d)$ and that Lu is well-defined in D. If u is nonnegative on D^c and has a negative global minimum at $x \in D$, then $Lu(x) \leq 0$. If the minimum is strict, then $Lu(x) < 0$.*

Example 3.4.5. Without the assumption that the minimum at x is strict, $Lu(x)$ is not necessarily strictly negative. Consider the Lévy measure $\delta_1 + \delta_{-1}$ on $\mathbb{R}$, let $D = (-2, 2)$ and let $u \in C_c^\infty(\mathbb{R})$ satisfy $0 \geq u \geq -1$, $u(x) = 0$ for $|x| > 2$, $u(x) = -1$ for $|x| < 3/2$. Clearly $Lu(0) = 0$.

However, by looking at the last iteration in the proof of Theorem 3.4.3, we can refine Proposition 3.4.4 in order to obtain $Lu < 0$ at a certain point.

Proposition 3.4.6. *Assume that $u \in C(\mathbb{R}^d)$ and that Lu is well-defined in D. If u is nonnegative on D^c and has a negative global minimum at $x \in D$, then there exists $x' \in D$ such that $u(x') = u(x)$ and $Lu(x') < 0$.*

Remark 3.4.7. According to the classical nomenclature, see, e.g., Gilbarg and Trudinger [76, page 15], the above results for the pointwise solutions should rather be called the *weak* maximum principles. For general symmetric Lévy measures we cannot expect the strong maximum principle to hold. Indeed, assume that $\nu(B(0,r)) = 0$ for some $r > 0$ and consider the function $u(x) = |x|(r/2 - |x|)_+$. Then $Lu \geq 0$ in $B(0, r/2)$, but u attains the (global) minimum at $x = 0$.

3.4.2 Barriers and supremum bounds

Lemma 3.4.8. *Let D be bounded and let ν be a symmetric Lévy measure. Then there exists a barrier, that is, a nonnegative function $w \in C_c(\mathbb{R}^d) \cap C^\infty(\overline{D})$ which satisfies*

$$\begin{cases} Lw(x) \geq 1, & x \in D, \\ w(x) \leq c_w, & x \in \mathbb{R}^d, \end{cases}$$

with c_w depending on ν and D. Furthermore, w is a weak solution of (DP) *with $f_w(x) = Lw(x) \in L^2(D)$.*

Taking our cue from [134, Section 5], we use different approaches depending on whether ν is compactly supported or not.

Proof. Assume that ν has bounded support and consider a sufficiently large r_1 so that $\nu(B(0,r_1)^c) = 0$ and let $r_2 = \sup\{|x| : x \in D\}$. For $R = r_1 + r_2 + 1$ and $x \in \mathbb{R}^d$, we let

$$\eta(x) = \left(1 - \frac{|x|^2}{R^2}\right)_+ .$$

By the smoothness of η in $B(0,R)$ and the choice of R, $L\eta(x)$ is well defined and $L\eta \in L^\infty(D)$:

$$\int_{\mathbb{R}^d} (2\eta(x) - \eta(x+y) - \eta(x-y))\, \mathrm{d}\nu(y) \lesssim \int_{\mathbb{R}^d} (1 \wedge |y|^2)\, \mathrm{d}\nu(y) \le c' < \infty.$$

In $B(0,R)$ the function η is smooth and strictly concave. Thus, there exists $\widetilde{c} > 0$ such that

$$2\eta(x) - \eta(x+y) - \eta(x-y) \ge \widetilde{c}, \quad x \in D,\ y \in B(0,r_1).$$

Therefore, if we let $0 < \varepsilon < r_1$ be such that $\nu(B(0,r_1) \setminus B(0,\varepsilon)) > 0$, then for every $x \in D$ we have

$$\begin{aligned} L\eta(x) &= \int_{\mathbb{R}^d} (2\eta(x) - \eta(x+y) - \eta(x-y))\, \mathrm{d}\nu(y) \\ &\ge \int_{B(0,r_1)\setminus B(0,\varepsilon)} (2\eta(x) - \eta(x+y) - \eta(x-y))\, \mathrm{d}\nu(y) \\ &\ge \widetilde{c}\nu(B(0,r_1) \setminus B(0,\varepsilon)). \end{aligned}$$

Thus, the function

$$w(x) = \frac{\eta(x)}{\widetilde{c}\nu(B(0,r_1) \setminus B(0,\varepsilon))}$$

is our desired barrier. Note that $w \in \widetilde{\mathcal{V}}_D$. Indeed, w is Lipschitz in B_{R-1} and bounded on $\mathbb{R}^d$, hence we have (cf. Lemma 3.2.1)

$$\int_D \int_{\mathbb{R}^d} (w(x) - w(x+y))^2\, \mathrm{d}\nu(y)\mathrm{d}x \le \overline{C} \int_D \int_{\mathbb{R}^d} (1 \wedge |y|^2)\, \mathrm{d}\nu(y)\mathrm{d}x < \infty.$$

Furthermore, all the calculations from the proof of Proposition 3.2.4 hold true with w in place of u. Hence w is a weak solution with $f_w := Lw \in L^2(D)$. This ends the case of compactly supported ν.

Now, assume that ν has unbounded support. Let $\varepsilon > 0$ and $D_\varepsilon = \{x \in D : d(x,D) < \varepsilon\}$, and consider $\eta_\varepsilon \in C_c^\infty(D_\varepsilon)$ such that $0 \le \eta_\varepsilon \le 1$ and $\eta_\varepsilon = 1$ in D. For $x \in D$ we have

$$L\eta_\varepsilon(x) = \int_{\mathbb{R}^d} (\eta_\varepsilon(x) - \eta_\varepsilon(y))\, \mathrm{d}\nu_x(y) \ge \int_{D_\varepsilon^c} \mathrm{d}\nu_x(y) = \nu(D_\varepsilon^c - x) =: \kappa^{D_\varepsilon}(x).$$

In particular, for every $x \in D$, we get $L\eta_\varepsilon(x) \ge \inf_{x\in D} \kappa^{D_\varepsilon}(x) =: c_\varepsilon$. Note that $c_\varepsilon > 0$, because ν has unbounded support and D is bounded. Furthermore, c_ε^{-1} decreases when ε does. The function $w_\varepsilon(x) = c_\varepsilon^{-1}\eta_\varepsilon(x)$ satisfies the desired conditions and $c_w = c_\varepsilon^{-1}$.

By Propositions 2.3.2 and 3.2.4, we know that the above barrier is also a weak solution with $f_{w_\varepsilon} := Lw_\varepsilon \in L^2(D)$. □

Considering variable ε in the latter construction is superfluous in terms of the sole existence of a barrier, but it will be useful in the discussion of the optimal constant in Corollary 3.4.10 below. We note in passing that in general the function η_ε is inappropriate for constructing barriers for ν with bounded support. Indeed, if $x \in D$, $d(x, D^c) > r_1$ (cf. the first line of the proof), then $L\eta_\varepsilon(x) = 0$ because $\eta_\varepsilon \equiv 1$ in D.

Now we will use the barriers to obtain L^∞ bounds for solutions.

Theorem 3.4.9. *Let ν be a symmetric Lévy measure and let D be bounded. Assume that u is a solution of* (3.2.3)*. Then there exists a constant c independent of f and g, such that*

$$\|u\|_{L^\infty(D)} \le c_w \|f\|_{L^\infty(D)} + \|g\|_{L^\infty(D^c)}. \tag{3.4.1}$$

Proof. We may assume that f and g are bounded. Define $v(x) = \|f\|_{L^\infty(D)} \cdot w(x) + \|g\|_{L^\infty(D^c)}$, where w is the appropriate barrier. Obviously, $v \ge u$ on D^c. Furthermore, we have

$$Lv(x) = \|f\|_{L^\infty(D)} \cdot Lw(x) =: f_v(x), \quad x \in D.$$

Since w is also a weak solution, we get that $\mathcal{E}(v, \phi) = \int_{\mathbb{R}^d} f_v \phi$ for every $\phi \in \widetilde{\mathcal{V}}^0_D$. Since $Lw \ge 1$ in D, we have $f_v \ge f$. Therefore, by Corollary 3.4.2, $v \ge u$. Since $w \le c_w$, we see that

$$u \le c_w \|f\|_{L^\infty(D)} + \|g\|_{L^\infty(D^c)}.$$

A similar argument using $-v$ shows that

$$u \ge -(c_w \|f\|_{L^\infty(D)} + \|g\|_{L^\infty(D^c)}),$$

which completes the proof. □

The above estimate works just as well for the strong solutions, given that they are continuous and thus enjoy the comparison principle. The construction of the barrier in the unbounded case immediately yields the following estimate for the constant in (3.4.1).

Corollary 3.4.10. *If u is a solution of* (3.2.3) *with ν having unbounded support, then we obtain the following constant c_w in Theorem 3.4.9:*

$$c_w = \lim_{\varepsilon \to 0^+} c_\varepsilon^{-1} = (\lim_{\varepsilon \to 0^+} \inf_{x \in D} \kappa^{D_\varepsilon}(x))^{-1}. \tag{3.4.2}$$

In the potential theory of Markov processes, the quantity κ^{D_ε} is often called the killing intensity. In the proof of [27, Lemma 7], Bogdan and Jakubowski give a slightly better estimate

$$c_w = (\inf_{x \in D} \kappa^D(x))^{-1} \tag{3.4.3}$$

for the process generated by the fractional Laplacian perturbed by gradient. This requires some explanation: in [27] we actually have the estimate for the tails of the exit time (cf. (2.2.4)): $\tilde{\mathbb{P}}(\tau_D > t) \le e^{-ct}$ with c_w as in (3.4.3) and $c = c_w^{-1}$ (the tilde notation and c come from [27]). This estimate integrated over t yields the following bound for the expected exit time:

$$\tilde{\mathbb{E}}^x \tau_D \le \frac{1}{c} = c_w = (\inf_{y \in D} \kappa^D(y))^{-1}.$$

The above estimate gives an analogue of (3.4.1) because of (3.1.1) and (2.2.9). We remark that the preceding sentence omits many details and the process considered by Bogdan and Jakubowski is not a pure-jump Lévy process, therefore the above discussion should rather be treated as a digression, which we conclude by showing that the constant (3.4.3) given in [27] may be strictly smaller than ours (3.4.2).

Example 3.4.11. Let $d = 1$ and $\nu = \sum_{k\in\mathbb{Z}\setminus\{0\}} \delta_k \frac{1}{k^2}$. If $D = (0,1)$, then $D_\varepsilon = (-\varepsilon, 1+\varepsilon)$ for $\varepsilon > 0$. We have $\inf\limits_{x\in D} \nu(D^c - x) = \frac{\pi^2}{3}$. On the other hand, $\inf\limits_{x\in D} \nu(D_\varepsilon^c - x) = \frac{\pi^2}{3} - 1$ for every $\varepsilon > 0$, because for $x \in (0,\varepsilon)$ we have $1 \notin D_\varepsilon^c - x$. Thus, we obtain

$$\frac{\pi^2}{3} - 1 = \lim_{\varepsilon\to 0^+} \inf_{x\in D} \nu(D_\varepsilon^c - x) < \inf_{x\in D} \nu(D^c - x) = \frac{\pi^2}{3}.$$

Chapter 4

Extension and trace operators, harmonic functions

4.1 Introduction

In the whole chapter we will assume that ν is unimodal and strictly positive, and in Sections 4.2 — 4.4 we will also stipulate that $\nu(\mathbb{R}^d) = \infty$, cf. the first paragraph of Subsection 2.2.2, and that $\mathbb{P}^x(\tau_D < \infty) = 1$ for $x \in \mathbb{R}^d$, cf. Remark 2.2.4. Sections 4.2 — 4.4 contain the material from the article by Bogdan, Grzywny, Pietruska-Pałuba and the author [21] and Section 4.5 is taken from the work of the author [137].

A serious development of extension operators began over one hundred years ago with the works of Lebesgue, Brouwer, Urysohn and Tietze on the extension of continuous functions given on a closed subset of a topological space. Incidentally, the earliest of these results, due to Lebesgue [114, p. 379], was obtained as a part of a study on the Dirichlet problem. Studies for more sophisticated properties soon followed: the extensions preserving the modulus of continuity and the Lipschitz constant by McShane [121] and Kirszbraun [103] and Whitney's extension theorem [160] for functions of class C^k.

The extension theorems for the classical Sobolev spaces grew to be a large theory with contributions by many great mathematicians. The first steps were made by Babič [8] and Nikol'skiĭ [124] who independently proved a version for C^1 domains. Generalizations for wider classes of sets were given, e.g., by Calderón [37] for Lipschitz domains, and Jones [96] for locally uniform domains. For many more references and a good overview of this subject, see Burenkov [34].

For the fractional Sobolev spaces (cf. Example 2.3.9), Jonsson and Wallin [97] give the extension result for n-sets in $\mathbb{R}^d$, $n \leq d$. This condition turned out to be optimal in a sense, because later Zhou [163] showed that the extension *domains* in $\mathbb{R}^d$ are exactly the d-sets. In particular, it is worth mentioning that for too irregular sets a continuous extension operator does not exist. A more accessible construction of extensions in fractional Sobolev spaces for Lipschitz sets is given by Di Nezza, Palatucci and Valdinoci [55, Section 5]. Section 4.5 below contains the author's contribution to this vein of research. Namely, we prove an extension theorem for the nonlocal Sobolev spaces $F_{2,2}(D^c)$, where D is a $C^{1,1}$ open set and ν is unimodal and satisfies a very mild scaling condition. Unlike many results for Lipschitz or smooth sets D, e.g., [8, 55], see also Hestenes [88], ours does not use a partition of unity. Instead we construct the extension *in vivo*, without dissecting the boundary. To this end, we use the reflection according to the normal vector of the closest point of the boundary, which behaves much like the sphere inversion does locally, see Definition 4.5.1. We must warn the reader that such method of extension is rather suboptimal in terms of the ratio of the difficulty of the construction and the admissible generality of D, compared to the methods using the partition of unity. However, we hope that the geometric analysis of the reflection operator in Lemma 4.5.2 will find further use in the future.

The extension problem for spaces $\mathcal{V}_D$ defined in Subsection 2.3.2 in terms of the finiteness of

$$\mathcal{E}_D[u] = \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} (u(x)-u(y))^2\nu(x,y)\,\mathrm{d}x\mathrm{d}y,$$

is completely different than the above. Indeed, in Remark 2.3.6 we have argued that if $\nu(\mathbb{R}^d) = \infty$, then $\mathcal{E}$ and $\mathcal{E}_D$ measure the smoothness of u. However, $\mathcal{E}_D$ is designed in order not to require smoothness on D^c, thus the exact behavior of $g = u|_{D^c}$ is being a puzzle to solve — it is certainly much different than the behavior of u in D. Here in fact, it would be more appropriate to ask what the *trace space* of $\mathcal{V}_D$ is, that is, how can we characterize the class $\{u|_{D^c} : u \in \mathcal{V}_D\}$. Note that this would be trivial for the spaces $F_{2,2}$ from the previous paragraph, because if $u \in F_{2,2}(\mathbb{R}^d)$, then we simply have $u|_{D^c} \in F_{2,2}(D^c)$. We remark that our topic is relatively fresh, as the spaces $\mathcal{V}_D$ have been studied only for the last decade. The first answer to the extension and trace problem for $\mathcal{V}_D$ was given by Dyda and Kassmann [62]. Namely, for $\Delta^{\alpha/2}$ they characterize the trace space by the elegant condition

$$\iint_{D^c\times D^c} \frac{(g(z)-g(w))^2}{(|z-w|+\delta_D(z)+\delta_D(w))^{d+\alpha}}\,\mathrm{d}z\mathrm{d}w < \infty. \tag{4.1.1}$$

We see that the kernel becomes singular only when the arguments are close to the boundary and close to each other, and so the increments of g need to be small only close to the boundary, as we would expect. The methods used by Dyda and Kassmann in [62] are purely analytic; the extension is constructed with the use of the Whitney decomposition and the *thickness* of the set D. We remark that [62] also contains the results for *nonquadratic* forms, cf. Subsection 6.6.3.

In Theorem 4.2.1 below, which is also the main result of [21], we propose an entirely different approach, which allows to consider more general ν. Namely, for $g\colon D^c \to \mathbb{R}$ we let

$$\mathcal{H}_D[g] = \tfrac{1}{2}\iint_{D^c\times D^c} (g(w)-g(z))^2\gamma_D(z,w)\,\mathrm{d}w\mathrm{d}z, \tag{4.1.2}$$

where γ_D is the interaction kernel given in (2.2.20). Accordingly, we define the space

$$\mathcal{X}_D = \{g\colon D^c\to\mathbb{R} :\ \mathcal{H}_D[g] < \infty\}, \tag{4.1.3}$$

which turns out to be the trace space for $\mathcal{V}_D$. We use the methods of the probabilistic potential theory, in particular the extension of $g \in \mathcal{X}_D$ is given by the Poisson integral $P_D[g]$. Notably, we obtain the following energy conservation principle:

$$\mathcal{E}_D[P_D[g]] = \mathcal{H}_D[g],$$

which we call the *Douglas identity*, by analogy with the classical situation [72, (1.2.18)], see the discussion following Corollary 4.2.3 and Section 6.1. In Theorem 4.2.5 we establish the estimates for γ_D when D is a bounded $C^{1,1}$ set. This in particular lets us compare the kernels in (4.1.1) and (4.1.2), see Example 4.2.6.

We remark that in Chapter 6 we provide a more general version of the Douglas identity for the so-called Sobolev–Bregman spaces, but we highlight the present case because of its significance for the Dirichlet problem, which will be seen in the following sections, and also with the aim to keep our development gradual. In order to avoid repetitions, we postpone the proof of Theorem 4.2.1 to Section 6.4. The original proof can be found in [21], but it is overridden by the proof of Theorem 6.4.1 below. Here we note that the arguments hinge on a suitable formula for variance (6.2.3) and on the Hardy–Stein identity in Proposition 6.3.2 inspired by the work of Bogdan, Dyda and Luks [20].

In Section 4.4 we analyze the harmonic functions of the operator L, that is, the functions which solve $Lu = 0$ in some sense — we propose various definitions. A standard example of a harmonic function is the Poisson integral $P_D[g]$, cf. Definition 4.4.1. In Theorem 4.4.9 we establish C^2 regularity for $P_D[g]$, which is essential for Theorem 4.2.1 and the results of Sections 6.3 and 6.4. Furthermore, in Theorem 4.4.14 we prove the equivalence of the definitions of harmonicity. The result complements the ones given in [19], [81] and by Chen [39], in the sense that it gives an additional insight into harmonic functions which are in $\mathcal{V}_D$. In particular, it implies that there are no *singular* harmonic functions in $\mathcal{V}_D$, cf., e.g., Bogdan [17]. Another consequence of Theorem 4.4.14 is the hypoellipticity for a class of nonlocal operators in Corollary 4.4.15.

Extension theorems for function spaces are interesting for their own sake, but they also have wide applications, the Dirichlet problem being only one of the many. In [163] we see a direct connection between the existence of the extension operator and the Sobolev embedding property. In general, continuous extensions help in obtaining certain properties for subsets of $\mathbb{R}^d$, by using the analogues for the whole space, where we have access to stronger methods, especially the invaluable Fourier transform, see, e.g., Mengesha [122], or [128, Section 3].

4.2 Extension and trace for $\mathcal{V}_D$

Here are additional assumptions on $\nu\colon [0,\infty) \to (0,\infty]$ which will sometimes be made in the sequel.

A1 ν is twice continuously differentiable and there is a constant c such that

$$|\nu'(r)|, |\nu''(r)| \le c\nu(r), \quad r > 1.$$

A2 There exist constants $\beta \in (0,2)$ and $C_2 > 0$ such that

$$\nu(\lambda r) \le C_2 \lambda^{-d-\beta} \nu(r), \qquad 0 < \lambda,\, r \le 1, \tag{4.2.1}$$

$$\nu(r) \le C_2 \nu(r+1), \qquad r \ge 1. \tag{4.2.2}$$

A3 There exist constants $\alpha \in (0,2)$ and $c > 0$ such that

$$\nu(\lambda r) \ge c \lambda^{-d-\alpha} \nu(r), \qquad 0 < \lambda,\, r \le 1. \tag{4.2.3}$$

Theorem 4.2.1. *Let $\emptyset \neq D \subset \mathbb{R}^d$ be open, D^c satisfy VDC, $|\partial D| = 0$, $\mathbb{P}^x(\tau_D < \infty) = 1$ for $x \in \mathbb{R}^d$, and let infinite unimodal ν satisfy* **A1**, **A2**.

(i) *If $g \in \mathcal{X}_D$, then $P_D[g] \in \mathcal{V}_D$ and $\mathcal{E}_D[P_D[g]] = \mathcal{H}_D[g]$.*

(ii) *If $u \in \mathcal{V}_D$, then $g = u|_{D^c} \in \mathcal{X}_D$ and $\mathcal{E}_D[u] \ge \mathcal{H}_D[g]$.*

Thus, under the assumptions in (i) we have

$$\frac{1}{2}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} (P_D[g](x)-P_D[g](y))^2\nu(x,y)\,\mathrm{d}x\mathrm{d}y = \frac{1}{2}\iint\limits_{D^c\times D^c}(g(w)-g(z))^2\gamma_D(z,w)\,\mathrm{d}w\mathrm{d}z. \quad (4.2.4)$$

We postpone the proof to Section 6.4 where we give a more general result from the recent work of Bogdan, Grzywny, Pietruska-Pałuba and the author [22]. Figure 4.1 presents an interpretation of the Douglas identity. Note that both forms describe the fact that two points (or particles) in D^c communicate by jumping to D, then moving in D along the trajectories of the process (X_t) started at the entrance point, cf. (2.2.8), and finally jumping out to the other point.

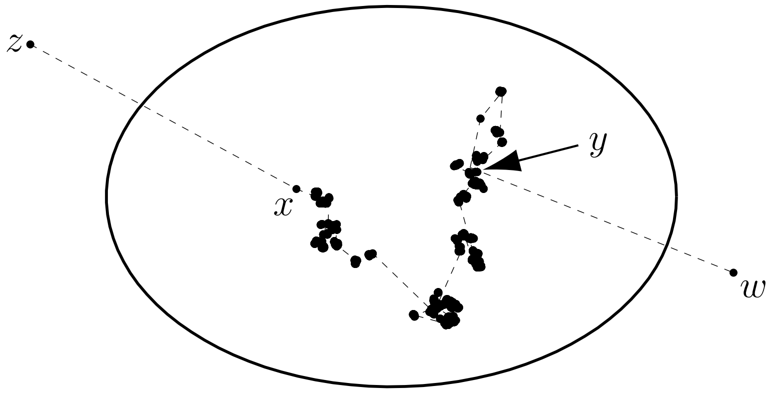

Figure 4.1: Illustration of the integrand $\nu(z,x)G_D(x,y)\nu(y,w)$ of γ_D defined by (2.2.20), but also of the only possibility of communication between $z,w\in D^c$ if there are no jumps from D^c to D^c, which is the case in $\mathcal{E}_D$.

In the setting of Theorem 4.2.1 we immediately obtain the following consequences.

Corollary 4.2.2. $\operatorname{Ext} g = P_D[g]$ *is a linear isometry from* $\mathcal{X}_D$ *into* $\mathcal{V}_D$ *and* $\operatorname{Tr} u = u|_{D^c}$ *is a linear contraction from* $\mathcal{V}_D$ *onto* $\mathcal{X}_D$*.* $\operatorname{Tr}\operatorname{Ext}$ *is the identity operator on* $\mathcal{X}_D$ *and* $\operatorname{Ext}\operatorname{Tr}$ *is a contraction on* $\mathcal{V}_D$*. Furthermore* $\operatorname{Tr} u = 0$ *characterizes* $u \in \mathcal{V}_D^0$*.*

Thus the Poisson integral and the restriction to D^c may serve as the extension and trace between the Sobolev spaces $\mathcal{V}_D$ and $\mathcal{X}_D$, correspondingly.

Corollary 4.2.3. *If* $P_D[|u|]<\infty$ *on* D*, in particular, if* $\mathcal{E}_{\mathbb{R}^d}[u]<\infty$*, then*

$$\frac{1}{2}\iint\limits_{D^c\times D^c}(u(w)-u(z))^2\left(\gamma_D(w,z)+\nu(w,z)\right)\mathrm{d}w\mathrm{d}z = \mathcal{E}_{\mathbb{R}^d}[P_D[u]].$$

Corollary 4.2.3 and the Douglas identity in Theorem 4.2.1 may be considered as analogues of the Douglas integral [72, (1.2.18)]. Such naming and setting is given in the context of general Dirichlet forms in the book of Chen and Fukushima [40, Sections 5.5-5.8 and 7.2], see also Chen, Fukushima and Ying [41], but we note that the form $(\mathcal{E}_D,\mathcal{V}^D)$ treated here is new. Let us comment shortly on this statement. The results of Section 7.2 in [40] are unrelated to our development due to the assumption of no jumps from D to D^c therein. Theorems 5.5.9 and 5.7.6 of [40] are the closest, as they yield our Corollary 4.2.3, but this does not imply that

(4.2.4) holds for all $g \in \mathcal{X}_D$. The trace part (ii) of Theorem 4.2.1 is even more elusive in this context, because in $\mathcal{V}_D$ we do not know a direct method of showing that the Poisson integrals P_D are well-defined, cf. the proof of Theorem 6.4.1. In [40, Theorem 3.4.2] the finiteness of the Poisson integrals is obtained for the functions from the extended Dirichlet space (see [40] for the definition) of the 'free' process (X_t), which is a much stronger condition than the finiteness of $\mathcal{H}_D$ in Lemma 4.4.6 below. We remark that nonlocal Douglas identities were studied also by Jacob and Schilling [94] in the more concrete context of subordinated operators, but for exterior conditions g given on ∂D, which is more in line with the censored-type forms $\mathcal{E}_D^{\text{cen}}$.

Example 4.2.4. In the setting of Example 2.2.7, let $u(x) = g(x)$ for $x \le 0$, and

$$u(x) = P_{(0,\infty)}[g](x) = \int_{-\infty}^{0} \frac{\sqrt{x} g(z)\,\mathrm{d}z}{\pi(x-z)\sqrt{|z|}}, \quad x > 0.$$

If the above integral is absolutely convergent, then by (4.2.4) we get

$$\iint\limits_{x>0 \text{ or } y>0} \frac{(u(x)-u(y))^2}{\pi(x-y)^2}\,\mathrm{d}x\mathrm{d}y = \iint\limits_{z<0 \text{ and } w<0} \frac{(g(z)-g(w))^2}{2\pi\sqrt{zw}(\sqrt{|z|}+\sqrt{|w|})^2}\,\mathrm{d}z\mathrm{d}w.$$

We note that $\mathcal{H}_D[g]$ in Theorem 4.2.1 may be finite even for rather rough functions. Indeed,

$$\begin{aligned}\mathcal{H}_D[g] &= \int_{D^c}\int_{D^c} (g(z)-g(w))^2 \gamma_D(z,w)\,\mathrm{d}z\mathrm{d}w \le 2\int_{D^c}\int_{D^c} g^2(z)\gamma_D(z,w)\,\mathrm{d}z\mathrm{d}w \\ &= 2\int_{D^c}\int_{D^c} g^2(z)\int_D \nu(z,x)P_D(x,w)\,\mathrm{d}x\mathrm{d}z\mathrm{d}w = 2\int_{D^c} g^2(z)\rho(z)\,\mathrm{d}z,\end{aligned}$$

where $\rho(z) = \int_D \nu(z,x)\,\mathrm{d}x$. In particular, if g is L^2-integrable and $\operatorname{dist}(D, \operatorname{supp} g) > 0$, then $\mathcal{H}_D[g] < \infty$ and so g has an extension $u \in \mathcal{V}_D$, which will prove useful for constructing various counterexamples below. On the other hand we note that $\mathcal{E}_{\mathbb{R}^d}[u] = \infty$ in general for such g. Similarly, if $L = \Delta^{\alpha/2}$ and D is a bounded $C^{1,1}$ set, then $\rho(z) \approx \delta_D(z)^{-\alpha}(1+|z|)^{-d}$, and so $\mathcal{H}_D[g] < \infty$ if g is merely bounded and $\alpha < 1$.

Below we propose sharp explicit estimates of $\gamma_D(z,w)$ for bounded open sets D of class $C^{1,1}$. To this end for $r > 0$ we let

$$K(r) = \int_{|z|\le r} \frac{|z|^2}{r^2}\nu(z)\,\mathrm{d}z, \qquad h(r) = K(r) + \nu(B_r^c) = \int_{\mathbb{R}^d}\left(\frac{|z|^2}{r^2}\wedge 1\right)\nu(z)\,\mathrm{d}z, \tag{4.2.5}$$

$$V(r) = \frac{1}{\sqrt{h(r)}}. \tag{4.2.6}$$

Note that $K, h > 0$. We also let $r(z,w) = |z-w| + \delta_D(z) + \delta_D(w)$. Here are the estimates of γ_D.

Theorem 4.2.5. *Let ν be unimodal and assume* **A2**, **A3** *and let D be a bounded $C^{1,1}$ set. Then,*

$$\gamma_D(z,w) \approx \begin{cases} \nu(\delta_D(w))\nu(\delta_D(z)), & \operatorname{diam}(D) \le \delta_D(z), \delta_D(w), \\ \nu(\delta_D(w))/V(\delta_D(z)), & \delta_D(z) < \operatorname{diam}(D) \le \delta_D(w), \\ \nu(r(z,w))V^2(r(z,w))/\left[V(\delta_D(z))\,V(\delta_D(w))\right], & \delta_D(z), \delta_D(w) < \operatorname{diam}(D). \end{cases}$$

The proof is very technical and we give it in the Appendix. As typical in the boundary potential theory, it is challenging to handle unbounded and less regular sets D, cf. Bogdan, Grzywny, and Ryznar [25]. In Theorem A.1.1 below we give estimates for $\gamma_H(z,w)$, where H is the half-space in dimensions $d \ge 3$. Other extensions are left for future.

Example 4.2.6. For $\Delta^{\alpha/2}$ we have $K(r) = cr^{-\alpha}$ and $V(r) = c'r^{\alpha/2}$ with some positive constants c, c'. Consequently, the estimates of Theorem 4.2.5 take on the following form:

$$\gamma_D(z,w) \approx \begin{cases} \delta_D(w)^{-d-\alpha}\delta_D(z)^{-d-\alpha}, & \mathrm{diam}(D) \le \delta_D(z), \delta_D(w), \\ \delta_D(w)^{-d-\alpha}\delta_D(z)^{-\alpha/2}, & \delta_D(z) < \mathrm{diam}(D) \le \delta_D(w), \\ r(z,w)^{-d}\delta_D(z)^{-\alpha/2}\delta_D(w)^{-\alpha/2}, & \delta_D(z), \delta_D(w) < \mathrm{diam}(D). \end{cases}$$

The examples of D which satisfy the assumptions of Theorem 4.2.1 are given in Section 2.1, see in particular Examples 2.1.6, 2.1.7 and 2.1.11. It is worth mentioning that Theorem 4.2.1 also allows some of the 0-order kernels.

Example 4.2.7. By inspection, **A1** and **A2** are satisfied when the Lévy density is

$$\nu(z) = \frac{1}{|z|^d \ln(2+|z|)^\alpha}, \quad z \in \mathbb{R}^d,$$

for some $\alpha > 1$.

4.3 Application to the Dirichlet problem

In order to apply the extension result of Corollary 4.2.2 for the weak solutions of the Dirichlet problem we need to ensure that $\mathcal{V}_D = \widetilde{\mathcal{V}}_D$. This is true in the present setting, provided that D is bounded.

Lemma 4.3.1. *We have $\mathcal{V}_D \subseteq L^2_{loc}(\mathbb{R}^d)$. As a consequence, if D is bounded, then $\mathcal{V}_D \subseteq L^2(D)$.*

Proof. Let $\emptyset \ne U \subseteq \mathbb{R}^d$ be open and bounded. For $u \in \mathcal{V}_D$ we have

$$\int_D \int_{\mathbb{R}^d} (u(x) - u(y))^2 \nu(x,y)\, \mathrm{d}x\mathrm{d}y < \infty.$$

In particular there is a point $y_0 \in D$ such that

$$\int_U (u(x) - u(y_0))^2 \nu(x, y_0)\, \mathrm{d}x < \infty.$$

Since ν is unimodal and strictly positive and U is bounded, we have $\nu(x, y_0) \ge c > 0$ for $x \in U$. Consequently, $\int_U (u(x) - u(y_0))^2\, \mathrm{d}x < \infty$. For every $a, b \in \mathbb{R}$ we have $a^2 \le 2(a-b)^2 + 2b^2$, hence

$$\int_U u(x)^2\, \mathrm{d}x \le 2\int_U (u(x) - u(y_0))^2\, \mathrm{d}x + 2|U|u(y_0)^2 < \infty.$$

For bounded D we can obviously take $U = D$. □

Thanks to the above result we obtain a constructive version of Theorem 3.1.1.

Corollary 4.3.2. *Assume that unimodal and infinite ν satisfies* **A1, A2**. *Let $D \subset \mathbb{R}^d$ be nonempty, open and bounded, and assume that D^c fulfills VDC. Then the Dirichlet problem* (DP) *has a unique solution for every $g \in \mathcal{X}_D$ and $f \in L^2(D)$.*

With the present assumptions we may prove that $\mathcal{V}_D$ is a Hilbert space as we have mentioned in Chapter 2. This fact was verified by Dipierro, Ros-Oton and Valdinoci [57, Proposition 3.1] for the fractional Laplacian. We present a short proof which uses only the fact that ν is locally bounded away from zero.

Lemma 4.3.3. *If D is bounded, then $\mathcal{V}_D$ is complete with the norm $\|\cdot\|_{\mathcal{V}_D}$.*

Proof. If $\emptyset \neq U \subset\subset D$, then

$$\begin{aligned}&\int_{D^c} u(y)^2\nu(y,U)\,\mathrm{d}y = \int_U\int_{D^c} u(y)^2\nu(x,y)\,\mathrm{d}y\mathrm{d}x\\ &\leq 2\int_U\int_{D^c}(u(x)-u(y))^2\nu(x,y)\,\mathrm{d}y\mathrm{d}x + 2\int_U\int_{D^c} u(x)^2\nu(x,y)\,\mathrm{d}y\mathrm{d}x\\ &\leq 4\mathcal{E}_D[u] + 2\int_U u(x)^2\nu(x,D^c)\,\mathrm{d}x \lesssim \|u\|^2_{\mathcal{V}_D}.\end{aligned}$$

The last inequality follows from the fact that $x \mapsto \nu(x,D^c)$ is bounded on U. Thus $\|\cdot\|_{\mathcal{V}^D}$ dominates the norm in $L^2((\mathbf{1}_D(y)+\nu(y,U)\mathbf{1}_{D^c}(y))\,\mathrm{d}y)$, in particular it is a norm. Furthermore, $y \mapsto \nu(y,U)$ is locally bounded from below (by a positive constant) on D^c. Therefore every Cauchy sequence in $\mathcal{V}^D$ has a subsequence that converges to some measurable u *a.e.* in $\mathbb{R}^d$. By Fatou's lemma, $\|u\|_{\mathcal{V}_D} < \infty$ and also $\|u_n - u\|_{\mathcal{V}_D} \to 0$ as $n \to \infty$, cf. [67, Lemma 2.3]. □

Lemma 4.3.4. *If* (4.2.2) *holds, then $\mathcal{V}^D \subset L^2(1\wedge\nu) \subset L^1(1\wedge\nu)$.*

Proof. For $D_1 \subseteq D_2 \subseteq \mathbb{R}^d$ we have $\mathcal{V}^{D_2} \subseteq \mathcal{V}^{D_1}$, so we may assume that D is bounded. Fix nonempty open $U \subset\subset D$ and $x_0 \in U$. By (4.2.2) we have $\nu(y,U) \approx \nu(y,x_0)$ for $y \in D^c$. The result follows from Lemma 4.3.1, the proof of Lemma 4.3.3 and the finiteness of the measure $1\wedge\nu(x)\,\mathrm{d}x$. □

4.4 Harmonic functions

This section is devoted to studying the *harmonic functions*, that is, the solutions of the Dirichlet problem (DP) with $f \equiv 0$. In fact, we introduce various definitions of a harmonicity and in the end, in Theorem 4.4.14, we show that they all coincide for functions in $\mathcal{V}_D$. As an essential part of the arguments we prove the second-order differentiability of harmonic functions under **A1**, see Theorem 4.4.9 below.

4.4.1 Harmonicity as the mean value property

Let L be the operator given by (2.3.1) and let (X_t) be the symmetric pure-jump Lévy process in $\mathbb{R}^d$ constructed in Subsection 2.2.1, both objects being associated with unimodal, infinite and strictly positive Lévy measure ν. As before, D denotes a fixed nonempty open subset of $\mathbb{R}^d$.

Definition 4.4.1. (i) We say that $u\colon \mathbb{R}^d \to \mathbb{R}$ is L-harmonic (or harmonic, if L is understood) in D if it has the mean value property, that is for all open $U \subset\subset D$ and $x \in U$,

$$u(x) = \mathbb{E}^x u(X_{\tau_U}).$$

(ii) We say that u is regular L-harmonic (or regular harmonic) in D if $u(x) = \mathbb{E}^x u(X_{\tau_D})$ for $x \in D$.
In (i) and (ii) we assume that the integrals are absolutely convergent.

The following fact is rather well-known, but we give a short proof for the sake of completeness.

Lemma 4.4.2. *If u is regular L-harmonic in D, then it is L-harmonic in D.*

Proof. Let $g\colon D^c \to [0,\infty]$. Let $u(x) = g(x)$ for $x \in D^c$ and $u(x) = \mathbb{E}^x g(X_{\tau_D})$ for $x \in D$. Thus, u is regular harmonic in D. Let U be an arbitrary open set such that $U \subset D$. Of course, $\tau_U \le \tau_D$. We have $\tau_D = \tau_U + \tau_D \circ \theta_{\tau_U}$ and $u(X_{\tau_D}) = u(X_{\tau_D}) \circ \theta_{\tau_U}$, where θ_{τ_U} is the shift operator introduced in Subsection 2.2.1. Let $x \in U$. By the tower rule of conditional expectations, the strong Markov property (2.2.2) of (X_t) and the regular harmonicity of u,

$$\begin{aligned} u(x) &= \mathbb{E}^x u(X_{\tau_D}) = \mathbb{E}^x[\mathbb{E}^x[u(X_{\tau_D}) \circ \theta_{\tau_U} | \mathcal{F}_{\tau_U}]] \\ &= \mathbb{E}^x[\mathbb{E}^{X_{\tau_U}}[u(X_{\tau_D})]] = \mathbb{E}^x[u(X_{\tau_U})]. \end{aligned}$$

In particular u is harmonic on D. The case of general (signed) u follows from the above by taking g equal to the positive and negative parts of u on D^c. □

Lemma 4.4.3. *If $u(x) = \mathbb{E}^x[u(X_{\tau_D}); X_{\tau_D -} \neq X_{\tau_D}]$ for all $x \in D$, then u is harmonic in D.*

The proof is similar to that of Lemma 4.4.2, so we skip it. Corollary 2.2.3 immediately yields the following result.

Remark 4.4.4. The proof of Lemma 4.4.2 in fact shows that $\{u(X_{\tau_U}), U \subset D\}$ is a martingale ordered by inclusion of open subsets of D; the martingale is closed by $u(X_{\tau_D})$ if u is regular harmonic.

Lemma 4.4.5. *If u is L-harmonic in D, then $u \in L^1_{loc}(\mathbb{R}^d)$.*

Proof. Let $0 < \varepsilon < d(x, D^c)$ and let $P_{B(x,\varepsilon)}$ be the Poisson kernel of $B(x,\varepsilon)$. Then we have $\int_{B(x,\varepsilon)^c} |u(z)| P_{B(x,\varepsilon)}(x,z)\,\mathrm{d}z < \infty$. By Ikeda–Watanabe formula (2.2.12), $P_{B(0,\varepsilon)}(0,z) > 0$ on $B(0,\varepsilon)^c$. By [82, Corollary 2.4], $z \mapsto P_{B(0,\varepsilon)}(0,z)$ is radially nonincreasing on $B(0,\varepsilon)^c$, so $P_{B(x,\varepsilon)}(x,\cdot)$ is locally bounded away from zero on $B(x,\varepsilon)^c$. The result easily follows by taking disjoint $B(x_1,\varepsilon_1)$, $B(x_2,\varepsilon_2)$ contained in D. □

The following result yields the well-definiteness of the Poisson integrals of functions in $\mathcal{X}_D$ and thus is substantial for Theorem 4.2.1. We present it in more general setting in Lemma 6.4.5, but we also give a proof of the present special case here. Unlike Lemma 6.4.5, the following result and its proof are prone to a generalization for the forms which use $|g(z) - g(w)|^p$ instead of $(g(z) - g(w))^2$. In such case we in fact obtain the finiteness of $P_D[|g|^p]$. The latter be used in Section 6.6.

Lemma 4.4.6. *Assume that* **A2** *holds. If $g \in \mathcal{X}_D$ and $x \in D$, then $\int_{D^c} g(z)^2 P_D(x,z)\,\mathrm{d}z < \infty$.*

Proof. By the definition of γ_D,

$$\mathcal{H}_D[u] = \tfrac{1}{2} \int_{D^c} \int_{D^c} \int_D (g(z) - g(w))^2 \nu(w,x) P_D(x,z)\,\mathrm{d}x\mathrm{d}z\mathrm{d}w < \infty. \tag{4.4.1}$$

Since $\nu > 0$, for almost all $(x,w) \in D \times D^c$ we obtain

$$\int_{D^c} g(z)^2 P_D(x,z)\,\mathrm{d}z \le 2 \int_{D^c} (g(w) - g(z))^2 P_D(x,z)\,\mathrm{d}z + 2g(w)^2 < \infty. \tag{4.4.2}$$

Thus $\int_{D^c} g(z)^2 P_D(x,z)\,\mathrm{d}z < \infty$ for almost every $x \in D$. **A2** lets us use the boundary Harnack principle given by Grzywny and Kwaśnicki in [82, (1.12)] to get this assertion for all $x \in D$. Indeed, let $n = 1,2,\ldots$, $u_n(x) = g_n(x) = g^2(x) \wedge n$ for $x \in D^c$ and $u_n(x) = \mathbb{E}^x[g_n(X_{\tau_D}); X_{\tau_D -} \neq X_{\tau_D}]$ otherwise. Similarly, we let $u(x) = g^2(x)$ if $x \in D^c$, elsewhere we let $u(x) = \mathbb{E}^x[g(X_{\tau_D})^2; X_{\tau_D -} \neq X_{\tau_D}]$. Clearly, $u = \lim u_n$. These functions are (finite and) regular harmonic on every $U \subset\subset D$. By Lemma 4.4.3 and [82, (1.12)] the functions u_n are uniformly in n locally bounded on D, because $u_n \le u$. It follows that u is locally bounded on D, in particular it is finite on D. □

The next result is due to Grzywny and Kwaśnicki [82]. Let $B_r = B(0,r)$ and recall that G_{B_r} and P_{B_r} are the Green function and the Poisson kernel of the ball B_r, cf. (2.2.7) and (2.2.18).

Lemma 4.4.7. *Let $0 \le q < r < \infty$. There is a radial kernel $\overline{P}_{q,r}(z)$, a constant $C = C(d,\nu,q,r) > 0$ and a probability measure $\mu_{q,r}$ on the interval $[q,r]$, such that*

$$\overline{P}_{q,r}(z) = \int_{[q,r]} P_{B_s}(z)\,\mu_{q,r}(\mathrm{d}s) = \int_{[q,r]}\int_{B_s} \nu(y,z)G_{B_s}(0,y)\,\mathrm{d}y\,\mu_{q,r}(\mathrm{d}s), \quad |z|>r, \tag{4.4.3}$$

$\overline{P}_{q,r} = 0$ in B_q, $0 \le \overline{P}_{q,r} \le C$ in $\mathbb{R}^d$, $\overline{P}_{q,r} = C$ in $B_r \setminus B_q$ and $\overline{P}_{q,r}$ decreases radially on B_r^c. Furthermore, $\overline{P}_{q,r}(z) \le P_{B_r}(z)$, for $|z|>r$, and if f is L-harmonic in B_r, then

$$f(0) = \int_{\mathbb{R}^d\setminus B_q} f(z)\overline{P}_{q,r}(z)\,\mathrm{d}z.$$

Corollary 4.4.8. *If f is L-harmonic in B_{2r}, then $f = f * \overline{P}_{0,r}$ in B_r.*

We will use Lemma 4.4.7 to prove that the Poisson extensions are twice continuously differentiable under the additional assumption **A1**. In the proof we closely follow the arguments from Theorem 1.7 and Remark 1.8 b) in [82] except that we do not assume the boundedness of u.

Theorem 4.4.9. *Suppose that ν satisfies **A1** and let $D \subset \mathbb{R}^d$ be an open set. If $u\colon \mathbb{R}^d \to \mathbb{R}$ is L-harmonic in D, e.g., if $u(x) = \int_{D^c} u(z)P_D(x,z)\,\mathrm{d}z$ for $x \in D$, then $u \in C^2(D)$.*

Proof. Note that **A1** yields (4.2.2). We are in a position to apply Lemma 4.4.7. Let $x \in D$, and let $r>0$ be such that $B_{2r}(x) \subset D$. Since $\nu(z)$ is continuous, we get from (4.4.3) that kernels $\overline{P}_{q,r}$ are continuous. By Corollary 4.4.8, u is continuous in $B_r(x)$. Next we fix a nonnegative smooth radial function κ such that $0 \le \kappa \le 1$, $\kappa \equiv 1$ in $B_{\frac{3}{2}r}$ and $\kappa \equiv 0$ outside B_{2r}. As in [82], we denote $\pi_r(z) = \overline{P}_{0,r}(z)\kappa(z)$ and $\Pi_r(z) = \overline{P}_{0,r}(z)(1-\kappa(z))$. Obviously, $u = \Pi_r * u + \pi_r * u$ in $B_r(x)$. In particular, both terms are well-defined. Iterating, we get

$$\begin{aligned} u &= \left(\Pi_r + \pi_r * \Pi_r + \pi_r^{*2} * \Pi_r + \ldots \pi_r^{*(k-1)} * \Pi_r + \pi_r^{*k}\right) * u \\ &= (\delta_0 + \pi_r + \pi_r^{*2} + \ldots + \pi_r^{*(k-1)}) * \Pi_r * u + \pi_r^{*k} * u. \end{aligned} \tag{4.4.4}$$

Using an argument based on the Fourier transform as in [82, Proof of Theorem 1.7], we get that for every N there is a sufficiently large k, such that the function π_r^{*k} is N times continuously differentiable. It is also compactly supported. Since $u \in L^1_{loc}(\mathbb{R}^d)$, it follows that $\pi_r^{*k} * u$ is N times continuously differentiable in D. For our purposes below, it suffices to take $N=2$.

We will now handle the first summand in (4.4.4). First, observe that if $\theta > r$, $|z| > \theta > r$, and $|\beta| \in \{1,2\}$, then

$$\left|\partial^\beta \overline{P}_{0,r}(z)\right| \le c_{\theta,r}\overline{P}_{0,r}(z). \tag{4.4.5}$$

Indeed, by the definition of $\overline{P}_{0,r}$ and the Ikeda–Watanabe formula we have

$$\overline{P}_{0,r}(z) = \int_{[0,r]} P_{B_s}(z)\,\mu_{0,r}(\mathrm{d}s) = \int_{[0,r]}\int_{B_s} \nu(y,z)G_{B_s}(0,y)\,\mathrm{d}y\,\mu_{0,r}(\mathrm{d}s)$$

and further

$$\partial^\beta \overline{P}_{0,r}(z) = \int_{[0,r]}\int_{B_s} \partial_z^\beta \nu(y,z)G_{B_s}(0,y)\,\mathrm{d}y\,\mu_{0,r}(\mathrm{d}s).$$

For z as above and $y \in B_s \subset B_r$ we have $|z-y| \ge \theta - r$. By **A1**,

$$|\partial^\beta \overline{P}_{0,r}(z)| \le c_{\theta,r}\int_{[0,r]}\int_{B_s} \nu(y,z)G_{B_s}(0,y)\,\mathrm{d}y\,\mu_{0,r}(\mathrm{d}s) = c_{\theta,r}\overline{P}_{0,r}(z).$$

Since $\operatorname{supp}\Pi_r \subset B^c_{\frac{3}{2}r}$, and κ is smooth, from the Leibniz rule and (4.4.5) we see that for all $z \in \mathbb{R}^d$, $|\partial^\beta \Pi_r(z)| \le c_r|\Pi_r(z)|$. Therefore if $|\beta| \le 2$, then

$$\int_{\mathbb{R}^d} |\partial^\beta \Pi_r(x-z)u(z)|\,\mathrm{d}z < \infty,$$

which allows to differentiate under the integral sign and so $\partial^\beta \Pi_r * u(x)$ is well-defined. Continuity of the derivative follows from the continuity of $\partial^\beta \nu$ and the dominated convergence. ☐

Lemma 4.4.10. *Assume* **A1**. *If u is L-harmonic in D, then $Lu = 0$ on D.*

Proof. By Theorem 4.4.9, $u \in C^2(D)$. Let $x \in U \subset\subset D$. Let $\varphi \in C^2_c(D)$ be such that $u = \varphi$ on U. Let $w = u - \varphi$. We recall that on $C^2_c(\mathbb{R}^d)$, L coincides with the Dynkin characteristic operator $\mathcal{U}$, see Lemma 2.3.4. Since $w = 0$ in a neighborhood of x, by Corollary 2.2.3 and the Ikeda–Watanabe formula (2.2.12) we get

$$\begin{aligned}\mathcal{U}w(x) &= \lim_{r\to 0^+} \frac{\mathbb{E}^x w(X_{\tau_{B(x,r)}})}{\mathbb{E}^x \tau_{B(x,r)}}\\ &= \lim_{r\to 0^+} \frac{1}{\mathbb{E}^x \tau_{B(x,r)}} \int_{B(x,r)^c}\int_{B(x,r)} G_{B(x,r)}(x,z)\nu(z,y)\,\mathrm{d}z w(y)\,\mathrm{d}y\\ &= \lim_{r\to 0^+} \frac{1}{\mathbb{E}^x \tau_{B(x,r)}} \int_{B(x,r)} G_{B(x,r)}(x,z)\int_{U^c} \nu(z,y)w(y)\,\mathrm{d}y\mathrm{d}z.\end{aligned}$$

By Lemma 4.4.5 we have $\int_{B(x,r)^c} \nu(x,y)|u(y)|\,\mathrm{d}y < \infty$ for $r > 0$. Since **A1** yields (4.2.2), it follows that $z \mapsto \int_{U^c} \nu(z,y)w(y)\,\mathrm{d}y$ is a bounded continuous function near x. Since $\mathbb{E}^x\tau_{B(x,r)} = \int_{B(x,r)} G_{B(x,r)}(x,z)\,\mathrm{d}z$, cf. (2.2.9), we see that the expression $G_{B(x,r)}(x,z)\,\mathrm{d}z / \int_{B(x,r)} G_{B(x,r)}(x,z)\,\mathrm{d}z$ converges weakly to the Dirac mass at x as $r \to 0^+$. Therefore, $\mathcal{U}w(x) = \int_{U^c} \nu(x,y)w(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} (w(y) - w(x))\nu(x,y)\,\mathrm{d}y = Lw(x)$. We get

$$Lu(x) = L\varphi(x) + Lw(x) = \mathcal{U}\varphi(x) + \mathcal{U}w(x) = \mathcal{U}u(x).$$

On the other hand, by the mean value property of u we get $\mathcal{U}u(x) = 0$. Therefore $Lu(x) = 0$. ☐

We should warn the reader that for operators L more general than those considered here, L-harmonic functions may lack sufficient regularity to calculate Lu pointwise, see remarks after Corollary 20 in Bogdan and Sztonyk [29].

4.4.2 Weak and distributional harmonicity

Definition 4.4.11. (i) We say that $u\colon \mathbb{R}^d \to \mathbb{R}$ is weakly harmonic in D, if $u \in \mathcal{V}^D$ and $\mathcal{E}_D(u,\varphi) = 0$ for every $\varphi \in \mathcal{V}^D_0$.

(ii) We say that $u \in L^1_{loc}(\mathbb{R}^d)$ is distributionally harmonic in D if $\int_{\mathbb{R}^d} uL\varphi = 0$ for every $\varphi \in C^\infty_c(D)$.

By Lemma 4.3.1 we see that for bounded D the weak harmonicity means that the function is a weak solution of the Dirichlet problem in the sense of Definition 3.2.3 with $f \equiv 0$. By Theorem 4.2.1, Theorem 3.1.1 and Lemma 3.2.5 we obtain the following.

Corollary 4.4.12. *Under the setting of Theorem 4.2.1, the Poisson integral $P_D[g]$ is weakly harmonic. Furthermore, for bounded D it is the only weakly harmonic function equal to g a.e. on D^c.*

Weak harmonicity implies the distributional harmonicity because of the following result.

Lemma 4.4.13. *If* (4.2.2) *holds, D is bounded, $u \in \mathcal{V}^D$, and $\varphi \in C_c^\infty(D)$, then*

$$\mathcal{E}_D(u,\varphi) = -\int_{\mathbb{R}^d} uL\varphi.$$

Proof. As in [19, Lemma 3.3] we obtain

$$\begin{aligned}
-\int_{\mathbb{R}^d} uL\varphi &= \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^d} u(x)\int_{|y-x|>\epsilon}(\varphi(x)-\varphi(y))\nu(x,y)\,\mathrm{d}y\mathrm{d}x\\
&= \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^d} \varphi(x)\int_{|y-x|>\epsilon}(u(x)-u(y))\nu(x,y)\,\mathrm{d}y\mathrm{d}x\\
&= \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^d} \varphi(y)\int_{|y-x|>\epsilon}(u(y)-u(x))\nu(x,y)\,\mathrm{d}y\mathrm{d}x\\
&= \tfrac{1}{2}\lim_{\epsilon\to 0^+}\iint_{|y-x|>\epsilon}(\varphi(x)-\varphi(y))(u(x)-u(y))\nu(x,y)\,\mathrm{d}y\mathrm{d}x = \mathcal{E}_D(u,\varphi).
\end{aligned}$$

Below we argue that interchanging the limit and the integrations in the above calculations is justified and that the conditions needed to extend the proof of [19, Lemma 3.3] to the present setting are satisfied. Indeed, in the last line we may use the dominated convergence theorem, the inequality $2|ab| \le a^2 + b^2$, and the fact that $u, \varphi \in \mathcal{V}^D$. Next, let $\epsilon > 0$. Arguing as in Proposition 2.3.2, for $x \in D$, we get that $|u(x)\int_{|x-y|>\epsilon}(\varphi(x)-\varphi(y))\nu(x,y)\,\mathrm{d}y|$ is bounded by $C|u(x)|\|\varphi\|_{C^2} \in L^1(D)$. Furthermore if we let $U = \operatorname{supp}\varphi$, then for $x \in D^c$ we have $|u(x)\int_{|x-y|>\epsilon}(\varphi(x)-\varphi(y))\nu(x,y)\,\mathrm{d}y| \le 2\|\varphi\|_\infty|u(x)|\nu(x,U)\,\mathrm{d}x \in L^1(D^c)$, by Lemma 4.3.4. By the dominated convergence theorem,

$$-\int_{\mathbb{R}^d} uL\varphi = \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^d} u(x)\int_{|y-x|>\epsilon}(\varphi(x)-\varphi(y))\nu(x,y)\,\mathrm{d}y\mathrm{d}x.$$

Further, note that $\int_{\mathbb{R}^d}|u\varphi| < \infty$. By (4.2.2) and Lemma 4.3.4, we get that

$$\int_{\mathbb{R}^d}\int_{|x-y|>\epsilon}|u(x)\varphi(y)|\nu(x,y)\,\mathrm{d}x\mathrm{d}y \lesssim \int_{\mathbb{R}^d}|\varphi(y)|\,\mathrm{d}y\int_{\mathbb{R}^d}|u(x)|(1\wedge\nu(0,x))\,\mathrm{d}x < \infty.$$

Thus the assumptions of [19, Lemma 3.3] are satisfied. The proof is complete. □

4.4.3 Equivalence of the definitions

By Theorem 4.2.1, if $g \in \mathcal{X}_D$, then its Poisson extension belongs to $\mathcal{V}_D$. In fact, the Poisson extension $P_D[g]$ is the only weak solution of (DP) with $f \equiv 0$, as we will see shortly.

We say that $\tilde{u}$ is a modification of u if $\tilde{u} = u$ *a.s.* Various definitions of harmonicity can be unified as follows.

Theorem 4.4.14. *Let $u \in \mathcal{V}^D$ and let D be bounded and have continuous boundary (see Definition 2.1.2). Under the assumptions of Theorem 4.2.1 the following statements are equivalent:*

(i) $\int_{\mathbb{R}^d} uL\varphi = 0$ *for every* $\varphi \in C_c^\infty(D)$ *(distributionally harmonic);*

(ii) $\mathcal{E}_D(u,\varphi) = 0$ *for every* $\varphi \in \mathcal{V}_0^D$ *(weakly harmonic);*

(ii') u is the minimizer of the form $\mathcal{E}_D$ among the functions a.e. equal to u on D^c;

(iii) u has a modification that is L-harmonic;

(iv) u has a modification that is regular L-harmonic, and $u = P_D[u]$ a.e.

Furthermore, any of the statements above yields $Lu(x) = 0$ in D.

Proof. First, (iv) implies (iii) by Lemma 4.4.2. Then we prove that (iii) implies (ii). Indeed, by Corollary 4.4.12 used for Lipschitz open $U \subset\subset D$ we get $\mathcal{E}_U(u, \varphi) = 0$ for every $\varphi \in C_c^\infty(U)$. By the dominated convergence theorem and the fact that $u \in \mathcal{V}^D$, we have $\mathcal{E}_D(u, \varphi) = 0$. Since for every $\varphi \in C_c^\infty(D)$ there exists $U \subset\subset D$ containing the support of φ, we get that $\mathcal{E}_D(u, \varphi) = 0$ for every $\varphi \in C_c^\infty(D)$. Then we use the density of smooth functions in $\mathcal{V}_0^D$, see [21, Theorem A.4] and cf. Fiscella, Servadei and Valdinoci [68]. The statements (ii) and (ii') are equivalent by Lemma 3.2.5. Further, (ii) implies (i) by Lemma 4.4.13. Finally, (i) implies (iv). Indeed, by [81, Theorem 1.1] u is harmonic and thus weakly harmonic. By the trace theorem $\operatorname{Tr} u \in \mathcal{X}_D$, and by the extension theorem $P_D[u] \in \mathcal{V}^D$. By Corollary 4.4.12, $P_D[u]$ is the unique weakly harmonic function equal to u *a.e.* on D^c. Hence $u = P_D[u]$ *a.e.* on $\mathbb{R}^d$. The statement $Lu = 0$ follows from Lemma 4.4.10. □

Theorem 4.4.9 allows for the following extension which may be regarded as a counterpart of the Weyl's lemma, or the hypoellipticity for nonlocal operators.

Corollary 4.4.15. *Assume that for every $r_0 > 0$ there exists $C(r_0)$ such that $|\nu^{(k)}(r)| \lesssim C(r_0)\nu(r)$ for $r > r_0$ and $k = 1, \ldots, n$. If $u \in L^1(1 \wedge \nu)$ is distributionally harmonic in D, then $u \in C^n(D)$.*

Proof. Adapt the proof of Theorem 4.4.9, starting from (4.4.4). □

4.5 Analytic approach to extension

Below we show an analytic approach to the extension problem for the spaces $F_{2,2}(D^c)$. The idea here is somewhat easier compared to the spaces $\mathcal{V}_D$, because usually the extension for the spaces of the type $F_{2,2}$ amounts to copying the values of the functions from the outside in an appropriate manner and verifying that the same property which was satisfied on D^c also holds true in D. In the following subsection we show that the reflection through the boundary of a $C^{1,1}$ set is a Lipschitz homeomorphism. This result enables us to obtain the appropriate estimates for the extension operator constructed with the use of that reflection in Subsection 4.5.2

4.5.1 Reflection through the boundary

Let D be a $C^{1,1}$ open set with the constants $r_0, \lambda > 0$, in the sense of Definition 2.1.2. Recall that by Lemma 2.1.4, D satisfies the interior and exterior ball condition, i.e., it is $C^{1,1}$ at scale r for some $r > 0$. Obviously, if $0 < s < r$, then D is also $C^{1,1}$ at scale s. Note, that by taking exterior and interior balls of radii strictly smaller than r, we avoid the situation in which either of the balls touches the boundary in more than one point. Thus, for every fixed $s \in (0, r)$, we obtain a bijective correspondence between the center of the interior ball of radius s and the point on the boundary that this ball is tangent to. We call this mapping $\psi_s : \partial D \to D$. We also get a similar bijection between the center of the exterior ball and the point on the boundary: $\chi_s : \partial D \to D^c$. We will denote by x_I (x_E) the center of a generic interior (exterior) ball tangent to ∂D at the point x. Recall that $\delta(x) = d(x, \partial D)$.

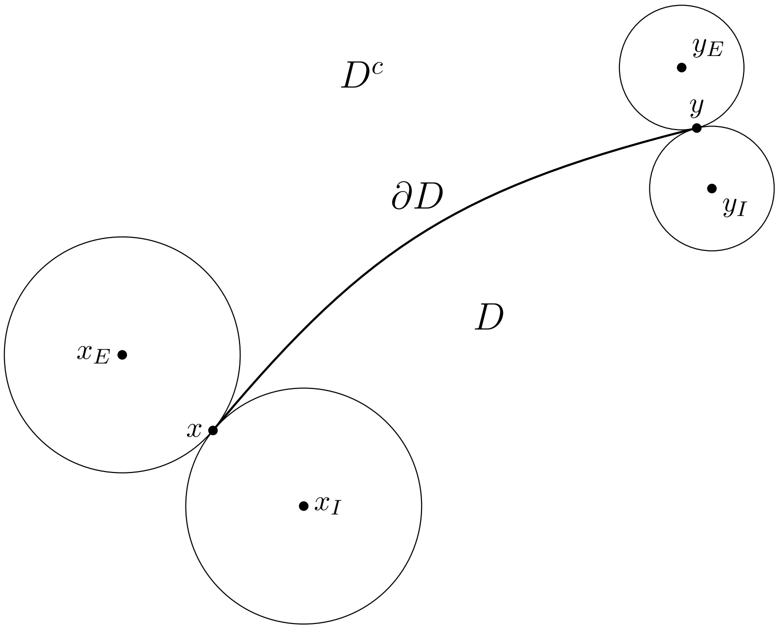

Figure 4.2: An illustration of the boundary of the $C^{1,1}$ set D and the related notation.

Definition 4.5.1. Let D be a $C^{1,1}$ open set with the constants λ, r_0 as in Definition 2.1.2, and r from Lemma 2.1.4. Let $V = \{x \in \mathbb{R}^d : d(x, \partial D) < \varepsilon = r \wedge \frac{1}{8\lambda} \wedge \frac{r_0}{3}\}$. We define the reflection operator $T\colon V \to V$ by the formula

$$\begin{cases} Tx_I = \chi_{\delta(x_I)} \circ \psi^{-1}_{\delta(x_I)}(x_I), & x_I \in D \cap V, \\ Tx_E = \psi_{\delta(x_E)} \circ \chi^{-1}_{\delta(x_E)}(x_E), & x_E \in \mathrm{Int}\,(D^c) \cap V, \\ Tx = x, & x \in \partial D. \end{cases}$$

From the construction we immediately get $T = T^{-1}$. The reasons for the choice of ε will be seen in the proof of Lemma 4.5.2. The reflection T, in general, does not preserve the distance between points, however we will prove that $|x - y| \approx |Tx - Ty|$ in V.

Lemma 4.5.2. *There exists a constant $C_T \geq 1$, such that $|x-y| \leq C_T|Tx-Ty|$ holds for every $x, y \in V$. As a consequence, $\frac{1}{C_T}|x - y| \leq |Tx - Ty| \leq C_T|TTx - TTy| = C_T|x - y|$.*

Proof. In Figure 4.2 we have $x_E = Tx_I$, $y_E = Ty_I$ and x and y are the corresponding points on the boundary. This will be our convention in the whole proof. Obviously, we may assume that $x \neq y$, because T preserves the distance for $x = y$. Below, by PQ we mean the line segment with endpoints P and Q and ΔPQR is the triangle with vertices P, Q, R. Note that under our notation x is the midpoint of $x_I x_E$. For $x \in \partial D$, let $U_x = B(x, r_0) \cap \partial D$. It suffices to consider three cases: first — when both points are in D, second — when one of them is in D, and the other is in $\mathrm{Int}\, D^c$, and third — when one of the points is on the boundary.

Case 1. Both points are in D. In this case we need to estimate $|x_E y_E|$ by means of $|x_I y_I|$ and vice versa. To this end, we make yet another division.

Case 1.1. $y \in U_x$. We will assume without any loss of generality that $|y_I y_E| \leq |x_I x_E|$. Let z_I and z_E be the orthogonal projections of y_I and y_E respectively, onto the unique line parallel

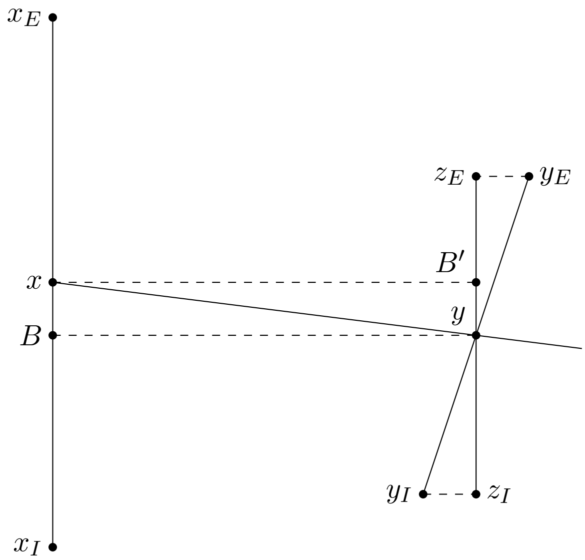

Figure 4.3: Projection of y_E and y_I. We have $\angle(xBy) = \angle(xB'y) = \frac{\pi}{2}$.

to $x_I x_E$ that contains y. Note that $|z_I z_E| \leq |x_I x_E|$. Further, we let B' be the projection of x onto $z_I z_E$ and B — the projection of y onto $x_I x_E$, so that both ΔxBy and $\Delta xB'y$ are right triangles, see Figure 4.3. The illustration is inevitably only two-dimensional, so we remark that the segment $y_E y_I$ need not belong to the plane generated by the segments $x_I x_E$ and $z_I z_E$, which is in fact the sole reason for considering z_I and z_E. In order to prove that $|x_I y_I| \approx |x_E y_E|$, we will first show that $|x_I z_I| \approx |x_E z_E|$. Recall that R_x and f_x are respectively the rigid motion and the $C^{1,1}$ function from Definition 2.1.2. By the Lagrange's mean value theorem there exists $\xi \in R_x(xB')$ such that $\frac{|B'y|}{|B'x|} = |\phi'(\xi)|$, where ϕ is the restriction of f_x to the unique line containing $R_x(xB')$. By the Lipschitz condition for the derivative, we get

$$\frac{|Bx|}{|By|} = \frac{|B'y|}{|B'x|} = |\phi'(\xi)| \leq \lambda |B'x|,$$

and, as a consequence,

$$|Bx| \leq \lambda |By|^2. \tag{4.5.1}$$

Assume, without loss of generality, that $|x_E z_E| \geq |x_I z_I|$ and let us look at Figure 4.4. The shape of the trapezoid may depend on positions of x and y, however the following arguments (in particular, the formula for $|x_I z_I|$ in (4.5.2) below) are independent of this shape. Recall the assumption $|x_I x_E| \geq |y_I y_E|$, and let $l = |x_I x_E| - |z_I z_E| > 0$, $h = |By|$, and $t = |x_E A|$. The ratio $\frac{|x_I z_I|^2}{|x_E z_E|^2}$ can be represented as a function of t:

$$\frac{|x_I z_I|^2}{|x_E z_E|^2} = \frac{(t-l)^2 + h^2}{t^2 + h^2} = 1 - \frac{l(2t-l)}{t^2 + h^2}. \tag{4.5.2}$$

Note that $|t - \frac{l}{2}| = |Bx|$ (recall that x and y bisect $x_I x_E$ and $z_I z_E$ respectively). The assumption $|x_I z_I| \leq |x_E z_E|$ yields $2t - l \geq 0$. Hence, from (4.5.1) and (4.5.2) we get that $2t - l \leq 2\lambda h^2$, thus

$$\frac{|x_I z_I|^2}{|x_E z_E|^2} \geq 1 - \frac{2l\lambda h^2}{t^2 + h^2} \geq 1 - 2l\lambda \geq \frac{1}{2}. \tag{4.5.3}$$

In the last inequality above we have used $l \le |x_I x_E| < 2\varepsilon$, and $\varepsilon \le \frac{1}{8\lambda}$. Thus,

$$|x_I z_I| \approx |x_E z_E|. \tag{4.5.4}$$

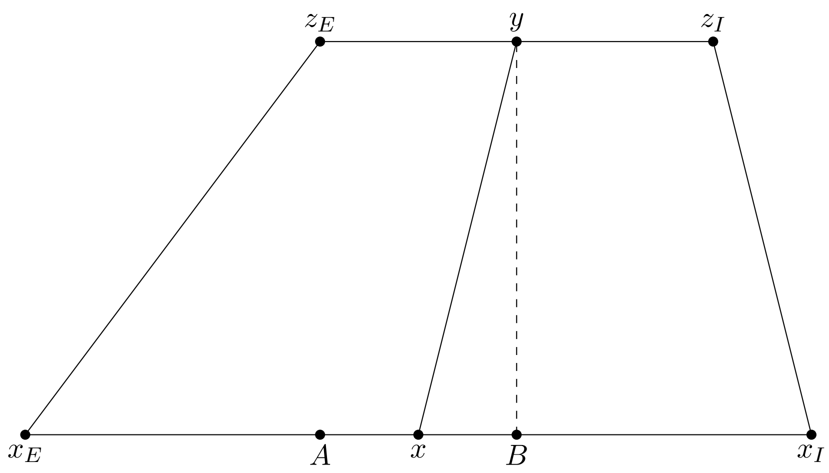

Figure 4.4: Projection of Figure 4.3 onto the plane (rotated by 90 degrees counterclockwise). Here Az_E and By are the heights of the trapezoid.

We now proceed with the estimates of $|x_I y_I|$. By the triangle inequality and (4.5.4),

$$\begin{aligned} |x_I y_I| \le |x_I z_I| + |z_I y_I| \approx |x_E z_E| + |z_I y_I| &\le |x_E y_E| + |y_E z_E| + |z_I y_I| \\ &= |x_E y_E| + 2|z_E y_E|. \end{aligned} \tag{4.5.5}$$

We claim that $|z_E y_E| \le |x_E y_E|$. By using the Lipschitz condition for ϕ', we get

$$\frac{|z_E y_E|}{|z_E y|} = \tan(\angle y_E y z_E) = |\phi'(x) - \phi'(B')| \le \lambda |B'x| \le \lambda |x_E z_E|. \tag{4.5.6}$$

Therefore,

$$|z_E y_E| \le \lambda |x_E z_E||z_E y| \le \lambda |x_E z_E||y_E y| \le \lambda\varepsilon |x_E z_E|. \tag{4.5.7}$$

Recall that by the definition of V we have $\varepsilon \le \frac{1}{2\lambda}$. Hence,

$$|z_E y_E| \le \frac{|x_E z_E|}{2}. \tag{4.5.8}$$

By (4.5.8), the triangle inequality, and (4.5.7) we get the claim:

$$|x_E y_E| \ge |x_E z_E| - |z_E y_E| \ge |x_E z_E| - \frac{|x_E z_E|}{2} = \frac{|x_E z_E|}{2} \ge |z_E y_E|. \tag{4.5.9}$$

By applying (4.5.9) to (4.5.5) we obtain

$$|x_I y_I| \lesssim |x_E y_E|.$$

Thanks to $|x_I z_I| \approx |x_E z_E|$, the reverse estimate is obtained similarly, by interchanging $|x_I y_I|$ and $|x_E y_E|$ in (4.5.5). Thus, Case 1.1. is resolved.

Case 1.2. $y \notin U_x$**.** In that situation $|yx| \geq r_0$. By the definition of V, we have $\varepsilon < \frac{r_0}{3}$ and as a consequence $|x_I x_E|, |y_I y_E| \leq \frac{r_0}{3}$. Hence, $|x_E y_E| \geq |xy| - |xx_E| - |yy_E| \geq r_0 - \frac{r_0}{3} - \frac{r_0}{3} = \frac{r_0}{3}$. Analogously $|x_I y_I| \geq \frac{r_0}{3}$. By using a similar argument we may show that $||x_I y_I| - |x_E y_E|| < 4\varepsilon < 2r_0$, hence $|x_I y_I|$ and $|x_E y_E|$ must be comparable. In the remaining cases we will not discuss the situation when x and y are far from each other as it can be resolved in exactly the same way.

Case 2. One point is inside D, one outside. We now compare $|x_I y_E|$ and $|x_E y_I|$. Once again we first project the situation on a plane with the assumption that $|z_I z_E| \leq |x_I x_E|$ and now we add that $|x_E z_I| \geq |x_I z_E|$.

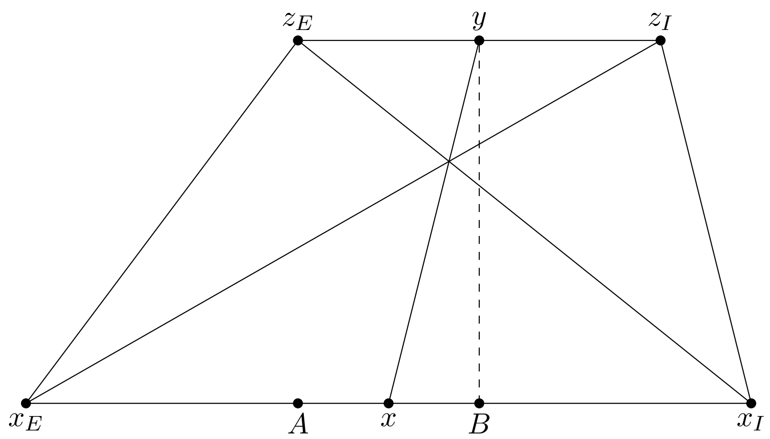

Figure 4.5: Illustration of the second case

We claim that $|x_E z_I| \lesssim |x_I z_E|$. Let l, h, t be the same as before, and denote $a = |z_E z_I|$, $b = |x_E x_I|$. Then, $|x_E z_I|^2 = (t+a)^2 + h^2$, and $|x_I z_E|^2 = (b-t)^2 + h^2$. Note that here we have the same condition on t as in the previous case: $2t - l \leq 2\lambda h^2$. Therefore, since $a + b \leq 4\varepsilon$,

$$\begin{aligned} |x_E z_I|^2 - |x_I z_E|^2 &= 2(a+b)t + a^2 - b^2 = (a+b)(2t - (b-a)) = (a+b)(2t - l) \\ &\leq (a+b)2\lambda h^2 \leq 8\varepsilon\lambda h^2 \leq 8\varepsilon\lambda |x_I z_E|^2. \end{aligned}$$

Thus we have obtained

$$|x_E z_I|^2 \leq |x_I z_E|^2 (1 + 8\varepsilon\lambda). \tag{4.5.10}$$

The claim is proved. Note that in the last inequality of (4.5.6), we can change $|x_E z_E|$ to $|x_E z_I|$. Therefore, in order to get that $|x_E y_I| \approx |x_I y_E|$ we can use the same approach as in Case 1.1.

Case 3. One point inside, one on the boundary. Note that in Case 1.1., when proving that $|x_I z_I| \approx |x_E z_E|$ we could as well assume that $|y_I y_E| = 0$. Therefore this situation can be handled in the same way. □

Corollary 4.5.3. *T is a bi-Lipschitz homeomorphism of V.*

In order to prove the continuity of the extension operator in the following subsection, we will frequently use the integration by substitution formula for T. This fact is well-known, it follows conveniently from the result of Hajłasz [85, Appendix], but we note that it has been known for much longer, see the book of Rado and Reichelderfer [132, V.2.3].

Lemma 4.5.4. *Let U and V be open subsets of $\mathbb{R}^d$ and assume that $T\colon U \to V$ is a bijection such that T and T^{-1} are Lipschitz with constant $c \geq 1$. Then T maps measurable sets to measurable sets and for every nonnegative measurable function $u\colon V \to \mathbb{R}$ we have*

$$(1/c)^d \int_V u(x)\,\mathrm{d}x \leq \int_U u(Tx)\,\mathrm{d}x \leq c^d \int_V u(x)\,\mathrm{d}x.$$

4.5.2 The extension operator

Let D be a $C^{1,1}$ open set. We will use the reflection T constructed in the previous subsection in order to construct extensions for functions from the space $F_{2,2}(D)$ defined in (2.3.18).

We let $W = V \cap D$ with V the same as in Definition 4.5.1. The extension of $g \in F_{2,2}(D^c)$ is done by mirroring the value from the reflected point, but since T is only defined near the boundary, we use a cut-off function, so that effectively we only need to prescribe the values in W. The details are given below.

Definition 4.5.5. Let $\varphi \in C^\infty(\mathbb{R}^n)$ satisfy $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in D^c, and $\varphi \equiv 0$ in $D \setminus W$. We define the extension operator $A\colon F_{2,2}(D^c) \to F_{2,2}(\mathbb{R}^d)$ by the formula $A(g) = \widetilde{g}$, where

$$\widetilde{g}(x) = \begin{cases} g(x), & x \in D^c, \\ g(Tx)\varphi(x), & x \in W, \\ 0, & x \in D \setminus W. \end{cases}$$

Theorem 4.5.6. *Let $C_T \geq 1$ be the Lipschitz constant for the reflection operator T of Definition 4.5.1, associated with a $C^{1,1}$ bounded open set D. Assume that ν is unimodal and suppose that we have*

$$\nu(\beta r) \lesssim \nu(r), \quad \beta \in [C_T^{-1} \wedge \tfrac{1}{3}, 1],\ r > 0. \tag{4.5.11}$$

Then the extension operator A from Definition 4.5.5 is continuous from $F_{2,2}(D^c)$ to $F_{2,2}(\mathbb{R}^d)$.

We note that the condition (4.5.11) yields the strict positivity of ν. However, it is mild enough to include even finite Lévy measures ν, but for such measures the extension problem becomes trivial because the L^2 norm dominates $\mathcal{E}$, cf. Remark 5.4.6.

Proof. Let $g \in F_{2,2}(D^c)$, i.e., $g \in L^2(D^c)$ and $\int_{D^c}\int_{D^c}(g(x) - g(y))^2\nu(x,y)\,\mathrm{d}y\mathrm{d}x < \infty$. In order to show that A is continuous, we need to estimate $\|\widetilde{g}\|^2_{L^2(D)} + \mathcal{E}[\widetilde{g}]$ by a multiple of $\|g\|^2_{F_{2,2}(D^c)}$. The estimate for the L^2 norm is straightforward:

$$\begin{aligned}\int_{\mathbb{R}^d} \widetilde{g}(x)^2\,\mathrm{d}x &= \int_{D^c} g(x)^2\,\mathrm{d}x + \int_W \widetilde{g}(x)^2\,\mathrm{d}x = \int_{D^c} g(x)^2\,\mathrm{d}x + \int_W g(Tx)^2\varphi(x)^2\,\mathrm{d}x \\ &\leq \int_{D^c} g(x)^2\,\mathrm{d}x + \int_W g(Tx)^2\,\mathrm{d}x \lesssim \int_{D^c} g(x)^2\,\mathrm{d}x + \int_{TW} g(x)^2\,\mathrm{d}x \leq 2\int_{D^c} g(x)^2\,\mathrm{d}x.\end{aligned}$$

In order to estimate $\mathcal{E}[\widetilde{g}]$, we split it into three parts, cf. Lemma 3.2.1:

$$2\mathcal{E}[\widetilde{g}] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\widetilde{g}(x) - \widetilde{g}(y))^2\nu(x,y)\,\mathrm{d}y\mathrm{d}x = \int_{D^c}\int_{D^c}(g(x) - g(y))^2\nu(x,y)\,\mathrm{d}y\mathrm{d}x \tag{4.5.12}$$

$$+\,2\int_D\int_{D^c}(\widetilde{g}(x) - g(y))^2\nu(x,y)\,\mathrm{d}y\mathrm{d}x \tag{a}$$

$$+\int_D\int_D(\widetilde{g}(x) - \widetilde{g}(y))^2\nu(x,y)\,\mathrm{d}y\mathrm{d}x. \tag{b}$$

There is nothing left to do in (4.5.12). Let us estimate (one half of) (**a**):

$$\int_D \int_{D^c} (\widetilde{g}(x) - g(y))^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x = \int_W \int_{D^c} (\widetilde{g}(x) - g(y))^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x \qquad (\mathbf{a1})$$

$$+ \int_{D\setminus W} \int_{D^c} g(y)^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x. \qquad (\mathbf{a2})$$

Recall that $\widetilde{g}(x) = g(Tx)\varphi(x)$ for $x \in W$. By the triangle inequality we have

$$|g(Tx)\varphi(x) - g(y)| \le |g(Tx)\varphi(x) - g(y)\varphi(x)| + |g(y)\varphi(x) - g(y)|,$$

hence (**a1**) is less than or equal to

$$\begin{aligned}
&\int_W \int_{D^c} (|g(Tx) - g(y)|\varphi(x) + |g(y) - g(y)\varphi(x)|)^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x \\
&\le 2\int_W \int_{D^c} g(y)^2 (1-\varphi(x))^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x \qquad (\mathbf{a11}) \\
&\quad + 2\int_W \int_{D^c} (g(Tx) - g(y))^2 \varphi(x)^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x. \qquad (\mathbf{a12})
\end{aligned}$$

Using that φ is smooth and $\varphi = 1$ on D^c, we get $|1-\varphi(x)| \lesssim d(x, D^c)$ for $x \in W$. Therefore we can estimate (**a11**) as follows

$$\begin{aligned}
\int_W \int_{D^c} g(y)^2 (1-\varphi(x))^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x &= \int_{D^c} g(y)^2 \int_W (1-\varphi(x))^2 \nu(x,y)\,\mathrm{d}x\mathrm{d}y \\
&\lesssim \int_{D^c} g(y)^2 \int_W d(x,D^c)^2 \nu(x,y)\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Note that if $c = \varepsilon \vee 1$, then we have $d(x, D^c) \le c(1 \wedge |x-y|)$ for every $x \in W$ and $y \in D^c$. Since ν is the density of a Lévy measure, we get

$$\begin{aligned}
\int_{D^c} g(y)^2 \int_W d(x,D^c)^2 \nu(x,y)\,\mathrm{d}x\mathrm{d}y &\le c^2 \int_{D^c} g(y)^2 \int_W (1 \wedge |x-y|^2)\nu(x,y)\,\mathrm{d}x\mathrm{d}y \\
&\le c^2 \int_{\mathbb{R}^d} (1\wedge |x|^2)\nu(x)\,\mathrm{d}x \int_{D^c} g(y)^2\,\mathrm{d}y \\
&\lesssim \int_{D^c} g(y)^2\,\mathrm{d}y.
\end{aligned}$$

Thus, (**a11**) is estimated. In (**a12**), the substitution $Tx \to x$ and Lemma 4.5.4 yield:

$$\begin{aligned}
&\int_W \int_{D^c} (g(Tx) - g(y))^2 \varphi(x)^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x \\
&\approx \int_{TW} \int_{D^c} (g(x) - g(y))^2 \varphi(Tx)^2 \nu(Tx,y)\,\mathrm{d}y\mathrm{d}x. \qquad (4.5.13)
\end{aligned}$$

For $x \in TW$ and $y \in D^c$ we have $|x-y| \le |x-Tx| + |Tx-y|$, and $|Tx-y| \ge d(Tx, D^c) = \frac{|x-Tx|}{2}$, hence $|x-y| \le 3|Tx-y|$. By the unimodality of ν and the assumption (4.5.11), we get $\nu(|Tx-y|) \le \nu(\frac{1}{3}|x-y|) \lesssim \nu(|x-y|)$. Therefore, the integral in (4.5.13) is estimated from above by

$$\int_{TW} \int_{D^c} (g(x) - g(y))^2 \varphi(Tx)^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x \le \|g\|^2_{F_{2,2}(D^c)}.$$

In order to estimate (**a2**), we note that for every $y \in D^c$ we have $d(y, D \setminus W) > \varepsilon$ with ε from Definition 4.5.1. Hence,

$$\int_{D^c} g(y)^2 \int_{D\setminus W} \nu(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{D^c} g(x)^2 \nu((D\setminus W)-x)\,\mathrm{d}x \le \nu(B(0,\varepsilon)^c) \int_{D^c} g(x)^2\,\mathrm{d}x.$$

Since $\widetilde{g} = 0$ on $D \setminus W$, the term (**b**) can be split as follows:

$$\int_D \int_D (\widetilde{g}(x) - \widetilde{g}(y))^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x = \int_W \int_W (g(Tx)\varphi(x) - g(Ty)\varphi(y))^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x \qquad (\mathbf{b1})$$
$$+ 2\int_W \int_{D\setminus W} g(Tx)^2 \varphi(x)^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x. \qquad (\mathbf{b2})$$

We use the triangle inequality in order to estimate (**b1**) from above by

$$2\int_W \int_W g(Tx)^2 (\varphi(x) - \varphi(y))^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x \qquad (\mathbf{b11})$$
$$+ 2\int_W \int_W (g(Tx) - g(Ty))^2 \varphi(y)^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x. \qquad (\mathbf{b12})$$

In (**b11**) we have

$$\int_W g(Tx)^2 \int_W (\varphi(x) - \varphi(y))^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x \lesssim \int_W g(Tx)^2 \int_W |x-y|^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x$$
$$\lesssim \int_{\mathbb{R}^d} (1 \wedge |y|^2)\nu(y)\,\mathrm{d}y \int_W g(Tx)^2\,\mathrm{d}x \approx \int_{\mathbb{R}^d} (1 \wedge |y|^2)\nu(y)\,\mathrm{d}y \int_{TW} g(x)^2\,\mathrm{d}x \lesssim \|g\|^2_{L^2(D^c)}.$$

We know that $C_T^{-1}|x-y| \le |Tx - Ty| \le C_T|x-y|$ holds for all $x, y \in W$. By substituting and by using the unimodality and (4.5.11), we can estimate the integral in (**b12**) as follows:

$$\int_W \int_W (g(Tx) - g(Ty))^2 \varphi(y)^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x$$
$$\lesssim \int_{TW} \int_{TW} (g(x) - g(y))^2 \nu(Tx, Ty)\,\mathrm{d}y\mathrm{d}x$$
$$\lesssim \int_{TW} \int_{TW} (g(x) - g(y))^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x \le \|g\|^2_{F_{2,2}(D^c)}.$$

Note that for every $x \in W$ and $y \in D \setminus W$ we have $\varphi(x)^2 \lesssim |x-y|^2$, thus (**b2**) is bounded from above by a multiple of

$$\int_W g(Tx)^2 \int_{D\setminus W} |x-y|^2 \nu(x,y)\,\mathrm{d}y\mathrm{d}x \le \int_{\mathbb{R}^d} (1 \wedge |y|^2)\nu(y)\,\mathrm{d}y \int_W g(Tx)^2\,\mathrm{d}x \approx \int_{TW} g(x)^2\,\mathrm{d}x.$$

This ends the proof. □

As a consequence we obtain another set of explicit conditions which guarantee the existence and uniqueness of solutions to the Dirichlet problem. We note that it is less exhaustive than Corollary 4.3.2, because of more restrictive assumptions on g.

Corollary 4.5.7. *Let ν and D satisfy the assumptions of Theorem 4.5.6. If $g \in F_{2,2}(D^c)$ and $f \in L^2(D)$, then the Dirichlet problem* (DP) *has a unique weak solution.*

Chapter 5

Triebel–Lizorkin spaces with reduction of integration domain

5.1 Introduction

The content of this chapter comes from the author's paper [138]. We assume that $D \subset \mathbb{R}^d$ is a **domain** and we let $p, q \in (1, \infty)$.

The structure of the Triebel–Lizorkin spaces may differ from that of the nonlocal Sobolev spaces, which calls for a slightly different class of kernels than the Lévy kernel ν. Namely, we let $K\colon \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty]$ be a unimodal kernel satisfying $\int_{\mathbb{R}^d} (1 \wedge |y|^q) K(0, y)\, \mathrm{d}y < \infty$, which for $q = 2$ reduces to the integrability condition for a Lévy measure. As a generalization of the spaces $F_{2,2}(D)$ from Subsection 2.3.2, we define the Triebel–Lizorkin spaces on D as follows:

$$F_{p,q}(D) := \Big\{ u \in L^p(D) : \int_D \Big(\int_D |u(x) - u(y)|^q K(x, y)\, \mathrm{d}y \Big)^{\frac{p}{q}} \mathrm{d}x < \infty \Big\}. \tag{5.1.1}$$

We endow $F_{p,q}(D)$ with the norm

$$\|u\|_{F_{p,q}(D)} = \|u\|_{L^p(D)} + \Big(\int_D \Big(\int_D |u(x) - u(y)|^q K(x, y)\, \mathrm{d}y \Big)^{\frac{p}{q}} \mathrm{d}x \Big)^{\frac{1}{p}}.$$

Originally, Lizorkin [118] and Triebel [152] defined their spaces for $D = \mathbb{R}^d$, by using Paley–Littlewood theory and the spaces on subsets $D \subset \mathbb{R}^d$ are defined, in a sense, as trace spaces, see Triebel [153, page 192], but this is slightly more involved than reducing all the integrations to D as we do above. Our definition of $F_{p,q}(D)$ for $K(x, y) = |x - y|^{-d-sq}$ coincides with the classical ones under various assumptions for D and the parameters p, q, s, d, see [128, 152, 153]. We acknowledge the fact that the original approach is widely used in analysis and its applications, also in the PDEs, see, e.g., [7, 33, 79]. However, in the sequel we will not discuss or use the original definition, as our main point of interest is the following Gagliardo-type seminorm

$$\Big(\int_D \Big(\int_D |u(x) - u(y)|^q K(x, y)\, \mathrm{d}y \Big)^{\frac{p}{q}} \mathrm{d}x \Big)^{\frac{1}{p}}, \tag{5.1.2}$$

which will be called the *full seminorm.* The seminorm (5.1.2) seems to be a more natural way of defining the spaces (compared to the classical approach of Triebel and Lizorkin) if the starting point is $p = q = 2$, e.g., fractional Sobolev spaces in nonlocal PDEs which was seen in the preceding chapters (mind the integration domain), or Dirichlet forms for Hunt processes

[18, 72]. It is also suitable for kernels K more general than $|x-y|^{-d-sq}$, which is of course of interest in the fields of nonlocal operators and stochastic processes.

In order to present the general goal of this chapter, we let $\theta \in (0,1]$ and we define the *truncated seminorm*:

$$\left(\int_D \left(\int_{B(x,\theta\delta(x))} |u(x)-u(y)|^q K(x,y)\,\mathrm{d}y \right)^{\frac{p}{q}} \mathrm{d}x \right)^{\frac{1}{p}}. \tag{5.1.3}$$

The problem, which we address below in various settings, is the comparability of the full and the truncated seminorm. Should the answer be positive, we call this occurrence a *comparability result.*

Here is our first comparability result. We recall that the uniform domains were defined in Subsection 2.1.3.

Theorem 5.1.1. *Assume that D is a uniform domain and that K satisfies* **B1***,* **B2** *and* **B3** *formulated in Section 5.2 below. Assume that $1 < q \le p < \infty$. Then for every $0 < \theta \le 1$,*

$$\begin{aligned} &\left(\int_D \left(\int_D |u(x)-u(y)|^q K(x,y)\,\mathrm{d}y \right)^{\frac{p}{q}} \mathrm{d}x \right)^{\frac{1}{p}} \\ \approx &\left(\int_D \left(\int_{B(x,\theta\delta(x))} |u(x)-u(y)|^q K(x,y)\,\mathrm{d}y \right)^{\frac{p}{q}} \mathrm{d}x \right)^{\frac{1}{p}}. \end{aligned}$$

The comparability constant depends on p, q, θ, D and the constants in assumptions **B2***,* **B3**.

This is a generalization of one of the results from the work of Prats and Saksman [128, Theorem 1.6], who proved it for the kernels of the form $K(x,y) = |x-y|^{-d-sq}$ for $s \in (0,1)$. We note that the restriction $p \ge q$ was overcome in the recent article by Prats [127]. As it turned out later, the comparability for the classical Triebel–Lizorkin spaces can also be recovered from the much earlier result of Seeger [140, Corollary 2].

In Theorem 5.1.1 we adapt the method of proof from [128] for a wide class of kernels of the form $K(x,y) = |x-y|^{-d}\phi(|x-y|)^{-q}$. The most technical assumption **B2** is tailored for the key Lemmas 5.2.2, 5.2.3, however in Subsection 5.4.2 we argue that it amounts to at least a power-type decay at 0, and for unbounded D at least a power-type growth at ∞, of ϕ.

A comparability result for very similar forms (identical to ours for $p = q = 2$) was established by Dyda [60, (13)] and was used to obtain Hardy inequalities for nonlocal operators. More recent results on the reduction of integration domain in fractional Sobolev spaces include Bux, Kassmann and Schulze [35] who consider certain cones with apex at x instead of $B(x, \delta(x))$, and Chaker and Silvestre [38] who give a more abstract restriction in the vein of the volume density condition, cf. (2.1.1) and see [38, Lemma 2.1]. Independently of the author, Kassmann and Wagner [102] have also proved comparability results which extend the ones from [128], allowing kernels with scaling conditions for $p = q = 2$. These articles provide plenty of applications of the results concerning the reduction of integration domain, see in particular, the references in [38]. Here we are more focused on the very phenomenon of the comparability.

Notably, we go beyond the uniform domains, where the methods used by Prats and Saksman in [128] are no longer available. Namely, in Section 5.6 we prove the following result for the infinite strip-like domains.

Theorem 5.1.2. *Assume that $p = q = 2$. Let $D = \mathbb{R}^k \times (0,1)^l \subseteq \mathbb{R}^{k+l}$ with $k, l > 0$. For $d = k + l$ let $K(x,y) = |x-y|^{-d-\alpha}$ with $\alpha \in (0,2)$. If $k - l - \alpha < -1$, then the seminorms* (5.1.2) *and* (5.1.3) *are comparable.*

We also construct a counterexample for $\alpha < 1$ and $k = l = 1$. This shows an intriguing interplay between the kernel and the width of the domain. Heuristically speaking, it can be seen both in Theorem 5.1.2 and in Subsection 5.4.2 that the comparability holds if the stochastic process corresponding to the jump kernel $K \cdot \mathbf{1}_{D\times D}$ and the shape of the domain D favor small jumps over large jumps. We remark that the connection between the jump kernel and the stochastic process on D is a very delicate matter. In Section 5.7 we present a short discussion of this subject and we place our comparability results in this context. To this end we introduce the basics of the theory of the Dirichlet forms.

Another object of our studies in this chapter is the 0-order kernel $K(x,y) \approx |x-y|^{-d}$, to which we devote Section 5.5. In Example 5.4.7 we show that in general the comparability does not hold for this K, but by establishing suitable counterparts of Lemmas 5.2.2 and 5.2.3, we obtain an estimate of (5.1.2) by a truncated seminorm with a slightly more singular kernel, see Theorem 5.5.1 below.

5.2 Assumptions and key lemmas

We remark that the geometric notions used in this chapter, in particular the uniform domains, were discussed in Subsection 2.1.3. In the sequel we will consider the exponents $1 < q \le p < \infty$ and the assumptions are subordinated to them. As usual, $p' = \frac{p}{p-1}$ is the Hölder conjugate of p. We also let

$$N(r) = \inf\{k \in \mathbb{N} : 2^k r > \operatorname{diam}(D)\}, \quad r > 0.$$

We have $N(r) = \infty$ for every $r > 0$ if and only if D is unbounded.
We assume that the kernel K is of the form $K(x,y) = |x-y|^{-d}\phi(|x-y|)^{-q}$, where $\phi\colon (0,\infty) \to (0,\infty)$ satisfies

B1 $(1 \wedge |y|^q)|y|^{-d}\phi(|y|)^{-q} \in L^1(\mathbb{R}^d)$,

B2 ϕ is increasing and there exists $c_2 > 0$ such that for $t_1 = \min(q, p - \frac{p}{q})$, $t_2 = \frac{1}{q-1}$ and for every $0 < r < \operatorname{diam}(D)$, we have

$$\sum_{k=1}^{N(r)} \frac{\phi(r)^{t_1}}{\phi(2^k r)^{t_1}} \le c_2$$

and

$$\sum_{k=1}^{\infty} \frac{\phi(2^{-k} r)^{t_2}}{\phi(r)^{t_2}} \le c_2.$$

B3 There exists $c_3 \ge 1$ such that for every $0 < r < 3\operatorname{diam}(D)$, we have $\phi(2r) \le c_3\phi(r)$.

In particular, we allow unbounded domains in which the scaling conditions **B2**, **B3** become global. Note that **B1** is a Lévy measure-type condition, which assures the finiteness of (5.1.2) for smooth, compactly supported u. If $q = 2$ and $\phi(r) = r^s$, $s \in (0,1)$, then K corresponds to the fractional Laplacian of order s and all the assumptions are satisfied. The conditions **B2** and **B3** imply certain scaling for K, see Subsection 5.4.2 for the details. The exponents t_1 and t_2 in **B2** stem from the five instances of usage of Lemma 5.2.2 and 5.2.3 in the proof of Theorem 5.1.1. Since ϕ is increasing, the bounds in **B2** hold for all larger exponents in place of t_1 and t_2. We note the following, frequently used below, consequence of **B3** and the monotonicity of ϕ: if $x \lesssim y$, then $\phi(y)^{-1} \lesssim \phi(x)^{-1}$.

Remark 5.2.1. The proofs appearing in this and the following section involve many '$\lesssim$' and '$\gtrsim$' signs. We would like to stress that any comparability for ϕ stems from **B2** and **B3**. In particular, for fixed p, q the constants can be chosen to depend only on the geometry of D (including the dimension) and the constants in **B2** and **B3** wherever ϕ is used.

The next lemma provides some inequalities for the non-centered Hardy–Littlewood maximal operator (denoted by M) with connection to the kernel K. It is inspired by the results of [128, Section 2] and Prats and Tolsa [129, Section 3]. Recall that for the dyadic cubes Q, S we let $D(Q,S) = l(Q) = d(Q,S) + l(S)$, cf. Subsection 2.1.3.

Lemma 5.2.2. *Let D be a domain with Whitney covering $\mathcal{W}$ and let ϕ satisfy* **B1**, **B2** *and* **B3**. *Assume that $g \in L^1_{loc}(\mathbb{R}^d)$ is nonnegative and $0 < r < 3\operatorname{diam}(D)$. For every $\eta \geq \min(q, p - \frac{p}{q})$, $Q \in \mathcal{W}$ and $x \in D$, we have*

$$\int_{D\cap\{|x-y|>r\}} \frac{g(y)\,\mathrm{d}y}{|x-y|^d\phi(|x-y|)^\eta} \lesssim \frac{Mg(x)}{\phi(r)^\eta}, \tag{5.2.1}$$

$$\sum_{S: D(Q,S)>r} \frac{\int_S g(y)\,\mathrm{d}y}{D(Q,S)^d\phi(D(Q,S))^\eta} \lesssim \frac{\inf_{x\in Q} Mg(x)}{\phi(r)^\eta}, \tag{5.2.2}$$

and

$$\sum_{S\in\mathcal{W}} \frac{l(S)^d}{D(Q,S)^d\phi(D(Q,S))^\eta} \lesssim \frac{1}{\phi(l(Q))^\eta}. \tag{5.2.3}$$

Proof. Let us look at (5.2.1). For clarity, assume that $D \ni x = 0$. Since $1/\phi$ is decreasing, we get

$$\begin{aligned}
\int_{D\cap\{|y|>r\}} \frac{\phi(r)^\eta g(y)\,\mathrm{d}y}{|y|^d\phi(|y|)^\eta} &\leq \sum_{k=1}^{N(r)} \int_{2^{k-1}r<|y|<2^k r} \frac{g(y)}{|y|^d}\frac{\phi(r)^\eta}{\phi(|y|)^\eta}\,\mathrm{d}y \\
&\lesssim \sum_{k=1}^{N(r)} \frac{\phi(r)^\eta}{\phi(2^{k-1}r)^\eta}\frac{1}{|B_{2^k r}|}\int_{2^{k-1}r<|y|<2^k r} g(y)\,\mathrm{d}y \\
&\leq \sum_{k=1}^{N(r)} \frac{\phi(r)^\eta}{\phi(2^{k-1}r)^\eta} Mg(0).
\end{aligned}$$

The sum is bounded with respect to r thanks to **B2**. In order to prove (5.2.2) note that if $D(Q,S) > r$, then for every $x \in Q$, $y \in S$, we have $|x-y| + r \lesssim D(Q,S)$. Therefore, by **B3** and the fact that ϕ is increasing, for every $x \in Q$ we have

$$\begin{aligned}
&\sum_{S: D(Q,S)>r} \frac{\phi(r)^\eta \int_S g(y)\,\mathrm{d}y}{D(Q,S)^d\phi(D(Q,S))^\eta} \lesssim \int_D \frac{\phi(r)^\eta g(y)\,\mathrm{d}y}{(|x-y|+r)^d\phi(|x-y|+r)^\eta} \\
&\leq \int_{D\cap\{|x-y|>r\}} \frac{\phi(r)^\eta g(y)\,\mathrm{d}y}{|x-y|^d\phi(|x-y|)^\eta} + \int_{|x-y|<r} \frac{\phi(r)^\eta g(y)\,\mathrm{d}y}{r^d\phi(r)^\eta} \\
&\lesssim \int_{D\cap\{|x-y|>r\}} \frac{\phi(r)^\eta g(y)\,\mathrm{d}y}{|x-y|^d\phi(|x-y|)^\eta} + \frac{1}{|B_r|}\int_{|x-y|<r} g(y)\,\mathrm{d}y.
\end{aligned}$$

The claim follows from the previous estimate. Since the constants in the inequalities do not depend on x, the same holds for the infimum.

Inequality (5.2.3) can be obtained by taking $g \equiv 1$ and $r = l(Q)$ in (5.2.2). In that case $D(Q,S) > r$ for every S, including Q. ☐

The following lemma is an extension of [128, (2.7),(2.8)].

Lemma 5.2.3. *Let* $\eta \geq \min(q, p - \frac{p}{q})$, $\kappa \geq \frac{1}{q-1}$, *assume that* **B2** *and* **B3** *hold, and assume that* $\mathcal{W}$ *is admissible. Then,*

$$\sum_{R:P\in \mathbf{Sh}_\rho(R)} \phi(l(R))^{-\eta} \lesssim \phi(l(P))^{-\eta}. \tag{5.2.4}$$

Furthermore, if $S \in \mathbf{Sh}_\rho(R)$, *then*

$$\sum_{P\in[S,R]} \phi(l(P))^{\kappa} \lesssim \phi(l(R))^{\kappa}. \tag{5.2.5}$$

Proof. Since the cubes are dyadic, we may and do assume in (5.2.4) that $l(P) = 2^{p_0}$ for some $p_0 \in \mathbb{Z}$. Every R which satisfies $P \in \mathbf{Sh}_\rho(R)$ must be at a distance from P smaller than a multiple of $l(R)$, therefore there can only be a bounded number c of such cubes R with a given side length. Furthermore, the considered cubes must be sufficiently large to contain P in its shadow, that is $l(R) \geq 2^{p_0-l_0}$ with $l_0 \in \mathbb{N}_0$ independent of p_0. We also obviously have $l(R) < \operatorname{diam}(D)$. Thus, the sum in the first assertion can be bounded from above as follows:

$$\sum_{R:P\in \mathbf{Sh}_\rho(R)} \phi(l(R))^{-\eta} \leq c \sum_{k=p_0-l_0}^{p_0+N(2^{p_0})} \phi(2^k)^{-\eta} = c \sum_{k=p_0-l_0}^{p_0} \phi(2^k)^{-\eta} + c \sum_{k=p_0+1}^{p_0+N(2^{p_0})} \phi(2^k)^{-\eta}.$$

The sums are estimated by a multiple of $\phi(2^{p_0})^{-\eta}$ using **B3** and **B2** respectively, which proves (5.2.4).

As in the proof of [128, (2.8)] we may deduce that if $S \in \mathbf{Sh}_\rho(R)$, then there is a bounded number c' of cubes $P \in [S, R]$ of a given side length. Furthermore, for every $P \in [S, R]$ we have $l(P) \leq 2^{r_0+l_0}$, where $l(R) = 2^{r_0}$ and l_0 is a fixed natural number independent of S and R. Therefore we estimate (5.2.5) as follows:

$$\sum_{P\in[S,R]} \phi(l(P))^{\kappa} \leq c' \sum_{k=-\infty}^{r_0+l_0} \phi(2^k)^{\kappa} = c' \sum_{k=-\infty}^{r_0} \phi(2^k)^{\kappa} + c' \sum_{k=r_0+1}^{r_0+l_0} \phi(2^k)^{\kappa}.$$

The first sum is bounded from above by a multiple of $\phi(2^{r_0})^{\kappa}$ because of the second assertion of **B2** and the second is handled by using **B3**. This ends the proof. □

5.3 Proof of Theorem 5.1.1

Proof of Theorem 5.1.1. Obviously it suffices to show that the truncated seminorm dominates the full one up to a multiplicative constant.

We will work with dual norms (cf. Hytönen et al. [91, Theorems 13.10 and 13.21]), namely

$$\sup_{\substack{g\geq 0 \\ \|g\|_{L^{p'}(L^{q'}(D))}\leq 1}} \int_D \int_D |u(x) - u(y)||x - y|^{-\frac{d}{q}} \phi(|x - y|)^{-1} g(x, y)\, \mathrm{d}y\mathrm{d}x. \tag{5.3.1}$$

From now on, g will be like in formula (5.3.1).

First let us take care of the case when x and y are close to each other. By the Hölder's inequality, we get

$$\sum_{Q\in\mathcal{W}} \int_Q \int_{2Q} \frac{|u(x) - u(y)| g(x, y)}{|x - y|^{\frac{d}{q}} \phi(|x - y|)} \mathrm{d}y\mathrm{d}x$$

$$\le \sum_{Q\in\mathcal{W}} \int_Q \Big(\int_{2Q} \frac{|u(x)-u(y)|^q}{|x-y|^d \phi(|x-y|)^q} \,\mathrm{d}y \Big)^{\frac{1}{q}} \Big(\int_{2Q} g(x,y)^{q'} \,\mathrm{d}y \Big)^{\frac{1}{q'}} \mathrm{d}x$$
$$\le \Big(\sum_{Q\in\mathcal{W}} \int_Q \Big(\int_{2Q} \frac{|u(x)-u(y)|^q}{|x-y|^d \phi(|x-y|)^q} \,\mathrm{d}y \Big)^{\frac{p}{q}} \mathrm{d}x \Big)^{\frac{1}{p}}.$$

What remains is the integral over $(D\times D)\setminus \bigcup_{Q\in\mathcal{W}} Q\times 2Q = \bigcup_{Q\in\mathcal{W}} Q\times (D\setminus 2Q) = \bigcup_{Q,S\in\mathcal{W}} Q\times (S\setminus 2Q)$. We claim that in this case $|x-y|\approx D(Q,S)$. Indeed, since $y\notin 2Q$, we immediately get $l(Q) \lesssim |x-y|$. Furthermore, if $l(S)\ge l(Q)$ and $|x-y|\le 2l(S)$, then $Q\subseteq 5S$, and by Definition 2.1.12 we get $l(Q)\ge \frac{1}{2} l(S)$ which proves the claim. Therefore, by **B3** we get

$$\sum_{Q,S} \int_Q \int_{S\setminus 2Q} \frac{|u(x)-u(y)| g(x,y)}{|x-y|^{\frac{d}{q}} \phi(|x-y|)} \,\mathrm{d}y\mathrm{d}x$$
$$\lesssim \sum_{Q,S} \int_Q \int_S \frac{|u(x)-u(y)| g(x,y)}{D(Q,S)^{\frac{d}{q}} \phi(D(Q,S))} \,\mathrm{d}y\mathrm{d}x. \tag{5.3.2}$$

Let $u_Q = \frac{1}{|Q|}\int_Q u(x)\,\mathrm{d}x$. By the triangle inequality (5.3.2) does not exceed

$$\sum_{Q,S} \int_Q \int_S \frac{|u(x)-u_Q| g(x,y)}{D(Q,S)^{\frac{d}{q}} \phi(D(Q,S))} \,\mathrm{d}y\mathrm{d}x \tag{A}$$
$$+\sum_{Q,S} \int_Q \int_S \frac{|u_Q-u_{Q_S}| g(x,y)}{D(Q,S)^{\frac{d}{q}} \phi(D(Q,S))} \,\mathrm{d}y\mathrm{d}x \tag{B}$$
$$+\sum_{Q,S} \int_Q \int_S \frac{|u_{Q_S}-u(y)| g(x,y)}{D(Q,S)^{\frac{d}{q}} \phi(D(Q,S))} \,\mathrm{d}y\mathrm{d}x. \tag{C}$$

Recall that Q_S is a fixed central cube in the admissible chain $[Q,S]$. By using Hölder's inequality and (5.2.3) we can estimate (**A**) from above by

$$\sum_{Q} \int_Q |u(x)-u_Q| \Big(\int_D g(x,y)^{q'} \,\mathrm{d}y \Big)^{\frac{1}{q'}} \Big(\sum_S \frac{l(S)^d}{D(Q,S)^d \phi(D(Q,S))^q} \Big)^{\frac{1}{q}} \mathrm{d}x$$
$$\lesssim \sum_{Q} \int_Q |u(x)-u_Q| \Big(\int_D g(x,y)^{q'} \,\mathrm{d}y \Big)^{\frac{1}{q'}} \frac{1}{\phi(l(Q))} \,\mathrm{d}x \tag{5.3.3}$$
$$\lesssim \Big(\sum_{Q} \int_Q \Big(\frac{|u(x)-u_Q|}{\phi(l(Q))} \Big)^p \mathrm{d}x \Big)^{\frac{1}{p}}.$$

Now, by the definition of u_Q, Jensen's inequality and **B3** we get

$$(\mathbf{A}) \lesssim \Big(\sum_{Q} \int_Q \Big(\int_Q \frac{|u(x)-u(y)|^q}{l(Q)^d \phi(l(Q))^q} \,\mathrm{d}y \Big)^{\frac{p}{q}} \mathrm{d}x \Big)^{\frac{1}{p}}$$
$$\lesssim \Big(\sum_{Q} \int_Q \Big(\int_Q \frac{|u(x)-u(y)|^q}{|x-y|^d \phi(|x-y|)^q} \,\mathrm{d}y \Big)^{\frac{p}{q}} \mathrm{d}x \Big)^{\frac{1}{p}}.$$

Let us consider (**B**). If we denote the successor of P in a chain $[Q,S]$ as $\mathcal{N}(P)$, then by the triangle inequality

$$(\mathbf{B}) \le \sum_{Q,S} \Bigg(\int_Q \int_S \frac{g(x,y)}{D(Q,S)^{\frac{d}{q}} \phi(D(Q,S))} \,\mathrm{d}y\mathrm{d}x \sum_{P\in[Q,Q_S)} |u_P - u_{\mathcal{N}(P)}| \Bigg).$$

Recall that $\mathcal{N}(P) \subseteq 5P$ and for every $P \in [Q, Q_S]$, $Q \in \mathbf{Sh}(P)$, cf. Subsection 2.1.3. For such P it is also true that $D(P,S) \approx D(Q,S)$, see [128, (2.6)]. Therefore, by **B3** we estimate (**B**) from above by a multiple of

$$\sum_P \int_P \int_{5P} \frac{|u(\xi)-u(\zeta)|}{|P||5P|} \,\mathrm{d}\xi \mathrm{d}\zeta \sum_{Q\in\mathbf{Sh}(P)} \int_Q \sum_S \int_S \frac{g(x,y)}{D(P,S)^{\frac{d}{q}} \phi(D(P,S))} \,\mathrm{d}y\mathrm{d}x.$$

By the Hölder's inequality and (5.2.3) this expression approximately less than or equal to

$$\sum_P \int_P \int_{5P} \frac{|u(\xi)-u(\zeta)|}{|P||5P|} \,\mathrm{d}\xi \mathrm{d}\zeta \int_{\mathbf{SH}(P)} \Big(\int_D g(x,y)^{q'} \,\mathrm{d}y \Big)^{\frac{1}{q'}} \frac{1}{\phi(l(P))} \,\mathrm{d}x. \tag{5.3.4}$$

Let $G(x) = \big(\int_D g(x,y)^{q'} \,\mathrm{d}y \big)^{\frac{1}{q'}}$. By [128, Lemma 2.7] we have $\int_{\mathbf{SH}(P)} G(x)\,\mathrm{d}x \lesssim \inf_{y\in P} MG(y) l(P)^d$. Using this, the Jensen's inequality, the Hölder's inequality and the fact that the maximal operator is continuous in $L^{p'}$, $p' > 1$, we obtain

$$\begin{aligned}
(\mathbf{B}) &\lesssim \sum_P \frac{1}{|P||5P|} \frac{l(P)^d}{\phi(l(P))} \int_P \int_{5P} |u(\xi)-u(\zeta)| MG(\zeta) \,\mathrm{d}\xi\mathrm{d}\zeta \\
&\lesssim \sum_P \int_P \frac{MG(\zeta)}{l(P)^{\frac{d}{q}} \phi(l(P))} \Big(\int_{5P} |u(\xi)-u(\zeta)|^q \,\mathrm{d}\xi \Big)^{\frac{1}{q}} \mathrm{d}\zeta \\
&\lesssim \Bigg(\sum_P \int_P \Big(\int_{5P} \frac{|u(\xi)-u(\zeta)|^q}{l(P)^d \phi(l(P))^q} \,\mathrm{d}\xi \Big)^{\frac{p}{q}} \mathrm{d}\zeta \Bigg)^{\frac{1}{p}}.
\end{aligned}$$

Since $|\xi - \zeta| \le 5l(P)$, (**B**) is estimated.

Now we will work on (**C**). Since $D(Q,S) \approx l(Q_S)$, by **B3** we obtain

$$(\mathbf{C}) \lesssim \sum_{Q,S} \int_Q \int_S \frac{|u_{Q_S} - u(y)| g(x,y)}{l(Q_S)^{\frac{d}{q}} \phi(l(Q_S))} \,\mathrm{d}y\mathrm{d}x.$$

Furthermore, for every admissible chain we have $Q, S \in \mathbf{Sh}(Q_S)$, therefore for every $Q, S \in \mathcal{W}$ we have

$$(Q_S, Q, S) \in \bigcup_{R\in\mathcal{W}} \{(R, P, P') : P, P' \in \mathbf{Sh}(R)\}.$$

Consequently, the following estimate holds:

$$(\mathbf{C}) \lesssim \sum_{R\in\mathcal{W}} \sum_{Q\in\mathbf{Sh}(R)} \sum_{S\in\mathbf{Sh}(R)} \int_Q \int_S \frac{|u_R - u(y)| g(x,y)}{l(R)^{\frac{d}{q}} \phi(l(R))} \,\mathrm{d}y\mathrm{d}x. \tag{5.3.5}$$

By Hölder's inequality the above expression does not exceed

$$\begin{aligned}
&\sum_{R\in\mathcal{W}} \frac{\Big(\int_{\mathbf{SH}(R)} |u_R - u(y)|^q \,\mathrm{d}y \Big)^{\frac{1}{q}}}{l(R)^{\frac{d}{q}} \phi(l(R))} \int_{\mathbf{SH}(R)} \Bigg(\int_{\mathbf{SH}(R)} g(x,y)^{q'} \,\mathrm{d}y \Bigg)^{\frac{1}{q'}} \mathrm{d}x \\
&\le \sum_{R\in\mathcal{W}} \frac{\Big(\int_{\mathbf{SH}(R)} |u_R - u(y)|^q \,\mathrm{d}y \Big)^{\frac{1}{q}}}{l(R)^{\frac{d}{q}} \phi(l(R))} \int_{\mathbf{SH}(R)} G(x)\,\mathrm{d}x.
\end{aligned}$$

By the last estimate of [128, Lemma 2.7], the fact that $\inf_R MG \le \frac{1}{l(R)^d}\int_R MG$ and the Hölder's inequality we get that

$$(\mathbf{C}) \lesssim \sum_{R\in\mathcal{W}} \frac{1}{l(R)^{\frac{d}{q}}\phi(l(R))}\left(\int_{\mathbf{SH}(R)} |u_R - u(y)|^q \,\mathrm{d}y\right)^{\frac{1}{q}} \int_R MG(\xi)\,\mathrm{d}\xi$$
$$\le \left(\sum_{R\in\mathcal{W}} \int_R \frac{1}{l(R)^{\frac{dp}{q}}\phi(l(R))^p}\Big(\int_{\mathbf{SH}(R)} |u_R - u(y)|^q \,\mathrm{d}y\Big)^{\frac{p}{q}} \mathrm{d}\xi\right)^{\frac{1}{p}} \|MG\|_{L^{p'}(D)}$$
$$\le \left(\sum_{R\in\mathcal{W}} \frac{l(R)^d}{l(R)^{\frac{dp}{q}}\phi(l(R))^p}\Big(\sum_{S\in\mathbf{Sh}(R)}\int_S |u_R - u(y)|^q \,\mathrm{d}y\Big)^{\frac{p}{q}}\right)^{\frac{1}{p}}.$$

Let $[S,R]$ be an admissible chain between S and R. Then, after using the inequality $|u_R - u(y)|^q \lesssim |u_R-u_S|^q + |u_S-u(y)|^q$, we get

$$(\mathbf{C})^p \lesssim \sum_{R\in\mathcal{W}} \frac{l(R)^d}{l(R)^{\frac{dp}{q}}\phi(l(R))^p}\left(\sum_{S\in\mathbf{Sh}(R)}\Big|\sum_{P\in[S,R)} u_P - u_{\mathcal{N}(P)}\Big|^q l(S)^d\right)^{\frac{p}{q}}$$
$$+ \sum_{R\in\mathcal{W}} \frac{l(R)^d}{l(R)^{\frac{dp}{q}}\phi(l(R))^p}\left(\sum_{S\in\mathbf{Sh}(R)}\int_S |u_S - u(y)|^q \,\mathrm{d}y\right)^{\frac{p}{q}} = (\mathbf{C1}) + (\mathbf{C2}).$$

If we write $u_P - u_{\mathcal{N}(P)} = (u_P - u_{\mathcal{N}(P)})\frac{\phi(l(P))^{\frac{1}{q}}}{\phi(l(P))^{\frac{1}{q}}}$, then by Hölder's inequality we estimate $(\mathbf{C1})$ from above by

$$\sum_{R\in\mathcal{W}} \frac{l(R)^d}{l(R)^{\frac{dp}{q}}\phi(l(R))^p}\left(\sum_{S\in\mathbf{Sh}(R)}\sum_{P\in[S,R)} \frac{|u_P - u_{\mathcal{N}(P)}|^q l(S)^d}{\phi(l(P))}\Big(\sum_{P\in[S,R)}\phi(l(P))^{\frac{q'}{q}}\Big)^{\frac{q}{q'}}\right)^{\frac{p}{q}}.$$

By Lemma 5.2.3

$$(\mathbf{C1}) \lesssim \sum_{R\in\mathcal{W}} \frac{l(R)^d}{l(R)^{\frac{dp}{q}}}\phi(l(R))^{\frac{p}{q}-p}\Big(\sum_{S\in\mathbf{Sh}(R)}\sum_{P\in[S,R)} \frac{|u_P - u_{\mathcal{N}(P)}|^q}{\phi(l(P))} l(S)^d\Big)^{\frac{p}{q}}.$$

Let us take ρ_2 large enough for $S \in \mathbf{Sh}^2(P) := \mathbf{Sh}_{\rho_2}(P)$ and $P \in \mathbf{Sh}^2(R)$ to hold. Then $\sum_{S\in\mathbf{Sh}(R)}\sum_{P\in[S,R)} \lesssim \sum_{P\in\mathbf{Sh}^2(R)}\sum_{S\in\mathbf{Sh}^2(P)}$. We denote the sum of the neighbors of P as U_P. Since $\sum_{S\in\mathbf{Sh}^2(P)} l(S)^d \lesssim l(P)^d$, we get that, up to a multiplicative constant, $(\mathbf{C1})$ does not exceed

$$\sum_{R\in\mathcal{W}} \frac{l(R)^d}{l(R)^{\frac{dp}{q}}}\phi(l(R))^{\frac{p}{q}-p}\left(\sum_{P\in\mathbf{Sh}^2(R)} \frac{(l(P)^{-d}\int_{U_P}|u_P - u(\xi)|\,\mathrm{d}\xi)^q}{\phi(l(P))} l(P)^d\right)^{\frac{p}{q}}.$$

Since $p \ge q$, we can use the Hölder's inequality with exponent $\frac{p}{q}$ to estimate this expression from above by

$$\sum_{R\in\mathcal{W}} \frac{l(R)^d}{l(R)^{\frac{dp}{q}}}\phi(l(R))^{\frac{p}{q}-p}\left(\sum_{P\in\mathbf{Sh}^2(R)} \frac{(l(P)^{-d}\int_{U_P}|u_P - u(\xi)|\,\mathrm{d}\xi)^p}{\phi(l(P))^{\frac{p}{q}}} l(P)^d\right)\left(\sum_{P\in\mathbf{Sh}^2(R)} l(P)^d\right)^{(1-\frac{q}{p})\frac{p}{q}}$$

$$\lesssim \sum_{R\in\mathcal{W}} \sum_{P\in\mathbf{Sh}^2(R)} \phi(l(R))^{\frac{p}{q}-p} \frac{(l(P)^{-d}\int_{U_P}|u_P-u(\xi)|\,\mathrm{d}\xi)^p l(P)^d}{\phi(l(P))^{\frac{p}{q}}}$$
$$\lesssim \sum_{P\in\mathcal{W}} \frac{(l(P)^{-d}\int_{U_P}|u_P-u(\xi)|\,\mathrm{d}\xi)^p l(P)^d}{\phi(l(P))^{\frac{p}{q}}} \sum_{R:P\in\mathbf{Sh}^2(R)} \phi(l(R))^{\frac{p}{q}-p}.$$

Furthermore, Lemma 5.2.3 and Jensen's inequality give

$$\begin{aligned}(\mathbf{C1}) &\lesssim \sum_{P\in\mathcal{W}} \frac{(l(P)^{-d}\int_{U_P}|u_P-u(\xi)|\,\mathrm{d}\xi)^p l(P)^d}{\phi(l(P))^p}\\ &\lesssim \sum_{P\in\mathcal{W}} \int_{U_P} \frac{|u_P-u(\xi)|^p}{\phi(l(P))^p}\,\mathrm{d}\xi \\ &\le \sum_{P\in\mathcal{W}} \int_{U_P} \left(\int_P \frac{|u(\zeta)-u(\xi)|^q}{l(P)^d\phi(l(P))^q}\,\mathrm{d}\zeta\right)^{\frac{p}{q}} \mathrm{d}\xi.\end{aligned} \tag{5.3.6}$$

Since $U_P \subseteq 5P$ we have finished estimating (**C1**).

Now we proceed with (**C2**). By Hölder's inequality

$$\begin{aligned}(\mathbf{C2}) &= \sum_{R\in\mathcal{W}} \frac{l(R)^{d(1-\frac{p}{q})}}{\phi(l(R))^p} \left(\sum_{S\in\mathbf{Sh}(R)} \int_S |u_S-u(\xi)|^q\,\mathrm{d}\xi \frac{l(S)^{d(1-\frac{q}{p})}}{l(S)^{d(1-\frac{q}{p})}}\right)^{\frac{p}{q}}\\ &\le \sum_{R\in\mathcal{W}} \frac{l(R)^{d(1-\frac{p}{q})}}{\phi(l(R))^p} \left(\sum_{S\in\mathbf{Sh}(R)} l(S)^d\right)^{\frac{p}{q}-1} \sum_{S\in\mathbf{Sh}(R)} \frac{(\int_S |u_S-u(\xi)|^q\,\mathrm{d}\xi)^{\frac{p}{q}}}{l(S)^{d(\frac{p}{q}-1)}}\\ &\lesssim \sum_{R\in\mathcal{W}} \sum_{S\in\mathbf{Sh}(R)} \frac{(\int_S |u_S-u(\xi)|^q\,\mathrm{d}\xi)^{\frac{p}{q}}}{l(S)^{d(\frac{p}{q}-1)}\phi(l(R))^p}.\end{aligned}$$

By rearranging and using Lemma 5.2.3 we obtain

$$(\mathbf{C2}) \lesssim \sum_{S\in\mathcal{W}} \frac{(\int_S |u_S-u(\xi)|^q\,\mathrm{d}\xi)^{\frac{p}{q}}}{l(S)^{d(\frac{p}{q}-1)}} \sum_{R:S\in\mathbf{Sh}(R)} \phi(l(R))^{-p} \lesssim \sum_{S\in\mathcal{W}} \left(\int_S \frac{|u_S-u(\xi)|^q}{l(S)^d}\,\mathrm{d}\xi\right)^{\frac{p}{q}} \frac{l(S)^d}{\phi(l(S))^p}.$$

Hence, by Jensen's inequality,

$$(\mathbf{C2}) \lesssim \sum_{S\in\mathcal{W}} \frac{l(S)^d}{\phi(l(S))^p} \int_S \frac{|u_S-u(\xi)|^p}{l(S)^d}\,\mathrm{d}\xi = \sum_{S\in\mathcal{W}} \int_S \frac{|u_S-u(\xi)|^p}{\phi(l(S))^p}\,\mathrm{d}\xi.$$

Thus we have arrived at the same situation as in (5.3.6) and the proof is finished (we may need to enlarge the constant $C_{\mathcal{W}}$ for small ρ). □

5.4 Examples of kernels

5.4.1 Positive examples

We will present some examples of kernels which satisfy **B2** and **B3**.

Example 5.4.1. Stable scaling is more than enough for **B2** to hold. Indeed, if we assume that there exist $\beta_1, \beta_2 \in (0,1)$ for which we have

$$\lambda^{\beta_1} \lesssim \frac{\phi(\lambda r)}{\phi(r)} \lesssim \lambda^{\beta_2}, \quad r > 0,\ \lambda \le 1,$$

then by the first inequality we get **B3** and by the second inequality the series in **B2** are geometric and independent of r.

Example 5.4.2. Assume that D is bounded. Let $\gamma \in (0,1)$, $\phi(r) = [\log(1+r)]^\gamma$ and let $R = \mathrm{diam}(D)$. Note that for $r > 0$ we have

$$1 \le \frac{\log(1+2r)}{\log(1+r)} \le 2.$$

Indeed, by looking at the derivative we see that the ratio is decreasing thus the inequalities result from its limits at 0^+ and at ∞. Therefore, ϕ satisfies **B3**. Furthermore for $r < R$ the lower bound can be replaced with a constant $c = c(R) > 1$, hence both series in **B2** become geometric thus it is satisfied.

5.4.2 O-regularly varying functions

Definition 5.4.3. We say that ϕ is O-regularly varying at infinity if there exist $a, b \in \mathbb{R}$ and $A, B, R > 0$ such that

$$A\left(\frac{r_2}{r_1}\right)^a \le \frac{\phi(r_2)}{\phi(r_1)} \le B\left(\frac{r_2}{r_1}\right)^b \tag{5.4.1}$$

holds whenever $R < r_1 < r_2$. Analogously, ϕ is O-regularly varying at zero if (5.4.1) holds for $0 < r_1 < r_2 < R$. The supremum of a and the infimum of b for which (5.4.1) is satisfied are called lower, respectively upper, Matuszewska indexes (or lower/upper indexes).

A nice short review of the O-regularly varying functions can be found in the work of Grzywny and Kwaśnicki [82, Appendix A], for further reading we refer to the book by Bingham, Goldie and Teugels [13].

Assume **B2** and **B3**. We will show that the assumptions enforce O-regular variation with positive lower index at 0, and for unbounded D also at infinity, by using Proposition A.1 of [82]. Note that by **B3** for $r > 0$, $k \in \mathbb{Z}$ and $z \in [2^{k-1}r, 2^k r]$ we have $\phi(z) \approx \phi(2^k r)$.

We first consider the regular variation at zero using [82, Proposition A.1 (c)]. Let $R = \mathrm{diam}(D)$ and $t_2 = \frac{1}{q-1}$. Then, for every $r \in (0,R)$ and $\eta \in \mathbb{R}$ we have

$$\int_0^r z^{-\eta}\phi(z)^{t_2}\frac{\mathrm{d}z}{z} \approx \sum_{k=1}^\infty \phi(2^{-k}r)^{t_2}(2^{-k}r)^{-\eta} = r^{-\eta}\phi(r)^{t_2}\sum_{k=1}^\infty \frac{\phi(2^{-k}r)^{t_2}}{\phi(r)^{t_2}}2^{k\eta}.$$

By **B2** the latter sum is finite for $\eta \le 0$, it is also bounded away from 0 because of **B3**. Therefore we obtain that ϕ^{t_2} (and thus, also ϕ) has to be O-regularly varying at 0 with some lower index $a_0 > 0$, that is

$$\frac{\phi(r_2)}{\phi(r_1)} \gtrsim \left(\frac{r_2}{r_1}\right)^{a_0/t_2}, \quad 0 < r_1 \le r_2 \le R.$$

The above condition yields a power-type decay at 0 for ϕ. This could also be obtained using the other summation condition from **B2** by applying [82, Proposition A.1 (d)]. Note that the obtained condition for ϕ yields (4.2.3) for K.

The behavior of ϕ at infinity only comes into play when D is unbounded, thus we assume that $\mathrm{diam}(D) = \infty$ for the remainder of this subsection. Let $r > 0$, $\eta \in \mathbb{R}$ and $t_1 = \min(q, p - \frac{p}{q})$. We have

$$\int_r^\infty z^{-\eta}\phi(z)^{-t_1}\frac{\mathrm{d}z}{z} \approx \sum_{k=1}^\infty \phi(2^k r)^{-t_1}(2^k r)^{-\eta} = r^{-\eta}\phi(r)^{-t_1}\sum_{k=1}^\infty \frac{\phi(r)^{t_1}}{\phi(2^k r)^{t_1}}2^{-k\eta}.$$

By **B2** and **B3** the sum is finite and bounded away from 0 if $\eta \geq 0$. Thus ϕ^{-t_1} is O-regularly varying at infinity with upper index $-a_\infty < 0$, which is equivalent to the O-regular variation with lower index a_∞ for ϕ^{t_1}:

$$\frac{\phi(r_2)}{\phi(r_1)} \gtrsim \left(\frac{r_2}{r_1}\right)^{a_\infty/t_1}, \quad R < r_1 \leq r_2 < \infty.$$

5.4.3 Negative examples

We will show some examples for which the seminorms (5.1.2) and (5.1.3) are not comparable. Assume for clarity that $p = q = 2$.

Example 5.4.4. Let $D = (0,1) \subset \mathbb{R}$ and let $K(x,y) \equiv 1$. Consider the function $u(x) = x^{-\gamma}$ with $\gamma \in (0, \frac{1}{2})$. A direct calculation shows that

$$\int_0^1\int_0^1 (u(x) - u(y))^2\,\mathrm{d}y\mathrm{d}x = 2\Big(\frac{1}{1-2\gamma} - \frac{1}{(1-\gamma)^2}\Big). \tag{5.4.2}$$

In particular, u belongs to the corresponding Sobolev space (actually the 'Sobolev space' is $L^2(D)$ in this case). Let $\varepsilon \in (0,1)$. We have

$$\begin{aligned}\int_0^1\int_{x-\varepsilon\delta(x)}^{x+\varepsilon\delta(x)} (u(x)-u(y))^2\,\mathrm{d}y\mathrm{d}x &\leq \int_0^1\int_{x(1-\varepsilon)}^{x(1+\varepsilon)} (u(x)-u(y))^2\,\mathrm{d}y\mathrm{d}x \\ =\frac{\varepsilon}{1-\gamma} - \frac{(1+\varepsilon)^{1-\gamma} - (1-\varepsilon)^{1-\gamma}}{(1-\gamma)^2} &+ \frac{(1+\varepsilon)^{1-2\gamma} - (1-\varepsilon)^{1-2\gamma}}{(1-2\gamma)(2-2\gamma)}.\end{aligned} \tag{5.4.3}$$

As $\gamma \to \frac{1}{2}^-$ the ratio of the right-hand side of (5.4.2) and (5.4.3) goes to infinity which shows that in this case the result from Theorem 5.1.1 does not hold.

Example 5.4.5. The preceding example gives an idea on how to show an analogous fact for any nonzero K such that $K(0,\cdot) \in L^1([0,1])$. On the restricted domain of integration we have $x \approx y$. Therefore $|\frac{1}{x^\gamma} - \frac{1}{y^\gamma}| \lesssim \frac{1}{x^\gamma}$, hence

$$\int_0^1\int_{B(x,\varepsilon\delta(x))}\Big(\frac{1}{x^\gamma} - \frac{1}{y^\gamma}\Big)^2 K(x,y)\,\mathrm{d}y\mathrm{d}x \lesssim \int_0^1 \frac{1}{x^{2\gamma}}\int_{B(x,\varepsilon\delta(x))} K(x,y)\,\mathrm{d}y\mathrm{d}x. \tag{5.4.4}$$

On the other hand, since K is nontrivial, there exists $\eta > 0$ such that for every $x \in (0,\eta)$ we have $\int_\eta^1 K(x,y)\,\mathrm{d}y \geq c > 0$. Therefore,

$$\begin{aligned}\int_0^1\int_0^1\Big(\frac{1}{x^\gamma} - \frac{1}{y^\gamma}\Big)^2 K(x,y)\,\mathrm{d}y\mathrm{d}x &\geq \int_0^{\eta/2}\int_\eta^1\Big(\frac{1}{x^\gamma} - \frac{1}{\eta^\gamma}\Big)^2 K(x,y)\,\mathrm{d}y\mathrm{d}x \\ &\gtrsim \int_0^{\eta/2}\frac{1}{x^{2\gamma}}\int_\eta^1 K(x,y)\,\mathrm{d}y\mathrm{d}x\end{aligned}$$

$$\gtrsim \int_0^{\eta/2} \frac{1}{x^{2\gamma}} \, \mathrm{d}x. \tag{5.4.5}$$

Note that the right-hand side of (5.4.4) is of the form $\int_0^1 \frac{u(x)}{x^{2\gamma}} \, \mathrm{d}x$ with $u(x)$ bounded and $\lim_{x \to 0^+} u(x) = 0$. Let us fix an arbitrarily small $\xi > 0$ and let ρ be sufficiently small so that $u(x) \le \xi$ for $x \in (0, \rho)$. If we separate $\int_0^1 = \int_0^\rho + \int_\rho^1$, then we see that the ratio of the right-hand side of (5.4.4) and (5.4.5) tends to 0 as $\gamma \to \frac{1}{2}^-$.

Remark 5.4.6. In previous examples the kernel was integrable. This means that

$$\int_D \int_D (u(x) - u(y))^2 K(x,y) \, \mathrm{d}y \mathrm{d}x \le 2 \int_D \int_D u(x)^2 K(x,y) \, \mathrm{d}y \mathrm{d}x \le 2 \|u\|^2_{L^2(D)} \|K(0,\cdot)\|_{L^1(\mathbb{R}^d)}.$$

Therefore, even though the quadratic forms (5.1.2) and (5.1.3) are incomparable, the Triebel–Lizorkin norm $\| \cdot \|_{F_{p,q}(D)}$ is comparable when we replace the full seminorm with the truncated one.

Example 5.4.7. For $K(x,y) = |x-y|^{-1}$ on $D = (0,1)$ the seminorms also fail to be comparable. Consider the functions $u_n(x) = n \wedge \frac{1}{x}$. Since

$$\int_0^1 \int_0^x (u(x) - u(y))^2 K(x,y) \, \mathrm{d}y \mathrm{d}x = \tfrac{1}{2} \int_0^1 \int_0^1 (u(x) - u(y))^2 K(x,y) \, \mathrm{d}y \mathrm{d}x,$$

we will assume that $y < x$ and work only with the integral on the left hand side. Note that for $u = u_n$, the integral over $(0, \frac{1}{n})^2$ vanishes. Therefore we have

$$\int_0^1 \int_0^x (u_n(x) - u_n(y))^2 K(x,y) \, \mathrm{d}y \mathrm{d}x = \int_{1/n}^1 \int_{1/n}^x \left(\frac{1}{x} - \frac{1}{y} \right)^2 K(x,y) \, \mathrm{d}y \mathrm{d}x \tag{5.4.6}$$

$$+ \int_{1/n}^1 \int_0^{1/n} \left(n - \frac{1}{x} \right)^2 K(x,y) \, \mathrm{d}y \mathrm{d}x. \tag{5.4.7}$$

We first compute the right-hand side of (5.4.6). Note that the integrand is equal to $\frac{(x-y)^2}{y^2 x^2} \cdot \frac{1}{x-y} = \frac{x-y}{y^2 x^2}$, so we get

$$\int_{1/n}^1 \int_{1/n}^x \frac{x-y}{y^2 x^2} \, \mathrm{d}y \mathrm{d}x = \int_{1/n}^1 \int_{1/n}^x \frac{1}{y^2 x} \, \mathrm{d}y \mathrm{d}x - \int_{1/n}^1 \int_{1/n}^x \frac{1}{y x^2} \, \mathrm{d}y \mathrm{d}x = n \log n - 2n + \log n + 2.$$

For (5.4.7) we only show the asymptotics. We have

$$\int_{1/n}^1 \int_0^{1/n} \left(n - \frac{1}{x} \right)^2 K(x,y) \, \mathrm{d}y \mathrm{d}x = \int_{1/n}^1 \left(n - \frac{1}{x} \right)^2 \left(\log x - \log \left(x - \frac{1}{n} \right) \right) \mathrm{d}x$$

$$= - n^2 \int_{1/n}^1 \left(1 - \frac{1}{nx} \right)^2 \log \left(1 - \frac{1}{nx} \right) \mathrm{d}x = -n \int_0^{1-1/n} \frac{t^2}{(1-t)^2} \log t \, \mathrm{d}t.$$

For $n > 2$ we split the latter integral according to $\int_0^{1-1/n} = \int_0^{1/2} + \int_{1/2}^{1-1/n}$. The first one converges, i.e., it is a (negative) constant. In the second one $t^2 \approx 1$ and $\frac{\log t}{1-t} \approx -1$, therefore

$$-n \int_0^{1-1/n} \frac{t^2}{(1-t)^2} \log t \, \mathrm{d}t \approx n \left(1 + \int_{1/2}^{1-1/n} \frac{\mathrm{d}t}{1-t} \right) = n(1 + \log n - \log 2). \tag{5.4.8}$$

Thus we get the desired asymptotics

$$\int_0^1 \int_0^1 (u_n(x) - u_n(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x \approx n \log n. \tag{5.4.9}$$

We next consider the truncated case. For clarity, assume that $\epsilon = \frac{1}{2}$. We have

$$\int_0^1 \int_{x/2}^x (u_n(x) - u_n(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x = \int_{2/n}^1 \int_{x/2}^x \left(\frac{1}{x} - \frac{1}{y}\right)^2 K(x,y)\,\mathrm{d}y\mathrm{d}x \tag{5.4.10}$$

$$+ \int_{1/n}^{2/n} \int_{1/n}^x \left(\frac{1}{x} - \frac{1}{y}\right)^2 K(x,y)\,\mathrm{d}y\mathrm{d}x \tag{5.4.11}$$

$$+ \int_{1/n}^{2/n} \int_{x/2}^{1/n} \left(n - \frac{1}{x}\right)^2 K(x,y)\,\mathrm{d}y\mathrm{d}x. \tag{5.4.12}$$

For the right-hand side of (5.4.10) and (5.4.11) we note that

$$\int_{2/n}^1 \int_{x/2}^x \left(\frac{1}{x} - \frac{1}{y}\right)^2 K(x,y)\,\mathrm{d}y\mathrm{d}x \leq \int_{2/n}^1 \int_{x/2}^x \frac{1}{y^2 x}\,\mathrm{d}y\mathrm{d}x = \frac{n}{2} - 1,$$

and

$$\int_{1/n}^{2/n} \int_{1/n}^x \left(\frac{1}{x} - \frac{1}{y}\right)^2 K(x,y)\,\mathrm{d}y\mathrm{d}x \leq \int_{1/n}^{2/n} \int_{1/n}^x \frac{1}{y^2 x}\,\mathrm{d}y\mathrm{d}x = n\log 2 - \frac{n}{2}.$$

The last integral (5.4.12) is estimated as follows

$$\int_{1/n}^{2/n} \int_{x/2}^{1/n} \left(n - \frac{1}{x}\right)^2 K(x,y)\,\mathrm{d}y\mathrm{d}x = \int_{1/n}^{2/n} \left(n - \frac{1}{x}\right)^2 \left(\log\frac{x}{2} - \log\left(x - \frac{1}{n}\right)\right)\mathrm{d}x$$

$$= -n^2 \int_{1/n}^{2/n} \left(1 - \frac{1}{nx}\right)^2 \left(\log\left(1 - \frac{1}{nx}\right) + \log 2\right)\mathrm{d}x \leq -n \int_0^{1/2} \frac{t^2}{(1-t)^2} \log t\,\mathrm{d}t \approx n.$$

To conclude, we get

$$\int_0^1 \int_{B(x,\delta(x)/2)} (u_n(x) - u_n(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x \lesssim n. \tag{5.4.13}$$

Since the ratio of the right-hand sides of (5.4.9) and (5.4.13) diverges as $n \to \infty$, our claim is proven.

5.5 The 0-order kernel

Even though the comparability may not hold for $K(x,y) = |x-y|^{-d}$, we are able to obtain the following embedding.

Theorem 5.5.1. *Let D be a bounded uniform domain. Then, for every $1 < q \leq p < \infty$ and $0 < \theta \leq 1$ we have*

$$\left(\int_D \left(\int_D \frac{|u(x) - u(y)|^q}{|x-y|^d}\,\mathrm{d}y\right)^{\frac{p}{q}} \mathrm{d}x\right)^{\frac{1}{p}} \tag{5.5.1}$$

$$\lesssim \left(\int_D \left(\int_{B(x,\theta\delta(x))} \frac{|u(x) - u(y)|^q}{|x-y|^d} (|\log|x-y|| \,\vee\, 1)^q\,\mathrm{d}y\right)^{\frac{p}{q}} \mathrm{d}x\right)^{\frac{1}{p}}. \tag{5.5.2}$$

The constant in the inequality depends only on p, q, θ, D.

In order to obtain this result we first prove an analogue of Lemma 5.2.2 for $K(x,y) = |x-y|^{-d}$, i.e. $\phi \equiv 1$. For now every integral is restricted to D by default.

Lemma 5.5.2. *Let D be a bounded domain with Whitney covering $\mathcal{W}$. Assume that $g \in L^1_{loc}(\mathbb{R}^d)$ and $0 < r < \operatorname{diam}(D)$. Then for every $Q \in \mathcal{W}$ and $x \in D$ we have*

$$\int_{|y-x|>r} \frac{g(y)\,\mathrm{d}y}{|y-x|^d} \lesssim Mg(x)(|\log r| \vee 1), \tag{5.5.3}$$

$$\sum_{S:D(Q,S)>r} \frac{\int_S g(y)\,\mathrm{d}y}{D(Q,S)^d} \lesssim \inf_{x\in Q} Mg(x)(|\log r| \vee 1), \tag{5.5.4}$$

and

$$\sum_{S\in\mathcal{W}} \frac{l(S)^d}{D(Q,S)^d} \lesssim |\log l(Q)| \vee 1. \tag{5.5.5}$$

Proof. Let $x \in D$. If we take $R = \operatorname{diam}(D)$, then proceeding as in Lemma 5.2.2 we get

$$\begin{aligned}\int_{|y-x|>r} \frac{g(y)\,\mathrm{d}y}{|y-x|^d} \leq \sum_{k=1}^{\lceil \log_2(R/r)\rceil} \int_{2^{k-1}r\leq|y-x|<2^k r} \frac{g(y)\,\mathrm{d}y}{|x-y|^d} &\lesssim Mg(x)\lceil \log_2(R/r)\rceil \\ &\lesssim Mg(x)(|\log r| \vee 1).\end{aligned}$$

As in the proof of Lemma 5.2.2, in order to prove (5.5.4) we use (5.5.3), and we are left with

$$\int_{|x-y|<r} \frac{g(y)\,\mathrm{d}y}{(|x-y|+r)^d} \lesssim \frac{1}{|B(x,r)|}\int_{B(x,r)} g(y)\,\mathrm{d}y \leq Mg(x)(|\log r| \vee 1).$$

Finally, (5.5.5) is obtained by taking $r = l(Q)$ and $g \equiv 1$. ☐

We will also use the following result similar to Lemma 5.2.3.

Lemma 5.5.3. *Let D be a bounded uniform domain with admissible Whitney decomposition $\mathcal{W}$ and let $\rho > 0$ and $\eta > 1$. Then, for every $S \in \mathcal{W}$ we have*

$$\sum_{R:S\in\mathbf{Sh}_\rho(R)} 1 \lesssim |\log l(S)| \vee 1. \tag{5.5.6}$$

If $S \in \mathbf{Sh}_\rho(R)$, then

$$\sum_{P\in[S,R)} (|\log l(P)| \vee 1)^{-\eta} \lesssim (|\log l(R)| \vee 1)^{1-\eta}. \tag{5.5.7}$$

Furthermore, for every $P \in \mathcal{W}$

$$\sum_{R:P\in\mathbf{Sh}_\rho(R)} (|\log l(R)| \vee 1)^{-\eta} \lesssim 1. \tag{5.5.8}$$

Proof. Throughout the proof we let $l(S) = 2^{s_0}$, $l(R) = 2^{r_0}$, $l(P) = 2^{p_0}$, whenever the cubes are fixed.

Arguing as in the proof of Lemma 5.2.3 we get that there is a limited number of cubes of a given side length contributing to the sum in (5.5.6) and the smallest of these cubes must have side length at least $2^{s_0-l_0}$ for some fixed natural number $l_0 \geq 0$. Therefore, if we let 2^{m_0} be the side length of the largest cube in $\mathcal{W}$, then we have

$$\sum_{R:S\in\mathbf{Sh}_\rho(R)} 1 \lesssim \sum_{k=s_0-l_0}^{m_0} 1 = m_0 - s_0 + l_0 + 1 \approx |\log l(S)| \vee 1.$$

As in Lemma 5.2.3, in (5.5.7) we have limited number of cubes of the same length and the cube length cannot be larger than $2^{r_0+l_0}$ and smaller than $2^{s_0-l_0}$ (l_0 may be different than above, but it does not depend on S and R). Therefore we estimate the sum in (5.5.7) as follows:

$$\sum_{P\in[S,R)} (|\log l(P)| \vee 1)^{-\eta} \lesssim \sum_{k=s_0-l_0}^{r_0+l_0} (|k| \vee 1)^{-\eta} \le \sum_{k=-\infty}^{r_0+l_0} (|k| \vee 1)^{-\eta}.$$

Since $\eta > 1$, the latter series is finite and it is of order $(|r_0| \vee 1)^{1-\eta}$, which proves (5.5.7).

In order to prove (5.5.8) we argue as above in terms of the numbers of the cubes, and because of $\eta > 1$ we get

$$\sum_{R:P\in\mathbf{Sh}_\rho(R)} (|\log l(R)| \vee 1)^{-\eta} \lesssim \sum_{k=p_0-l_0}^{m_0} (|k| \vee 1)^{-\eta} \le \sum_{k=-\infty}^{m_0} (|k| \vee 1)^{-\eta} = c < \infty.$$

□

Proof of Theorem 5.5.1. We proceed as in Theorem 5.1.1 starting with 1 in place of ϕ. The integrals over $Q \times 2Q$ are trivially estimated, because the kernel in (5.5.2) is larger than the one in (5.5.1).

In (**A**) and (**B**) the modification is quite straightforward. Lemma 5.2.2 is used in (5.3.3) and (5.3.4) respectively. Using Lemma 5.5.2 instead, we get respectively $(|\log l(Q)| \vee 1)^{\frac{1}{q}}$ and $(|\log l(P)| \vee 1)^{\frac{1}{q}}$. The remaining arguments are conducted with $(|\log r| \vee 1)^{-\frac{1}{q}}$ in place of $\phi(r)$. Note that $(|\log r| \vee 1)^{-\frac{1}{q}} \approx (|\log 2r| \vee 1)^{-\frac{1}{q}}$. We remark that this yields estimates for (**A**) and (**B**) which are better than the ones in the statement, in fact both expressions are bounded from above by

$$\left(\int_D \left(\int_{B(x,\theta\delta(x))} \frac{|u(x)-u(y)|^q}{|x-y|^d} (|\log|x-y|| \vee 1)\,\mathrm{d}y \right)^{\frac{p}{q}} \mathrm{d}x \right)^{\frac{1}{p}}. \tag{5.5.9}$$

Notice the lack of exponent q in the logarithmic term. At this point we distinguish between the case $p = q$ and $p \neq q$. In the former case the test functions g from (5.3.1) are defined by the condition $\int_D \int_D g(x,y)^{p'}\,\mathrm{d}y\mathrm{d}x \le 1$, therefore (**C**) can be estimated exactly as (**A**) and (**B**) because we can interchange the roles of Q, S and x, y, using Tonelli's theorem. Thus, in this case we in fact obtain an estimate better than postulated, as the whole expression in (5.5.1) is approximately bounded from above by (5.5.9).

For the remainder of the proof we assume that $p > q$. The procedure for (**C**) is also similar to the one in the proof of Theorem 5.1.1, but the computations are slightly different in terms of the exponents, therefore we give the details. There are no essential changes up to the moment of splitting into (**C1**) and (**C2**), thus we make it our starting point. As in the proof of Theorem 5.1.1 we get

$$\begin{aligned}(\mathbf{C2}) &= \sum_{R\in\mathcal{W}} l(R)^{d(1-\frac{p}{q})} \left(\sum_{S\in\mathbf{Sh}(R)} \int_S |u_S - u(\xi)|^q\,\mathrm{d}\xi \frac{l(S)^{d(1-\frac{q}{p})}}{l(S)^{d(1-\frac{q}{p})}} \right)^{\frac{p}{q}} \\ &\lesssim \sum_{R\in\mathcal{W}} \sum_{S\in\mathbf{Sh}(R)} l(S)^{d(1-\frac{p}{q})} \left(\int_S |u_S - u(\xi)|^q\,\mathrm{d}\xi \right)^{\frac{p}{q}}.\end{aligned}$$

We rearrange, use (5.5.6) and then Jensen's inequality twice to obtain:

$$(\mathbf{C2}) \lesssim \sum_{S\in\mathcal{W}} l(S)^{d(1-\frac{p}{q})} \left(\int_S |u_S - u(\xi)|^q\,\mathrm{d}\xi \right)^{\frac{p}{q}} \left(\sum_{R:S\in\mathbf{Sh}(R)} 1 \right)$$

$$\begin{aligned}
&\lesssim \sum_{S\in\mathcal{W}} l(S)^d(|\log l(S)|\vee 1)\Big(\frac{1}{l(S)^d}\int_S |u_S-u(\xi)|^q\,\mathrm{d}\xi\Big)^{\frac{p}{q}}\\
&\le \sum_{S\in\mathcal{W}} (|\log l(S)|\vee 1)\int_S |u_S-u(\xi)|^p\,\mathrm{d}\xi\\
&\le \sum_{S\in\mathcal{W}} \int_S\Big(\int_S \frac{|u(\zeta)-u(\xi)|^q}{l(S)^d}(|\log l(S)|\vee 1)^{\frac{q}{p}}\,\mathrm{d}\zeta\Big)^{\frac{p}{q}}\mathrm{d}\xi,
\end{aligned}$$

and thus (**C2**) is estimated, since $\frac{q}{p}<1<q$.

In order to estimate (**C1**) we write $|u_P-u_{\mathcal{N}(P)}| = |u_P-u_{\mathcal{N}(P)}|\frac{|\log l(P)|\vee 1}{|\log l(P)|\vee 1}$ and we use Hölder's inequality with exponent q and (5.5.7):

$$\begin{aligned}
(\mathbf{C1}) \le &\sum_{R\in\mathcal{W}} l(R)^{d(1-\frac{p}{q})}\Bigg[\sum_{S\in\mathbf{Sh}(R)}\Big(\sum_{P\in[S,R)}|u_P-u_{\mathcal{N}(P)}|^q(|\log l(P)|\vee 1)^q l(S)^d\Big)\\
&\times\Big(\sum_{P\in[S,R)}(|\log l(P)|\vee 1)^{-q'}\Big)^{\frac{q}{q'}}\Bigg]^{\frac{p}{q}}\\
\lesssim &\sum_{R\in\mathcal{W}} l(R)^{d(1-\frac{p}{q})}(|\log l(R)|\vee 1)^{-\frac{p}{q}}\Big(\sum_{S\in\mathbf{Sh}(R)}\sum_{P\in[S,R)}|u_P-u_{\mathcal{N}(P)}|^q(|\log l(P)|\vee 1)^q l(S)^d\Big)^{\frac{p}{q}}.
\end{aligned}$$

By rearranging as in the proof of Theorem 5.1.1 and by using Hölder's and Jensen's inequalities we further estimate (**C1**) from above by a multiple of

$$\begin{aligned}
&\sum_{R\in\mathcal{W}} l(R)^{d(1-\frac{p}{q})}(|\log l(R)|\vee 1)^{-\frac{p}{q}}\\
&\times\Big(\sum_{P\in\mathbf{Sh}^2(R)}\sum_{S\in\mathbf{Sh}^2(P)}\Big(\int_{U_P}\frac{|u_P-u(\xi)|}{l(P)^d}\,\mathrm{d}\xi\Big)^q(|\log l(P)|\vee 1)^q l(S)^d\Big)^{\frac{p}{q}}\\
\lesssim &\sum_{R\in\mathcal{W}} l(R)^{d(1-\frac{p}{q})}(|\log l(R)|\vee 1)^{-\frac{p}{q}}\Big(\sum_{P\in\mathbf{Sh}^2(R)}\Big(\int_{U_P}\frac{|u_P-u(\xi)|}{l(P)^d}\,\mathrm{d}\xi\Big)^q(|\log l(P)|\vee 1)^q l(P)^d\Big)^{\frac{p}{q}}\\
\le &\sum_{R\in\mathcal{W}}\sum_{P\in\mathbf{Sh}^2(R)}(|\log l(R)|\vee 1)^{-\frac{p}{q}}(|\log l(P)|\vee 1)^p\int_{U_P}|u_P-u(\xi)|^p\,\mathrm{d}\xi.
\end{aligned}$$

We rearrange once more and use (5.5.8) (recall that $p>q$) and Jensen's inequality to get that, up to a multiplicative constant, (**C1**) does not exceed

$$\begin{aligned}
&\sum_{P\in\mathcal{W}}(|\log l(P)|\vee 1)^p\int_{U_P}|u_P-u(\xi)|^p\,\mathrm{d}\xi\Big(\sum_{R:P\in\mathbf{Sh}^2(R)}(|\log l(R)|\vee 1)^{-\frac{p}{q}}\Big)\\
\lesssim &\sum_{P\in\mathcal{W}}(|\log l(P)|\vee 1)^p\int_{U_P}|u_P-u(\xi)|^p\,\mathrm{d}\xi\\
\lesssim &\sum_{P\in\mathcal{W}}\int_{U_P}\Big(\int_P\frac{|u(\zeta)-u(\xi)|^q}{l(P)^d}(|\log l(P)|\vee 1)^q\,\mathrm{d}\zeta\Big)^{\frac{p}{q}}\mathrm{d}\xi.
\end{aligned}$$

This finishes the proof. □

Since the kernel in (5.5.2) is significantly larger than the one in (5.5.1), it is plausible that the converse inequality is not true. We will show the existence of a counterexample when $D = (0,1)$, $p = q = 2$. For an open interval $I \subseteq \mathbb{R}$ we let

$$F_0(I) = \left\{u \in L^2(I) : \int_I \int_I \frac{(u(x)-u(y))^2}{|x-y|}\,\mathrm{d}y\mathrm{d}x < \infty\right\},$$

$$F_{\log}(I) = \left\{u \in L^2(I) : \int_I \int_I \frac{(u(x)-u(y))^2}{|x-y|}(|\log|x-y|| \vee 1)\,\mathrm{d}y\mathrm{d}x < \infty\right\}.$$

We note that in $F_{\log}(I)$ the logarithm is in power 1. This suffices for our present purpose, because $q > 1$ in Theorem 5.5.1.

Theorem 5.5.4. *For every $\theta \in (0,1]$, there exists $u \in F_0(0,1) \cap L^\infty(0,1)$ such that*

$$\int_0^1 \int_{B(x,\theta\delta(x))} (u(x)-u(y))^2|x-y|^{-1}(|\log|x-y|| \vee 1)\,\mathrm{d}y\mathrm{d}x = \infty. \tag{5.5.10}$$

Proof.

Step 1. First, note that the finiteness of the left hand side of (5.5.10) implies that $u \in F_{\log}(\frac{n}{2n+1}, \frac{n+1}{2n+1})$ for a sufficiently large $n \in \mathbb{N}$. Indeed, if $\theta \geq \frac{1}{n}$ for some natural number $n \geq 2$, then

$$\int_0^1 \int_{B(x,\theta\delta(x))} (\ldots) \geq \int_0^1 \int_{B(x,\delta(x)/n)} (\ldots) \geq \int_{\frac{n}{2n+1}}^{\frac{n+1}{2n+1}} \int_{B\left(x,\frac{1}{2n+1}\right)} (\ldots) \geq \int_{\frac{n}{2n+1}}^{\frac{n+1}{2n+1}} \int_{\frac{n}{2n+1}}^{\frac{n+1}{2n+1}} (\ldots). \tag{5.5.11}$$

We fix a number n for which (5.5.11) is satisfied.

Step 2. In order to construct the counterexample we will use the asymptotics of the Fourier expansions of functions in $F_0(I)$ and $F_{\log}(I)$. We adopt the following convention for the Fourier coefficients of an integrable function u on an interval (a,b):

$$\widehat{u}(m) = \frac{1}{b-a}\int_a^b u(x)e^{-\frac{2\pi i m x}{b-a}}\,\mathrm{d}x, \quad m \in \mathbb{Z}.$$

Below, by $\widehat{u}(m)$ we mean the Fourier coefficient on $(0,1)$. Let u satisfy $u(x+1) = u(x)$ for $x \in \mathbb{R}$. Let $K(x,y)$ be equal to $|x-y|^{-1}$ (resp. $|x-y|^{-1}(|\log|x-y|| \vee 1)$). We claim that given $u \in L^\infty(0,1)$, it belongs to $F_0(0,1)$ (resp. $F_{\log}(0,1)$) if and only if

$$\int_0^1 \int_0^1 (u(x)-u(x-h))^2 K(0,h)\,\mathrm{d}h\mathrm{d}x < \infty.$$

Indeed, we have

$$\begin{aligned}\int_0^1 \int_0^1 (u(x)-u(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x &= 2\int_0^1 \int_0^x (u(x)-u(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x \\ &= 2\int_0^1 \int_0^x (u(x)-u(x-h))^2 K(0,h)\,\mathrm{d}h\mathrm{d}x.\end{aligned}$$

Therefore, it suffices to verify that $\int_0^1 \int_x^1 (u(x)-u(x-h))^2 K(0,h)\,\mathrm{d}h\mathrm{d}x < \infty$ for bounded u. Clearly we can assume that $K(x,y) = |x-y|^{-1}(|\log|x-y|| \vee 1)$. Then,

$$\int_0^1 \int_x^1 (u(x)-u(x-h))^2 K(0,h)\,\mathrm{d}h\mathrm{d}x \lesssim \int_0^1 \int_x^1 \frac{(-\log h) \vee 1}{h}\,\mathrm{d}h\mathrm{d}x$$

$$= \int_0^{1/e} \int_x^{1/e} \frac{-\log h}{h} \, \mathrm{d}h\mathrm{d}x + \int_0^{1/e} \int_{1/e}^1 \frac{1}{h} \, \mathrm{d}h\mathrm{d}x + \int_{1/e}^1 \int_x^1 \frac{1}{h} \, \mathrm{d}h\mathrm{d}x.$$

All the integrals are finite, therefore the claim is proved.

By Parseval's identity and Tonelli's theorem we get

$$\int_0^1 K(0,h) \int_0^1 (u(x) - u(x-h))^2 \, \mathrm{d}x\mathrm{d}h = \int_0^1 K(0,h) \sum_{m\in\mathbb{Z}} |\widehat{u}(m)|^2 |1 - e^{2\pi i m h}|^2 \, \mathrm{d}h$$
$$= \sum_{m\in\mathbb{Z}} |\widehat{u}(m)|^2 \int_0^1 |1 - e^{2\pi i m h}|^2 K(0,h) \, \mathrm{d}h = 2 \sum_{m\in\mathbb{Z}} |\widehat{u}(m)|^2 \int_0^1 (1 - \cos(2\pi m h)) K(0,h) \, \mathrm{d}h.$$

Now let us inspect the remaining integrals for both cases of K. For $m \neq 0$ we have

$$\int_0^1 \frac{1 - \cos(2\pi m h)}{h} \, \mathrm{d}h = \int_0^{|m|} \frac{1 - \cos(2\pi h)}{h} \, \mathrm{d}h \approx \log|m|.$$

In the logarithmic case

$$\int_0^1 \frac{1 - \cos(2\pi m h)}{h} (-\log h \vee 1) \, \mathrm{d}h = \int_0^{|m|} \frac{1 - \cos(2\pi h)}{h} (-\log \frac{h}{|m|} \vee 1) \, \mathrm{d}h \approx \log^2|m|.$$

To summarize, for bounded functions we can characterize $F_0(0,1)$ by

$$\sum_{m\in\mathbb{Z}, m\neq 0} |\widehat{u}(m)|^2 \log|m| < \infty \tag{5.5.12}$$

and $F_{\log}(0,1)$ by

$$\sum_{m\in\mathbb{Z}, m\neq 0} |\widehat{u}(m)|^2 \log^2|m| < \infty. \tag{5.5.13}$$

The same characterizations hold for $I = (\frac{n}{2n+1}, \frac{n+1}{2n+1})$ and the respective Fourier expansion.

Step 3. We give an example of $u \in F_0(0,1) \cap L^\infty(0,1)$ for which (5.5.12) is satisfied and (5.5.13) is not. For $m = (2n+1)2^l$, $l = 1, 2, \ldots$, we put $\widehat{u}(m) = \frac{1}{l^{3/2}}$. For other m we let $\widehat{u}(m) = 0$. Note that u is $\frac{1}{2n+1}$–periodic. Therefore the j-th Fourier coefficient of u on $(\frac{n}{2n+1}, \frac{n+1}{2n+1})$ is equal to its $(2n+1) \cdot j$-th Fourier coefficient on $(0,1)$. Since $(\widehat{u}(m))_{m\in\mathbb{Z}}$ is summable, u is bounded. Furthermore $l^{-3}\log[(2n+1)2^l] = l^{-2}\log 2 + l^{-3}\log(2n+1)$ and $l^{-3}\log^2(2^l) \approx l^{-1}$. Therefore (5.5.12) is satisfied and (5.5.13) is not. By (5.5.11), the proof is finished.

□

5.6 Comparability in non-uniform domains

In this section we examine the strip $\mathbb{R} \times (0,1)$ which is a non-uniform domain. We will show that the comparability fails for fractional Sobolev spaces with $\alpha < 1$. Then we prove that for $\alpha > 1$ and slightly more general kernels the comparability holds. Later, we present a higher-dimensional case in which the comparability may also hold for $\alpha < 1$ in non-uniform domains. For clarity of the presentation, we assume that $p = q = 2$.

Example 5.6.1. Let $D = \mathbb{R} \times (0,1)$ and let $K(x,y) = |x-y|^{-2-\alpha}$. Recall that D is not uniform, cf. Subsection 2.1.3.

We will show for $\alpha \in (0,1)$ the comparability does not hold. Consider a sequence of functions (u_n) given by the formula $u_n(x_1,x_2) = (1-\frac{|x_1|}{n})\vee 0$. Since u_n are constant on the second variable, for every $\xi \in (0,1)$ we have

$$\int_D\int_D \frac{(u_n(x)-u_n(y))^2}{|x-y|^{2+\alpha}}\,\mathrm{d}y\mathrm{d}x = \int_{\mathbb{R}}\int_{\mathbb{R}} (u_n(x_1,\xi)-u_n(y_1,\xi))^2 \int_0^1\int_0^1 |x-y|^{-2-\alpha}\,\mathrm{d}y_2\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}x_1.$$

Let the integral over $(0,1)\times(0,1)$ be called $\kappa(x_1,y_1)$. We claim that $\kappa(x_1,y_1)$ is comparable with $|x_1-y_1|^{-2-\alpha}$ if $|x_1-y_1|\ge 1$ and with $|x_1-y_1|^{-1-\alpha}$ otherwise. Indeed, we have $|x-y|\approx |x_1-y_1|+|x_2-y_2|$. If $|x_1-y_1|\ge 1$, then

$$\int_0^1\int_0^1 |x-y|^{-2-\alpha}\,\mathrm{d}y_2\mathrm{d}x_2 \approx |x_1-y_1|^{-2-\alpha}\int_0^1\int_0^1 \mathrm{d}y_2\mathrm{d}x_2 = |x_1-y_1|^{-2-\alpha}.$$

For $|x_1-y_1|<1$ note that for fixed $a>0$,

$$\begin{aligned} a^{1+\alpha}&\int_0^1\int_0^1 (a+|x_2-y_2|)^{-2-\alpha}\,\mathrm{d}y_2\mathrm{d}x_2 \approx a^{1+\alpha}\int_0^1\int_0^{x_2}(a+x_2-y_2)^{-2-\alpha}\,\mathrm{d}y_2\mathrm{d}x_2\\ &= \frac{a^{1+\alpha}}{1+\alpha}\int_0^1 (a^{-1-\alpha}-(a+x_2)^{-1-\alpha})\,\mathrm{d}x_2 = \frac{1}{1+\alpha}-\frac{1}{1+\alpha}\int_0^1 (1+\frac{x_2}{a})^{-1-\alpha}\,\mathrm{d}x_2.\end{aligned}$$

For $a=|x_1-y_1|<1$ we have $x_2/a > x_2$, so the latter integral is bounded from above by $c\in(0,1)$. Thus the whole expression is approximately equal to a positive constant which proves our claim.

The shape of D grants that for every $\theta\in(0,1]$ we have

$$\int_D\int_{B(x,\theta\delta(x))} \frac{(u_n(x)-u_n(y))^2}{|x-y|^{2+\alpha}}\,\mathrm{d}y\mathrm{d}x \le \int_{\mathbb{R}}\int_{B(x_1,1)} (u_n(x_1,\xi)-u_n(y_1,\xi))^2\kappa(x_1,y_1)\,\mathrm{d}y_1\mathrm{d}x_1.$$

To simplify the notation we will write $u_n(x_1)=u_n(x_1,\xi)$ for some fixed $\xi\in(0,1)$, $x\in\mathbb{R}$. Since u_n is Lipschitz with constant $1/n$, we have

$$\begin{aligned} &\int_{\mathbb{R}}\int_{B(x_1,1)} (u_n(x_1)-u_n(y_1))^2\kappa(x_1,y_1)\,\mathrm{d}y_1\mathrm{d}x_1\\ \approx&\int_{\mathbb{R}}\int_{B(x_1,1)} (u_n(x_1)-u_n(y_1))^2|x_1-y_1|^{-1-\alpha}\,\mathrm{d}y_1\mathrm{d}x_1\\ =&\int_{-n-1}^{n+1}\int_{B(x_1,1)} (u_n(x_1)-u_n(y_1))^2|x_1-y_1|^{-1-\alpha}\,\mathrm{d}y_1\mathrm{d}x_1\\ \lesssim&\frac{1}{n^2}\int_{-n-1}^{n+1}\int_{B(x_1,1)} |x_1-y_1|^{1-\alpha}\,\mathrm{d}y_1\mathrm{d}x_1 \approx \frac{1}{n}.\end{aligned}$$

Thanks to the fact that $\alpha<1$, the full seminorm is significantly greater as $n\to\infty$:

$$\begin{aligned} &\int_{\mathbb{R}}\int_{\mathbb{R}} (u_n(x_1)-u_n(y_1))^2\kappa(x_1,y_1)\,\mathrm{d}y_1\mathrm{d}x_1 \gtrsim \int_{-\frac{n}{2}}^0\int_{-\infty}^{-n} |x_1-y_1|^{-2-\alpha}\,\mathrm{d}y_1\mathrm{d}x_1\\ =&\int_{-\frac{n}{2}}^0 \frac{1}{1+\alpha}\frac{1}{(x_1+n)^{1+\alpha}}\,\mathrm{d}x_1 \ge \frac{1}{1+\alpha}\frac{n/2}{n^{1+\alpha}} \approx \frac{1}{n^\alpha}.\end{aligned}$$

Lemma 5.6.2. *Let $D=\mathbb{R}\times(0,1)$. If $f\colon \mathbb{R}^2\to[0,\infty)$ is radial, then $\int_D (1\vee|x|)f(x)\,\mathrm{d}x \approx \int_{\mathbb{R}^2} f(x)\,\mathrm{d}x<\infty$ with a constant independent of f.*

Proof. Note that for $n \in \mathbb{N}$ the area of $D \cap (B_n \setminus B_{n-1})$ is comparable to the $1/n$-th of the area of the annulus $B_n \setminus B_{n-1}$. Therefore by the rotational symmetry of f we get

$$\int_D (1 \vee |x|) f(x)\,\mathrm{d}x \approx \sum_{n\in\mathbb{N}} \int_{D\cap(B_n\setminus B_{n-1})} n f(x)\,\mathrm{d}x \approx \sum_{n\in\mathbb{N}} \int_{B_n\setminus B_{n-1}} f(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} f(x)\,\mathrm{d}x.$$

□

The case of $\alpha \in (1,2)$ is included in the following result.

Theorem 5.6.3. *Let $D = \mathbb{R} \times (0,1)$. Assume that K satisfies* **B1**, **B2**, **B3** *and $\sum_{n\geq 1} \int_{B(0,n)^c} K(0,x)\,\mathrm{d}x < \infty$. Then the seminorms* (5.1.2) *and* (5.1.3) *are comparable.*

Proof. We split the domain D into open unit cubes Q_n centered in $(n, 1/2)$, $n \in \mathbb{Z}$, so that we have $D \subseteq \bigcup_{n\in\mathbb{Z}} \overline{Q_n}$. If we let $L_n = \mathrm{Int}[\overline{Q_{n-1} \cup Q_n \cup Q_{n+1}}]$, then L_n is a uniform domain, hence by Theorem 5.1.1

$$\int_{L_n}\int_{L_n} (u(x)-u(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x \approx \int_{L_n}\int_{B(x,\theta\delta(x))} (u(x)-u(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x$$

with the constant independent of n. Therefore for every $0 < \theta \leq 1$,

$$\begin{aligned}\int_D\int_{B(x,\theta\delta(x))} (u(x)-u(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x &\approx \sum_{n\in\mathbb{Z}} \int_{L_n}\int_{L_n} (u(x)-u(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x \\ &\approx \sum_{n\in\mathbb{Z}} \int_{Q_n}\int_{L_n} (u(x)-u(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x, \qquad (5.6.1)\end{aligned}$$

so it suffices to show that the latter expression is comparable with the integral over $D \times D$. We have

$$\int_D\int_D (u(x)-u(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x = \sum_{i,j\in\mathbb{Z}} \int_{Q_i}\int_{Q_j} (u(x)-u(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x$$
$$\approx \sum_{i\in\mathbb{Z}}\sum_{j+1<i} \int_{Q_i}\int_{Q_j} (u(x)-u(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x + \sum_{i\in\mathbb{Z}} \int_{Q_i}\int_{L_i} (u(x)-u(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x.$$

Clearly it suffices to estimate the first summand. Since the cubes are far apart, we have $|x-y| \approx |i-j|$ for $x \in Q_i$, $y \in Q_j$. Hence

$$\begin{aligned}&\sum_{i\in\mathbb{Z}}\sum_{j+1<i} \int_{Q_i}\int_{Q_j} (u(x)-u(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x \\ \lesssim &\sum_{i\in\mathbb{Z}}\sum_{j+1<i} \int_{Q_i}\int_{Q_j} (u(x)-u_{Q_i})^2 K(x,y)\,\mathrm{d}y\mathrm{d}x \\ &+ \sum_{i\in\mathbb{Z}}\sum_{j+1<i} \int_{Q_i}\int_{Q_j} (u(y)-u_{Q_j})^2 K(x,y)\,\mathrm{d}y\mathrm{d}x \\ &+ \sum_{i\in\mathbb{Z}}\sum_{j+1<i}\sum_{j\leq n<i} \int_{Q_i}\int_{Q_j} (u_{Q_{n+1}}-u_{Q_n})^2 |x-y| K(x,y)\,\mathrm{d}y\mathrm{d}x. \qquad (5.6.2)\end{aligned}$$

In this inequality we have used $(a_1 + \ldots + a_m)^2 \leq m(a_1^2 + \ldots + a_m^2)$ and $|Q_i| = |Q_j| = 1$. For the first term we use Jensen's inequality and the fact that the sum over j is uniformly bounded with respect to i and $x \in Q_i$:

$$\sum_{i\in\mathbb{Z}} \int_{Q_i} (u(x)-u_{Q_i})^2 \sum_{j+1<i} \int_{Q_j} K(x,y)\,\mathrm{d}y\mathrm{d}x \lesssim \sum_{i\in\mathbb{Z}} \int_{Q_i}\int_{Q_i} (u(y)-u(x))^2\,\mathrm{d}y\mathrm{d}x.$$

The latter expression does not exceed (5.6.1). The second term can be estimated in a similar way after changing the order of summation.

By Lemma 5.6.2 the additional assumption on K is equivalent to

$$\sum_{n\geq 1}\int_{B(0,n)^c\cap D}|x|K(0,x)\,\mathrm{d}x<\infty.$$

We change the order of summation and use that fact to estimate (5.6.2):

$$\begin{aligned}&\sum_{i\in\mathbb{Z}}\sum_{j+1<i}\sum_{j\leq n<i}(u_{Q_{n+1}}-u_{Q_n})^2\int_{Q_i}\int_{Q_j}|x-y|K(x,y)\,\mathrm{d}y\mathrm{d}x\\ =&\sum_{n\in\mathbb{Z}}(u_{Q_{n+1}}-u_{Q_n})^2\sum_{i>n}\sum_{\substack{j+1<i\\ j\leq n}}\int_{Q_i}\int_{Q_j}|x-y|K(x,y)\,\mathrm{d}y\mathrm{d}x\\ \lesssim&\sum_{n\in\mathbb{Z}}(u_{Q_{n+1}}-u_{Q_n})^2\leq\sum_{n\in\mathbb{Z}}\int_{Q_n}\int_{Q_{n+1}}(u(x)-u(y))^2\,\mathrm{d}y\mathrm{d}x\\ \lesssim&\sum_{n\in\mathbb{Z}}\int_{Q_n}\int_{Q_{n+1}}(u(x)-u(y))^2K(x,y)\,\mathrm{d}y\mathrm{d}x.\end{aligned}$$

□

Proof of Theorem 5.1.2. The idea is similar as above. We split D into a family of unit cubes $(Q_i)_{i\in\mathbb{Z}^k}$ and we let $L_i=\mathrm{Int}\big[\bigcup\{\overline{Q_j}\colon B(x_{Q_i},\sqrt{d})\cap Q_j\neq\emptyset\}\big]$. By Theorem 5.1.1, for $0<\theta\leq 1$ we have

$$\int_D\int_{B(x,\theta\delta(x))}(u(x)-u(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}y\mathrm{d}x\approx\sum_{i\in\mathbb{Z}^k}\int_{Q_i}\int_{L_i}(u(x)-u(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}y\mathrm{d}x.$$

For $i=(i_1,\ldots,i_k),\ j=(j_1,\ldots,j_k)$ and $m\in\mathbb{N}_0$, we say that $j>i+m$ if $j_1>i_1+m,\ldots,j_k>i_k+m$. By $j>m$ we mean $j>0+m$ and $j\geq i+m$ is defined by replacing *all* the inequalities by weak ones. By the radial symmetry of $|x-y|^{-d-\alpha}$ it suffices to show that under our assumptions on l and α we have

$$\sum_{i\in\mathbb{Z}^k}\int_{Q_i}\int_{L_i}(u(x)-u(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}y\mathrm{d}x\gtrsim\sum_{i\in\mathbb{Z}^k}\int_{Q_i}\sum_{j>i+1}\int_{Q_j}(u(x)-u(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}y\mathrm{d}x.$$

In order to perform a decomposition similar to the one which (5.6.2) appears in, we fix a method of communication from Q_i to Q_j, $j>i$: first we move on the coordinate i_1 until we reach j_1, and then we do the same with the next coordinates. The set of indexes of the cubes connecting Q_i and Q_j in the way presented above, with Q_i included and Q_j excluded, will be called $i\to j$. Note that $|i\to j|\approx|i-j|$. Let $\mathcal{N}(Q)$ be the successor of Q on the way from Q_i to Q_j. As before, we have $|i-j|\approx|x-y|$ for $x\in Q_i$, $y\in Q_j$, therefore

$$\begin{aligned}&\sum_{i\in\mathbb{Z}^k}\int_{Q_i}\sum_{j>i+1}\int_{Q_j}(u(x)-u(y))^2|x-y|^{-d-\alpha}\,\mathrm{d}y\mathrm{d}x\\ \lesssim&\sum_{i\in\mathbb{Z}^k}\int_{Q_i}\sum_{j>i+1}\int_{Q_j}(u(x)-u_{Q_i})^2|x-y|^{-d-\alpha}\,\mathrm{d}y\mathrm{d}x\\ &+\sum_{i\in\mathbb{Z}^k}\int_{Q_i}\sum_{j>i+1}\int_{Q_j}(u(y)-u_{Q_j})^2|x-y|^{-d-\alpha}\,\mathrm{d}y\mathrm{d}x\end{aligned}$$

$$+\sum_{i\in\mathbb{Z}^k}\int_{Q_i}\sum_{j>i+1}\int_{Q_j}\sum_{n\in i\to j}(u_{Q_n}-u_{\mathcal{N}(Q_n)})^2|x-y|^{-d-\alpha+1}\,\mathrm{d}y\mathrm{d}x.$$

The first two terms can be handled as in the previous theorem. In the latter we change the order of summation and we get that up to a constant it does not exceed

$$\sum_{n\in\mathbb{Z}^k}\left(\int_{L_n}|u_{Q_n}-u(\xi)|\,\mathrm{d}\xi\right)^2\sum_{j\geq n}\sum_{\substack{i\leq n\\ i+1<j}}\int_{Q_i}\int_{Q_j}|x-y|^{-d-\alpha+1}\,\mathrm{d}y\mathrm{d}x.$$

To finish the proof we note that the double sum over i,j does not depend on n, hence we take $n=(1,\ldots,1)$ (in short, $n=1$) and we estimate as follows:

$$\sum_{j\geq 1}\sum_{\substack{i\leq 1\\ i+1<j}}\int_{Q_i}\int_{Q_j}|x-y|^{-d-\alpha+1}\,\mathrm{d}y\mathrm{d}x\approx\sum_{j\geq 1}\int_{Q_j}\int_{B(y,|j|)^c\cap D}|x-y|^{-d-\alpha+1}\,\mathrm{d}x\mathrm{d}y$$

$$=\sum_{j\geq 1}\int_{Q_j}\sum_{m=0}^{\infty}\int_{(B(0,2^{m+1}|j|)\setminus B(0,2^m|j|))\cap D}|x|^{-d-\alpha+1}\,\mathrm{d}x\mathrm{d}y\approx\sum_{j\geq 1}\int_{Q_j}\sum_{m=0}^{\infty}(2^m|j|)^k(2^m|j|)^{-d-\alpha+1}\,\mathrm{d}y$$

$$\approx\sum_{j\geq 1}|j|^{k-d-\alpha+1}=\sum_{j\geq 1}|j|^{-l-\alpha+1}\approx\sum_{j\in\mathbb{Z}^k\setminus\{0\}}|j|^{-l-\alpha+1},$$

which is finite provided that $k-l-\alpha<-1$. □

5.7 Truncated seminorms as the Dirichlet forms

In this section we indicate how our comparability results can be applied to prove the existence of Markov stochastic processes corresponding to the truncated seminorms (5.1.3). Hereafter we work with the nonlocal Sobolev spaces, i.e. $p=q=2$. We will discuss several cases which depend on various results concerning Sobolev spaces and censored/reflected Markov processes, each with its own assumptions. Therefore we refrain from formulating any *theorems* here, as they would be unnecessarily complicated.

We begin by introducing necessary notions concerning the Dirichlet forms with the aim to make this section self-contained. For further reading we refer to [72, Chapter 1.1].

In this paragraph we slightly abuse the notation by letting $\mathcal{E}$ be a generic quadratic form with domain $D[\mathcal{E}]\subseteq L^2(D)$ for some $D\subseteq\mathbb{R}^d$. Let $\mathcal{E}_1[u]=\mathcal{E}[u]+\|u\|^2_{L^2(D)}$. We say that $(\mathcal{E},D[\mathcal{E}])$ (this pair will also be called form below) is *closed* if, with respect to $\mathcal{E}_1$, every Cauchy sequence has a limit in $D[\mathcal{E}]$. We say that the form is *closable* if it has a closed extension. This is equivalent to the fact that for every sequence (u_n) contained in $D[\mathcal{E}]$, which is Cauchy with respect to $\mathcal{E}$ (the meaning is usual even though $\mathcal{E}$ only yields a pseudometric) and converges to 0 in $L^2(D)$, we have $\mathcal{E}[u_n]\to 0$ as $n\to\infty$. The form $\mathcal{E}$ is *Markovian*, if for every $\varepsilon>0$ there exists a nondecreasing $\phi_\varepsilon\colon\mathbb{R}\to(-\varepsilon,1+\varepsilon)$, Lipschitz with constant 1, such that $\phi_\varepsilon(t)=t$ for $t\in[0,1]$ and, in addition, if $u\in D[\mathcal{E}]$, then $\phi_\varepsilon(u)\in D[\mathcal{E}]$ and $\mathcal{E}[\phi_\varepsilon(u)]\leq\mathcal{E}[u]$. A quadratic form which is both Markovian and closed is called a *Dirichlet form*. The set $\mathcal{C}\subseteq D[\mathcal{E}]\cap C_c(D)$ is a *core* of the form $\mathcal{E}$, if it is dense in $D[\mathcal{E}]$ with respect to $\mathcal{E}_1$ norm and dense in $C_c(D)$ in L^∞ norm. The form $\mathcal{E}$ is *regular* if it possesses a core.

Recall from the Introduction that

$$\mathcal{E}_D^{\mathrm{cen}}[u]=\int_D\int_D(u(x)-u(y))^2K(x,y)\,\mathrm{d}y\mathrm{d}x.$$

Additionally, for $\theta \in (0,1]$ we let

$$\mathcal{E}_D^{\mathrm{red}}[u] = \int_D \int_{B(x,\theta\delta(x))} (u(x)-u(y))^2 K(x,y)\,\mathrm{d}y\mathrm{d}x.$$

The symbol $\mathcal{E}_D^{\mathrm{cen}}$ refers to the censored stable processes introduced by Bogdan, Burdzy and Chen [18]. There, the kernel was that of the fractional Laplacian: $K(x,y) = c|x-y|^{-d-\alpha}$. Censored processes for more general K corresponding to a class of subordinated Brownian motions were studied by Wagner [157].

As an introductory digression we remark that the form of the type $\mathcal{E}_D^{\mathrm{cen}}$ is well-understood if $D = \mathbb{R}^d$ — it is then equal to $\mathcal{E}$ from Subsection 2.3.2. The advantage here is in large due to the fact that $C_c^\infty(\mathbb{R}^d)$ is dense in $F_{2,2}(\mathbb{R}^d)$, see Section A.2. It is also well-known that $C_c^\infty(\mathbb{R}^d)$ is dense in $C_c(\mathbb{R}^d)$ with the supremum norm, hence $C_c^\infty(\mathbb{R}^d)$ is a core for $(\mathcal{E}, F_{2,2}(\mathbb{R}^d))$. The form is closed by Lemma 4.3.3 and to see that it is Markovian it suffices to consider the *unit contraction* $\phi(u) = 0 \vee (u \wedge 1)$. Note that $|\phi(u)(x) - \phi(u)(y)| \le |u(x)-u(y)|$ for every function u and $x, y \in \mathbb{R}^d$. The forms on proper subsets $D \subset \mathbb{R}^d$ are more difficult to handle as we will see shortly.

We will consider $\mathcal{E}_D^{\mathrm{cen}}$ and $\mathcal{E}_D^{\mathrm{red}}$ in two contexts. In the first one, our starting point is the space $C_c^\infty(D)$. Following the arguments of [18, page 93] we will show that $(\mathcal{E}_D^{\mathrm{cen}}, C_c^\infty(D))$ is closable and Markovian for arbitrary Lévy kernel K: in order to prove the closability, we use the L^2 convergence of (u_n) to extract a subsequence which converges a.e. to 0. Then we use Fatou's lemma and the assumption that (u_n) is Cauchy with respect to $\mathcal{E}_D^{\mathrm{cen}}$, as in Lemma 4.3.3. We note that the unit contraction from the previous paragraph is not appropriate for proving that the form is Markovian, because $\phi(u)$ need not be smooth. Instead we note that there exist smooth contractions ϕ_ε with values in $(-\varepsilon, 1+\varepsilon)$, such that $\phi_\varepsilon(t) = t$ for $t \in [0,1]$. Then we have $\phi_\varepsilon(u) \in C_c^\infty(D)$ for all $u \in C_c^\infty(D)$, and the structure of the form immediately yields $\mathcal{E}_D^{\mathrm{cen}}[\phi(u)] \le \mathcal{E}_D^{\mathrm{cen}}[u]$, so $(\mathcal{E}_D^{\mathrm{cen}}, C_c^\infty(D))$ is Markovian. Thus, if we let

$$\mathcal{F} := \text{`Completion of } C_c^\infty(D) \text{ with respect to } (\mathcal{E}_D^{\mathrm{cen}})_1\text{'},$$

then, being the smallest closed extension of $(\mathcal{E}_D^{\mathrm{cen}}, C_c^\infty(D))$, $(\mathcal{E}_D^{\mathrm{cen}}, \mathcal{F})$ is closed and Markovian, that is, a Dirichlet form, see [72, Theorem 3.1.1]. Furthermore, by construction, it is obvious that $C_c^\infty(D)$ is a core for $(\mathcal{E}_D^{\mathrm{cen}}, \mathcal{F})$, hence the form is regular and by [72, Theorem 7.2.1] to every regular Dirichlet form corresponds a Hunt process. Thus, when $\mathcal{E}_D^{\mathrm{cen}}$ is comparable to $\mathcal{E}_D^{\mathrm{red}}$ we obtain the existence of a Hunt process with the Dirichlet form $(\mathcal{E}_D^{\mathrm{red}}, \mathcal{F})$. However, we note that the above arguments may also be used directly with $\mathcal{E}_D^{\mathrm{red}}$, and independently of comparability results, we obtain a regular Dirichlet form $(\mathcal{E}_D^{\mathrm{red}}, \mathcal{F}^{\mathrm{red}})$, where

$$\mathcal{F}^{\mathrm{red}} := \text{`Completion of } C_c^\infty(D) \text{ with respect to } (\mathcal{E}_D^{\mathrm{red}})_1\text{'}.$$

The second approach is by considering the domain corresponding to the so-called active reflected form, that is, $F_{2,2}(D)$, cf. [18]. Here the situation becomes more tedious, since in general $C_c^\infty(D)$ (or even $C_c(D)$) need not be dense in $F_{2,2}(D)$. For some K and D the density holds true, see e.g., [18, Corollary 2.6] and [157, Corollary 2.9]. In that case we get that $\mathcal{F} = F_{2,2}(D)$ and when the comparability holds, we also have $\mathcal{F}^{\mathrm{red}} = F_{2,2}(D)$. Then, the form $(\mathcal{E}_D^{\mathrm{red}}, F_{2,2}(D))$ is a regular Dirichlet form and there exists an associated Hunt process. If the density does not hold, the technical remedy is to change the reference set to $\overline{D}$, cf. [18, Remark 2.1]. By Lemma A.2.1 we have the density of $C_c^\infty(\mathbb{R}^d)$ in $F_{2,2}(\mathbb{R}^d)$, and for sufficiently regular K and D there exist the extension (and trace) operators between $F_{2,2}(D)$ and $F_{2,2}(\mathbb{R}^d)$, see Section 4.5 or [55, 97, 163]. These operators let us use the density result for the whole space in order to prove that $C_c^\infty(\overline{D})$ is dense in $F_{2,2}(D)$ with respect to $(\mathcal{E}_D^{\text{cen}})_1$. Then we obtain the existence of a process on $\overline{D}$ corresponding to the regular Dirichlet form $(\mathcal{E}_D^{\text{cen}}, F_{2,2}(D))$ and the comparability yields the existence of the process corresponding to $(\mathcal{E}_D^{\text{red}}, F_{2,2}(D))$.

The second approach seems more interesting in terms of applying the comparability results, as we build regular Dirichlet forms from the truncated form $\mathcal{E}_D^{\text{red}}$ on the well-established Sobolev/Triebel–Lizorkin space $F_{2,2}(D)$, which is then its natural domain.

Chapter 6

Hardy–Stein and Douglas identities in nonlinear setting

6.1 Introduction

Most of the contents of this chapter may be found in the recent preprint of Bogdan, Grzywny, Pietruska-Pałuba and the author [22]. The proof of Lemma 6.3.1 originally appeared in [21] and Subsection 6.6.3 was not published. In the whole chapter we assume that ν is an infinite, unimodal Lévy measure. We will also stipulate that $\mathbb{P}^x(\tau_D < \infty) = 1$ for $x \in \mathbb{R}^d$, cf. Remark 2.2.4.

In 1931 Douglas [59] established a connection of the energy of the harmonic function u on the unit disc $B(0,1)$ with the 'energy' of its boundary trace g, regarded as a function on $[0, 2\pi)$:

$$\int_{B(0,1)} |\nabla u(x)|^2 \mathrm{d}x = \frac{1}{8\pi} \iint_{[0,2\pi)\times[0,2\pi)} \frac{(g(\eta) - g(\xi))^2}{\sin^2((\eta - \xi)/2)} \mathrm{d}\eta \mathrm{d}\xi. \tag{6.1.1}$$

The formula arose in the study of the so-called Plateau problem — the problem of existence of minimal surfaces posed by Lagrange. It holds true provided that the left-hand side is finite — for details see, e.g., Chen and Fukushima [40, (2.2.60)]. Thus, under the integrability condition, (6.1.1) is valid for the solutions of the Dirichlet problem,

$$\begin{cases} \Delta u = 0 & \text{in } B(0,1), \\ u = g & \text{in } \partial B(0,1). \end{cases}$$

In Theorem 4.2.1 we have announced the following nonlocal variant of the Douglas identity.

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} (P_D[g](x) - P_D[g](y))^2 \nu(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{D^c}\int_{D^c} (g(z) - g(w))^2 \gamma_D(z,w)\,\mathrm{d}z\mathrm{d}w, \tag{6.1.2}$$

where $g\colon D^c \to \mathbb{R}$, $\mathcal{H}_D[g] < \infty$, $u = P_D[g]$ is the Poisson integral of g and γ_D is the interaction kernel defined in Subsection 2.2.2. Noteworthy, $P_D[g]$ is a harmonic function of L, so the identity (6.1.2) explains the energy of a harmonic function by the energy of its external values, see also the discussion following Corollary 4.2.3.

The above 'quadratic' Douglas identity has major consequences for the understanding of the Dirichlet problem for L. As we have mentioned in Chapter 4, it strongly relies on the Hardy–Stein identity given by Bogdan, Dyda and Luks [20], precisely, on its special case for $p = 2$. In fact, [20, (14)] provides a more general version of the Hardy–Stein identity as a tool for characterizing the Hardy spaces H^p. Namely, for $p > 1$, functions u harmonic in $D \subseteq \mathbb{R}^d$, we have

$$\mathbb{E}^x |u(X_{\tau_U})|^p = |u(x)|^p + \int_U G_U(x,y) \int_{\mathbb{R}^d} c_{d,\alpha} \frac{F_p(u(y),u(z))}{|y-z|^{d+\alpha}} \,\mathrm{d}z\mathrm{d}y, \quad U \subset\subset D,\ x \in U. \tag{6.1.3}$$

Here,

$$F_p(a,b) = |b|^p - |a|^p - pa^{\langle p-1\rangle}(b-a), \quad a,b \in \mathbb{R}$$

and $a^{\langle p-1\rangle} = |a|^{p-1}\operatorname{sgn}(a)$. Note that $F_2(a,b) = (b-a)^2$. The formula (6.1.3) motivates the following version of Douglas identity for the case $p \neq 2$, which is one of the main results of this chapter:

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} F_p(P_D[g](x), P_D[g](y))\nu(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{D^c}\int_{D^c} F_p(g(z),g(w))\gamma_D(z,w)\,\mathrm{d}z\mathrm{d}w. \tag{6.1.4}$$

The formula (NDI) is obtained from the above by symmetrization, see (6.2.9) and (6.2.11) below. Similarly to the case $p = 2$, (6.1.4) depends on suitable assumptions on ν and D. A precise statement is given in Theorem 6.4.1, where we also obtain a corresponding result for the trace. Theorem 4.2.1 is then retrieved as a special case, see Remark 6.4.2. The function spaces determined by the finiteness of the form on the left-hand side of (6.1.4) will be called Sobolev–Bregman spaces, see Section 6.2 below. The name is motivated by the fact that F_p is an example of a *Bregman divergence*, see Sprung [147], or Bregman [30] for the original contribution. Apart from the Hardy–Stein identity mentioned above, Bregman divergences are commonly used in the theory of entropy inequalities, see, e.g., Wang [158]. They are also an important tool for statistical learning and its applications, see Nielsen and Nock [123], or Frigyik, Gupta and Srivastava [71] and the references therein. We note that the Sobolev–Bregman spaces admit many characterizations which stem from various numerical inequalities, see (6.2.6) below.

No analogue of (6.1.4) seems to exist in the literature for $p \neq 2$, even for $\Delta^{\alpha/2}$. However, related nonlinear forms $\int u^{\langle p-1\rangle}Lu$, see also (6.2.14) and (6.6.3) below, appear often in the literature concerning Markovian semigroups of operators on L^p spaces. This is because for $p \in (1,\infty)$ the dual space of L^p is $L^{p/(p-1)}$ and for $u \in L^p$ we have $u^{\langle p-1\rangle} \in L^{p/(p-1)}$, and $\int |u|^p = \int |u^{\langle p-1\rangle}|^{p/(p-1)} = \int u^{\langle p-1\rangle}u$. Therefore, in view of the Lumer–Phillips theorem, $u^{\langle p-1\rangle}$ yields a linear functional on L^p appropriate for testing *dissipativity* of generators, see, e.g., Pazy [125, Section 1.4]. In this connection we note that Davies [49, Chapter 2 and 3] gives some fundamental calculations with forms and powers. For the semigroups generated by local operators we refer to Langer and Maz'ya [112] and Sobol and Vogt [146, Theorem 1.1]. Liskevich and Semenov [116] use the L^p setting to analyze perturbations of Markovian semigroups. For nonlocal operators we refer to Farkas, Jacob and Schilling [66, (2.4)], and to the monograph of Jacob [93, (4.294)].

In Section 6.5 we refine the Hardy–Stein and Douglas identities in order to incorporate non-harmonic functions which are sufficiently regular. The augmented Douglas identity in Theorem 6.5.4 gives the energy of a non-harmonic function in terms of the energy of the exterior condition and a fairly explicit remainder term. Thanks to such representation we are able to show that for $p \neq 2$ the harmonic function $P_D[g]$ need not minimize the energy form given by the left-hand side of (6.1.4), contrary to the case $p = 2$, see Example 6.5.5 below.

In Section 6.6 we give, for $p \geq 2$, the following result for Poisson integrals $u = P_D[g]$ and the more usual integral forms based on the p-increments of functions:

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} |u(x)-u(y)|^p\nu(x,y)\,\mathrm{d}x\mathrm{d}y \leq c \iint\limits_{D^c\times D^c} |g(w)-g(z)|^p\gamma_D(w,z)\,\mathrm{d}w\mathrm{d}z\,. \tag{6.1.5}$$

It follows that $g \mapsto P_D[g]$ is an extension operator for nonlocal Sobolev-type spaces $\mathcal{W}_D^p$, defined by the finiteness of the left-hand side. In the remainder of Section 6.6 we compare the spaces $\mathcal{V}_D^p$, $\mathcal{W}_D^p$ and the fractional Sobolev-type spaces studied by Dyda and Kassmann in [62].

6.2 Function F_p and related function spaces

We will use the already announced, convenient notation of the *French power*:

$$x^{\langle\kappa\rangle} := |x|^\kappa \operatorname{sgn}(x) \quad x \in \mathbb{R},\ \kappa > 0.$$

Clearly, the function $x \mapsto x^{\langle\kappa\rangle}$ is antisymmetric: $(-x)^{\langle\kappa\rangle} = -x^{\langle\kappa\rangle}$. Furthermore, we have the following rule for the derivatives:

$$(|x|^\kappa)' = \kappa x^{\langle\kappa-1\rangle} \quad \text{and} \quad (x^{\langle\kappa\rangle})' = \kappa|x|^{\kappa-1}, \quad x \neq 0.$$

Let $p > 1$ and recall that

$$F_p(a,b) = |b|^p - |a|^p - pa^{\langle p-1\rangle}(b-a), \quad a,b \in \mathbb{R}.$$

As the second-order Taylor remainder of the convex function $|x|^p$, F_p is nonnegative. In fact,

$$F_p(a,b) \ \approx\ (b-a)^2(|b| \vee |a|)^{p-2}, \quad a,b \in \mathbb{R}, \tag{6.2.1}$$

see [20, Lemma 6]. In particular, for $p \geq 2$ we have

$$F_p(a,b) \approx (b-a)^2(|a|^{p-2} + |b|^{p-2}), \quad a,b \in \mathbb{R}. \tag{6.2.2}$$

Recall that if X is a random variable with the first moment finite and $a \in \mathbb{R}$, then

$$\mathbb{E}(X-a)^2 = \mathbb{E}(X-\mathbb{E}X)^2 + (\mathbb{E}X-a)^2 = \operatorname{Var} X + (\mathbb{E}X-a)^2. \tag{6.2.3}$$

Here we do not exclude the case $\mathbb{E}X^2 = \infty$, in which case both sides of (6.2.3) are infinite, hence equal. This variance formula has the following analogue for F_p.

Lemma 6.2.1. *Let $p > 1$. Suppose that X is a random variable such that $\mathbb{E}|X| < \infty$. Then,*

(i) $\mathbb{E}F_p(\mathbb{E}X, X) = \mathbb{E}|X|^p - |\mathbb{E}X|^p \geq 0$,

(ii) $\mathbb{E}F_p(a, X) = F_p(a, \mathbb{E}X) + \mathbb{E}F_p(\mathbb{E}X, X) \geq \mathbb{E}F_p(\mathbb{E}X, X), \quad a \in \mathbb{R}$,

(iii) $\mathbb{E}F_p(a, X) = \mathbb{E}F_p(b, X) + F_p(a,b) + (pa^{\langle p-1\rangle} - pb^{\langle p-1\rangle})(b - \mathbb{E}X), \quad a,b \in \mathbb{R}$.

Proof. The verification is elementary, but we present it to emphasize that the finiteness of the first moment suffices. We have

$$\mathbb{E}F_p(\mathbb{E}X, X) = \mathbb{E}\Big[|X|^p - |\mathbb{E}X|^p - p(\mathbb{E}X)^{\langle p-1\rangle}(X - \mathbb{E}X)\Big] = \mathbb{E}|X|^p - |\mathbb{E}X|^p,$$

where $\mathbb{E}|X|^p = \infty$ is permitted, too. The expression in (i) is nonnegative by Jensen's inequality or because F_p is nonnegative. For all $a \in \mathbb{R}$ we have,

$$\begin{aligned}\mathbb{E}F_p(a, X) &= \mathbb{E}\Big[|X|^p - |a|^p - pa^{\langle p-1\rangle}(X-a)\Big]\\ &= \mathbb{E}\Big[|X|^p - |\mathbb{E}X|^p - p(\mathbb{E}X)^{\langle p-1\rangle}(X - \mathbb{E}X)\Big] + |\mathbb{E}X|^p - |a|^p - pa^{\langle p-1\rangle}(\mathbb{E}X - a)\end{aligned}$$

$$= \mathbb{E}F_p(\mathbb{E}X, X) + F_p(a, \mathbb{E}X) \geq \mathbb{E}F_p(\mathbb{E}X, X),$$

as claimed in (ii). Finally, for all $a, b \in \mathbb{R}$ the right-hand side of (iii) is

$$\mathbb{E}|X|^p - |b|^p - pb^{\langle p-1\rangle}(\mathbb{E}X - b) + |b|^p - |a|^p - pa^{\langle p-1\rangle}(b-a) + (pa^{\langle p-1\rangle} - pb^{\langle p-1\rangle})(b - \mathbb{E}X),$$

which simplifies to the left-hand side of (iii). Needless to say, (ii) is a special case of (iii). □

We next propose a simple lemma concerning the p-th moments of random variables, which is another generalization of (6.2.3).

Lemma 6.2.2. *For every $p \geq 1$ there exist constants $0 < c_p \leq \widetilde{c}_p$ such that for every random variable X with $\mathbb{E}|X| < \infty$ and every number $a \in \mathbb{R}$,*

$$c_p\left(\mathbb{E}|X - \mathbb{E}X|^p + |\mathbb{E}X - a|^p\right) \leq \mathbb{E}|X-a|^p \leq \widetilde{c}_p\left(\mathbb{E}|X - \mathbb{E}X|^p + |\mathbb{E}X - a|^p\right). \tag{6.2.4}$$

Proof. If $\mathbb{E}|X|^p = \infty$, then all the sides of (6.2.4) are infinite. Otherwise, by convexity,

$$\mathbb{E}|X-a|^p = \mathbb{E}|(X - \mathbb{E}X) + (\mathbb{E}X - a)|^p \leq 2^{p-1}\left(\mathbb{E}|X - \mathbb{E}X|^p + |\mathbb{E}X - a|^p\right).$$

For the lower bound we make two observations: $|\mathbb{E}X - a|^p \leq \mathbb{E}|X-a|^p$ (Jensen's inequality), and

$$\mathbb{E}|X - \mathbb{E}X|^p = \mathbb{E}|(X-a) - (\mathbb{E}X - a)|^p \leq 2^{p-1}\left(\mathbb{E}|X-a|^p + |\mathbb{E}X - a|^p\right) \leq 2^p\mathbb{E}|X-a|^p.$$

Adding the two, we get that $|\mathbb{E}X - a|^p + \mathbb{E}|X - \mathbb{E}X|^p \leq (1+2^p)\mathbb{E}|X-a|^p$. □

The function $F_p(a,b)$ is not symmetric in a, b, but the right-hand side of (6.2.1) is, so it is natural to consider the symmetrized version of F_p, given by the formula:

$$H_p(a,b) = \tfrac{1}{2}(F_p(a,b) + F_p(b,a)) = \tfrac{p}{2}(b^{\langle p-1\rangle} - a^{\langle p-1\rangle})(b-a), \quad a, b \in \mathbb{R}. \tag{6.2.5}$$

We can relate H_p to a 'quadratic' expression as follows.

Lemma 6.2.3. *For every $p > 1$ we have $F_p(a,b) \approx H_p(a,b) \approx (b^{\langle p/2\rangle} - a^{\langle p/2\rangle})^2$.*

Proof. The first comparison follows from (6.2.1): we have $F_p(a,b) \approx F_p(b,a)$, hence $F_p \approx H_p$. As for the second statement, if either a or b are equal to 0, then the expressions coincide up to constants depending on p. If $a, b \neq 0$, then $a = tb$ with $t \neq 0$. Using this representation we see that the second comparison is equivalent to the following:

$$(t^{\langle p-1\rangle} - 1)(t-1) \approx (t^{\langle p/2\rangle} - 1)^2, \quad t \in \mathbb{R}.$$

The latter holds because both sides are continuous and positive except at $t = 1$; at infinity both are power functions with the leading term $|t|^p$, and at $t = 1$ their ratio converges to a positive constant. □

Summarizing, by (6.2.1) and Lemma 6.2.3 for each $p \in (1, \infty)$ we have

$$F_p(a,b) \approx H_p(a,b) \approx (b-a)^2(|b| \vee |a|)^{p-2} \approx (b^{\langle p/2\rangle} - a^{\langle p/2\rangle})^2, \quad a, b \in \mathbb{R}. \tag{6.2.6}$$

It is hard to trace down the first occurrence of such comparisons in the literature. The one-sided inequality $|b^{p/2} - a^{p/2}|^2 \leq \frac{p^2}{4(p-1)}(b-a)(b^{p-1} - a^{p-1})$ for $a, b \geq 0$ can be found in connection with logarithmic Sobolev inequalities, e.g., in Davies [49, (2.2.9)] for $2 < p < \infty$, and Bakry [9, p. 39] for $p > 1$. The opposite inequality $(b-a)(b^{p-1}-a^{p-1}) \le (b^{p/2}-a^{p/2})^2$ with $a, b > 0$ and $p > 1$ appears, e.g., in [116, Lemma 2.1].

In fact the following inequalities hold for all $p \in (1,\infty)$ and $a, b \in \mathbb{R}$:

$$\frac{4(p-1)}{p^2}(b^{\langle p/2\rangle} - a^{\langle p/2\rangle})^2 \le (b-a)(b^{\langle p-1\rangle} - a^{\langle p-1\rangle}) \le 2(b^{\langle p/2\rangle} - a^{\langle p/2\rangle})^2. \tag{6.2.7}$$

Indeed, if a and b have opposite signs then it is enough to consider $b = t \ge 1$ and $a = -1$, and to compare $(t+1)(t^{p-1}+1) = t^p + t^{p-1} + t + 1$ with $(t^{p/2}+1)^2 = t^p + 2t^{p/2} + 1$. We have $t^{p/2} = \sqrt{t^{p-1}t} \le (t^{p-1}+t)/2$, which verifies the left-hand side inequality in (6.2.7) with constant 1, which is better than $4(p-1)/p^2$. We further get the right-hand side inequality in (6.2.7), and the constant 2 suffices, because $t^{p-1} + t - (t^p + 1) = (1-t)(t^{p-1}-1) \le 0$. Note that the constant 2 is not optimal for individual values of p, e.g., for $p = 2$, but the constant 1 does not suffice for $p \in (1,2) \cup (2,\infty)$ because then $1 \vee (p-1) > p/2$, and so $t^{p-1} + t > 2t^{p/2}$ for large t.

If a and b have the same sign, then we may assume $b = ta$, $a > 0$, $t \ge 1$, and consider the quotient

$$H(t) = \frac{(t^{p-1}-1)(t-1)}{(t^{p/2}-1)^2} = 1 - \frac{t(t^{(p-2)/2}-1)^2}{(t^{p/2}-1)^2} = 1 - h(s)^2,$$

where $s = \sqrt{t}$, $h(s) = s(s^{p-2}-1)/(s^p-1)$. We see that $h(s)$ is strictly positive for $p > 2$, $s > 1$ and negative for $p \in (1,2)$. We claim that it decreases in the former case and increases in the latter. The sign of the derivative of h is the same as the sign of the function $l(s) = -s^{2p-2} + (p-1)s^p - (p-1)s^{p-2} + 1$. Now, since $l(1) = 0$, the sign of l on $(1,\infty)$ is in turn equal to the sign of $l'(s) = (p-1)s^{p-3}(-2s^p + ps^2 - (p-2))$, and further equal to the sign of $-2p(s^{p-1} - s)$. Since the last function is negative on $(1,\infty)$ if $p > 2$ and positive for $p \in (1,2)$, the claim is proved. Consequently, the function $s \mapsto h(s)^2$ is decreasing on $(1,\infty)$, so we get

$$\lim_{t\to 1^+} H(t) = \frac{4(p-1)}{p^2} < H(t) < 1, \quad t > 1,$$

and (6.2.7) follows. The above also shows that the constant $4(p-1)/p^2$ in (6.2.7) cannot be improved.

We would like to note that for $p \ne 2$, $F_p(a+t, b+t)$ is *not* comparable with $F_p(a,b)$. Indeed, for $a, r > 0$ one has $F_p(a, a+r) \approx r^2(a \vee (a+r))^{p-2} = r^2(a+r)^{p-2}$, which is not comparable with $F_p(0,r) = r^2$ for large values of a. The following lemma compares $F_p(a,b)$ with the more usual p-increments and is rather well-known, see, e.g., Zeidler [162, p. 503].

Lemma 6.2.4. *If $p \ge 2$ then $F_p(a,b) \gtrsim |b-a|^p$, and if $1 < p \le 2$, then $|b-a|^p \gtrsim F_p(a,b)$.*

Proof. If $a = b$, then the inequalities are trivial, so assume that $a \ne b$ and consider the quotient

$$\frac{F_p(a,b)}{|b-a|^p} \approx \frac{(|a| \vee |b|)^{p-2}}{|b-a|^{p-2}}.$$

Both parts of the statements now follow from the inequality $|b-a|^r \le 2^r(|a| \vee |b|)^r$, $r > 0$. □

In analogy to the form $\mathcal{E}_D$ given in (2.3.10), for $u \colon \mathbb{R}^d \to \mathbb{R}$ we define

$$\mathcal{E}_D^{(p)}[u] := \frac{1}{p} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} F_p(u(x), u(y))\nu(x,y)\,\mathrm{d}x\mathrm{d}y. \tag{6.2.8}$$

By the symmetry of ν and (6.2.5),

$$\begin{aligned}\mathcal{E}_D^{(p)}[u] &= \tfrac{1}{p} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} H_p(u(x),u(y))\nu(x,y)\,\mathrm{d}x\mathrm{d}y \\ &= \tfrac{1}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} (u(y)^{\langle p-1\rangle} - u(x)^{\langle p-1\rangle})(u(y)-u(x))\nu(x,y)\,\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{6.2.9}$$

Of course, $\mathcal{E}_D^{(2)} = \mathcal{E}_D$. For $D = \mathbb{R}^d$ we have

$$\mathcal{E}_{\mathbb{R}^d}^{(p)}[u] = \tfrac{1}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} (u(y)^{\langle p-1\rangle} - u(x)^{\langle p-1\rangle})(u(y)-u(x))\nu(x,y)\,\mathrm{d}x\mathrm{d}y. \tag{6.2.10}$$

Clearly, for $p=2$ we retrieve $\mathcal{E}$ — the classical Dirichlet form of the operator L.

Let $g\colon D^c \to \mathbb{R}$. With (4.1.2) in mind, we use the following form to quantify the increments of g:

$$\begin{aligned}\mathcal{H}_D^{(p)}[g] &= \tfrac{1}{p} \iint\limits_{D^c\times D^c} F_p(g(w),g(z))\gamma_D(w,z)\,\mathrm{d}w\mathrm{d}z = \tfrac{1}{p} \iint\limits_{D^c\times D^c} H_p(g(w),g(z))\gamma_D(w,z)\,\mathrm{d}w\mathrm{d}z \\ &= \tfrac{1}{2} \iint\limits_{D^c\times D^c} (g(z)^{\langle p-1\rangle} - g(w)^{\langle p-1\rangle})(g(z)-g(w))\gamma_D(w,z)\,\mathrm{d}w\mathrm{d}z. \end{aligned} \tag{6.2.11}$$

The corresponding function spaces are given by

$$\mathcal{V}_D^p := \{u\colon \mathbb{R}^d \to \mathbb{R} \mid \mathcal{E}_D^{(p)}[u] < \infty\}, \tag{6.2.12}$$

and

$$\mathcal{X}_D^p := \{g\colon D^c \to \mathbb{R} \mid \mathcal{H}_D^{(p)}[g] < \infty\}. \tag{6.2.13}$$

We call them Sobolev–Bregman spaces, since they involve the Bregman divergence. In view of (6.2.9) for all $u\colon \mathbb{R}^d \to \mathbb{R}$ we have

$$\mathcal{E}_D^{(p)}[u] = \mathcal{E}_D(u^{\langle p-1\rangle}, u), \tag{6.2.14}$$

where

$$\mathcal{E}_D(v,u) := \tfrac{1}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} (v(x)-v(y))(u(x)-u(y))\nu(x,y)\,\mathrm{d}x\mathrm{d}y,$$

if the integral is well defined, which is the case in (6.2.14) for $v = u^{\langle p-1\rangle}$. For clarity we also note that by (6.2.7), (6.2.9) and (6.2.11), we have the *comparisons*

$$\mathcal{E}_D^{(p)}[u] \approx \mathcal{E}_D[u^{\langle p/2\rangle}], \tag{6.2.15}$$

and

$$\mathcal{H}_D^{(p)}[g] \approx \mathcal{H}_D[g^{\langle p/2\rangle}], \tag{6.2.16}$$

for all $u\colon \mathbb{R}^d \to \mathbb{R}$ and $g\colon D^c \to \mathbb{R}$ with the comparability constants depending only on p. Below, however, we focus on genuine *equalities*.

6.3 The Hardy–Stein identity

The main goal of this section is to prove the Hardy–Stein identity for the harmonic functions. To this end, we will often use the results of Subsection 4.4.1, which we now briefly recall. The L-harmonic functions are understood as the functions which satisfy the mean value property as in Definition 4.4.1. We have shown that if u is L-harmonic in D, then $u \in L^1_{loc}(\mathbb{R}^d) \cap C^2(D)$, $Lu(x)$ can be computed pointwise for $x \in D$ as in (2.3.1) and $Lu(x) = 0$ for $x \in D$. We also recall that the Harnack inequality holds for L-harmonic functions, see Grzywny and Kwaśnicki [82, Theorem 1.9]; the assumptions of that theorem follow from the assumption **A2** given in Section 4.2.

We first prove the following Dynkin-type lemma (in which u need not be harmonic). Recall that L is given in (2.3.1) in the positive definite version. This will result in many negative signs below.

Lemma 6.3.1. *Assume that* (4.2.2) *of* **A2** *holds, that is, $\nu(r+1) \approx \nu(r)$ for $r > 1$. Let $U \subset\subset D$ be open and Lipschitz. Let $u \in C^2(\overline{U})$ and $\int_{\mathbb{R}^d} |u(y)|(1 \wedge \nu(y))\,\mathrm{d}y < \infty$. Then Lu is bounded on $\overline{U}$ and for every $x \in \mathbb{R}^d$,*

$$\mathbb{E}^x u(X_{\tau_U}) - u(x) = -\int_U G_U(x,y) Lu(y)\,\mathrm{d}y. \tag{6.3.1}$$

Proof. Both sides of (6.3.1) are equal to zero for $x \notin U$, so let $x \in \overline{U}$. To prove that $Lu(x)$ is bounded on $\overline{U}$ we choose $\varepsilon > 0$ so small that u is C^2 on $\overline{U} + B_{2\varepsilon}$ (recall that $B_r = B(0,r)$). In particular u and its second-order partial derivatives D^2u are bounded on $\overline{U} + B_\varepsilon$. As usual by Taylor's formula,

$$\begin{aligned}
|Lu(x)| &= \left| \tfrac{1}{2} \int_{\mathbb{R}^d} (2u(x) - u(x+y) - u(x-y))\nu(y)\,\mathrm{d}y \right| \\
&\le \tfrac{1}{2} \sup_{\substack{\xi \in \overline{U}+B_\epsilon \\ |\beta|=2}} |\partial^\beta u(\xi)| \int_{B_\epsilon} |y|^2 \nu(y)\,\mathrm{d}y + \tfrac{1}{2}\int_{B_\epsilon^c} |(2u(x) - u(x+y) - u(x-y)|\, \nu(y)\,\mathrm{d}y \\
&\le c_\epsilon \int_{B_\epsilon} |y|^2 \nu(y)\,\mathrm{d}y + |u(x)|\nu(B_\epsilon^c) + \int_{B_\epsilon^c} |u(x+y)|\nu(y)\,\mathrm{d}y.
\end{aligned}$$

We only need to estimate the last integral. Let $R = \varepsilon + \sup_{x\in U} |x|$. Then,

$$\begin{aligned}
\int_{B_\epsilon^c} |u(x+y)|\nu(y)\,\mathrm{d}y &= \int_{B(x,\epsilon)^c} |u(z)|\nu(x,z)\,\mathrm{d}z \\
&= \int_{B(x,\epsilon)^c \cap B_{2R}} |u(z)|\nu(x,z)\,\mathrm{d}z + \int_{B(x,\epsilon)^c \cap B_{2R}^c} |u(z)|\nu(x,z)\,\mathrm{d}z.
\end{aligned} \tag{6.3.2}$$

The first integral in (6.3.2) does not exceed $\nu(\epsilon) \int_{B_{2R}} |u(z)|\,\mathrm{d}z < \infty$. For the second integral we note that $x \in \overline{U}$, $z \notin B_{2R}$, imply $|z - x| \ge |z| - |x| \ge |z| - R$. From the assumption $\nu(r) \approx \nu(r+1)$, we get that there is $c_R > 0$ such that $\nu(z,x) \le c_R \nu(z)$ and so the integral is bounded by $c_R \int_{B_{2R}^c} |u(z)|\nu(z)\,\mathrm{d}z < \infty$.

Collecting all the bounds together we see that

$$\begin{aligned}
|Lu(x)| \le c_\epsilon &\int_{\mathbb{R}^d} (|y|^2 \wedge 1)\nu(y)\,\mathrm{d}y + 2\|u\|_{L^\infty(U)} \nu(B_\epsilon^c) \\
&+ \nu(\epsilon) \int_{B_{2R}} |u(z)|\,\mathrm{d}z + c_R \int_{B_{2R}^c} |u(z)|\nu(z)\,\mathrm{d}z < \infty.
\end{aligned}$$

For the second part of the statement, let $u = \varphi + h$, where $\varphi \in C_c^2(\mathbb{R}^d)$ and $h = 0$ in a neighborhood of $\overline{U}$. Note that $h \in C^2(\overline{U})$. We claim that (6.3.1) holds with φ and h in place of u. Recall that $-L$ coincides with the generator of the semigroup of X_t for functions from $C_c^2(\mathbb{R}^d)$. Therefore, by Dynkin's formula [64, (5.8)],

$$\mathbb{E}^x\varphi(X_{\tau_U}) - \varphi(x) = -\mathbb{E}^x \int_0^{\tau_U} L\varphi(X_t)\,\mathrm{d}t = -\int_0^\infty \mathbb{E}^x\left[L\varphi(X_t); \tau_U > t\right]\mathrm{d}t. \tag{6.3.3}$$

Here the change of the order of integration is justified because $L\varphi$ is bounded on $\overline{U}$ and $\mathbb{E}^x\tau_U \le \mathbb{E}^x\tau_{B_R} < \infty$ for sufficiently large $R > 0$, cf. [131, Theorem 1, (3.1)]. As in (2.2.5), we let p^U denote the transition density of the process killed upon leaving U. Since $L\varphi$ is measurable and bounded on $\overline{U}$, for every $t > 0$ we have

$$\mathbb{E}^x\left[L\varphi(X_t); \tau_U > t\right] = \int_U p_t^U(x,y)L\varphi(y)\,\mathrm{d}y.$$

Therefore,

$$\mathbb{E}^x\varphi(X_{\tau_U}) - \varphi(x) = -\int_0^\infty \int_U p_t^U(x,y)L\varphi(y)\,\mathrm{d}y\mathrm{d}t = -\int_U G_U(x,y)L\varphi(y)\,\mathrm{d}y,$$

which proves the claim for φ.

Let $x \in U$. By the Ikeda–Watanabe formula (2.2.12) we have

$$\begin{aligned}\mathbb{E}^x h(X_{\tau_U}) - h(x) &= \int_{U^c} h(z)P_U(x,z)\,\mathrm{d}z = \int_{U^c} h(z)\int_U G_U(x,y)\nu(y,z)\,\mathrm{d}y\mathrm{d}z\\ &= \int_U G_U(x,y)\int_{U^c} h(z)\nu(y,z)\,\mathrm{d}z\mathrm{d}y = \int_U G_U(x,y)\int_{\mathbb{R}^d}(h(z)) - h(y))\nu(y,z)\,\mathrm{d}z\mathrm{d}y\\ &= -\int_U G_U(x,y)Lh(y)\,\mathrm{d}y.\end{aligned}$$

The use of Fubini's theorem is justified as follows: for $z \in \operatorname{supp} h$ we have $|h(z)|P_U(x,z) \approx |h(z)|\nu(x,z)$ (cf. (2.2.17)), which is integrable by $\nu(r) \approx \nu(r+1)$ for $r > 1$ and the integrability assumption on u. □

The following Hardy–Stein formula extends [20, Lemma 8] and [21, Lemma 4.12], where it was proved, for the fractional Laplacian and $p > 1$, and for unimodal operators L and $p = 2$, respectively.

Proposition 6.3.2. *Assume that* **A1** *and* **A2** *hold. If* $u\colon \mathbb{R}^d \to \mathbb{R}$ *is* L*-harmonic in* D, $p > 1$ *and* $U \subset\subset D$ *is open Lipschitz, then*

$$\mathbb{E}^x|u(X_{\tau_U})|^p = |u(x)|^p + \int_U G_U(x,y)\int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y, \quad x \in U. \tag{6.3.4}$$

Proof. As a guideline, the result follows by taking $\phi = |u|^p$ in the Dynkin formula (6.3.1). We combine the methods of [20] and [21]. By Theorem 4.4.9 if u is harmonic in D, then $u \in C^2(D)$. Thus, in particular, $|u|^p$ is bounded in a neighborhood of $\overline{U}$. Let $x \in U$. Consider the complementary cases:

$$\text{(i)} \int_{U^c} |u(z)|^p\nu(x,z)\,\mathrm{d}z = \infty, \quad \text{or} \quad \text{(ii)} \int_{U^c} |u(z)|^p\nu(x,z)\,\mathrm{d}z < \infty.$$

Since $|u|^p$ is bounded in a neighborhood of $\overline{U}$, by using (2.2.17) we may reformulate this dichotomy as

$$\text{(i)} \quad \mathbb{E}^x|u(X_{\tau_U})|^p = \infty, \quad \text{or} \quad \text{(ii)} \quad \mathbb{E}^x|u(X_{\tau_U})|^p < \infty,$$

In case (i), we show that the right-hand side of (6.3.4) is infinite as well. Assume first that $|u| > 0$ on a subset of U of positive measure. Pick $y \in U$ satisfying $|u(y)| > 0$ and let $A = \{z \in U^c : |u(z)| \geq (2 + \sqrt{2})|u(y)|\}$. Now, since $x, y \in U$ are fixed and ν is positive, continuous and satisfies (4.2.2) of **A2**, we have $\nu(x,z) \approx \nu(y,z)$ for $z \in U^c$. Therefore, by (i),

$$\int_{U^c} |u(z)|^p \nu(y,z)\,\mathrm{d}z = \infty$$

as well. Furthermore,

$$\int_{U^c \setminus A} |u(z)|^p \nu(y,z)\,\mathrm{d}z \approx \int_{U^c \setminus A} |u(z)|^p \nu(x,z)\,\mathrm{d}z \leq (2+\sqrt{2})^p |u(y)|^p \nu(x,U^c) < \infty,$$

and consequently we must have

$$\int_A |u(z)|^p \nu(y,z)\,\mathrm{d}z = \infty.$$

By the definition of A, for $z \in A$ we have

$$(u(z) - u(y))^2 \geq \frac{1}{2} u(z)^2 \quad \text{and} \quad |u(z)| \geq |u(y)|. \tag{6.3.5}$$

By (6.2.1) and (6.3.5) we therefore obtain

$$\begin{aligned}\int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z &\approx \int_{\mathbb{R}^d} (u(z)-u(y))^2 (|u(y)| \vee |u(z)|)^{p-2} \nu(y,z)\,\mathrm{d}z \\ &\geq \int_A (u(z)-u(y))^2 |u(z)|^{p-2} \nu(y,z)\,\mathrm{d}z \geq \tfrac{1}{2} \int_A |u(z)|^p \nu(y,z)\,\mathrm{d}z = \infty.\end{aligned}$$

This is true for all points y in a set of positive Lebesgue measure, which proves that the right-hand side of (6.3.4) is infinite. If, on the other hand, $u \equiv 0$ in U, then $F_p(u(y),u(z)) = c|u(z)|^p$ for all $z \in \mathbb{R}^d$, $y \in U$, and by (i) the right-hand side of (6.3.4) is infinite again.

We now consider the case (ii). Then $\mathbb{E}^x |u(X_{\tau_U})|^p < \infty$ and the integrability condition of Lemma 6.3.1 is satisfied for $\phi = |u|^p$. We will first prove (6.3.4) for $p \geq 2$. Then ϕ is of class C^2 on D, so we are in a position to use Lemma 6.3.1 and we get

$$\mathbb{E}^x |u(X_{\tau_U})|^p = |u(x)|^p - \int_U G_U(x,y) L|u|^p(y)\,\mathrm{d}y, \quad x \in U. \tag{6.3.6}$$

The integral on the right-hand side is absolutely convergent. Furthermore, since u is L-harmonic,

$$\begin{aligned} L|u|^p(y) &= L|u|^p(y) - p u(y)^{\langle p-1 \rangle} Lu(y) \\ &= -\lim_{\epsilon \to 0+} \int_{|z-y|>\epsilon} (|u(z)|^p - |u(y)|^p - p u(y)^{\langle p-1\rangle}(u(z)-u(y)))\nu(y,z)\,\mathrm{d}z \\ &= -\int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z \leq 0. \end{aligned}$$

Inserting this to (6.3.6) gives the statement.

When $p \in (1,2)$, the function $\mathbb{R} \ni r \mapsto |r|^p$ is not twice differentiable, so the above argument needs to be modified. We work under the assumption (ii) and we follow the proof of [20, Lemma 3]. Consider $\varepsilon \in \mathbb{R}$ and the function $\mathbb{R}^d \ni x \mapsto (x^2 + \varepsilon^2)^{p/2}$. Let

$$F_p^{(\varepsilon)}(a,b) = (b^2+\varepsilon^2)^{p/2} - (a^2+\varepsilon^2)^{p/2} - pa(a^2+\varepsilon^2)^{(p-2)/2}(b-a)\,, \quad a,b \in \mathbb{R}. \tag{6.3.7}$$

Since $1 < p < 2$, by [20, Lemma 6],

$$0 \le F_p^{(\varepsilon)}(a,b) \le \frac{1}{p-1} F_p(a,b)\,, \qquad \varepsilon, a, b \in \mathbb{R}\,, \tag{6.3.8}$$

Let $\varepsilon > 0$. We note that $(u^2+\varepsilon^2)^{p/2} \in C^2(D)$. Also, the integrability condition in Lemma 6.3.1 is satisfied for $\phi = (u^2+\varepsilon^2)^{p/2}$ since it is satisfied for $\phi = |u|^p$ by (ii), and

$$(u^2+\varepsilon^2)^{p/2} \le (|u|+\varepsilon)^p \le 2^{p-1}(|u|^p + \varepsilon^p), \tag{6.3.9}$$

Furthermore, $\mathbb{E}_x(u(X_{\tau_U})^2+\varepsilon^2)^{p/2} < \infty$. As in the first part of the proof,

$$\begin{aligned} L(u^2+\varepsilon^2)^{p/2}(y) &= L(u^2+\varepsilon^2)^{p/2}(y) - pu(y)(u(y)^2+\varepsilon^2)^{(p-2)/2} Lu(y) \\ &= -\int_{\mathbb{R}^d} F_p^{(\varepsilon)}(u(y),u(z))\nu(y,z)\,\mathrm{d}z, \end{aligned} \tag{6.3.10}$$

therefore by Lemma 6.3.1,

$$\mathbb{E}_x(u(X_{\tau_U})^2+\varepsilon^2)^{p/2} = (u(x)^2+\varepsilon^2)^{p/2} + \int_U G_U(x,y) \int_{\mathbb{R}^d} F_p^{(\varepsilon)}(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y. \tag{6.3.11}$$

From the dominated convergence theorem the left-hand side of (6.3.11) goes to $\mathbb{E}_x|u(X_{\tau_U})|^p < \infty$ as $\varepsilon \to 0^+$. Of course, $F_p^{(\varepsilon)}(a,b) \to F_p(a,b)$ as $\varepsilon \to 0^+$. Furthermore, by Fatou's lemma and (6.3.11),

$$\begin{aligned} \int_U G_U(x,y) \int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y &\le \liminf_{\varepsilon\to 0^+} \int_U G_U(x,y) \int_{\mathbb{R}^d} F_p^{(\varepsilon)}(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y \\ &= \mathbb{E}_x|u(X_{\tau_U})|^p - |u(x)|^p < \infty. \end{aligned}$$

By (6.3.8) and the dominated convergence theorem, we obtain (6.3.4) for $p \in (1,2)$. □

As a consequence, we obtain the the Hardy–Stein identity for D, generalizing and strengthening [20, (16)] and [21, Theorem 2.1].

Proposition 6.3.3. *Let $p > 1$ be given and assume that* **A1** *and* **A2** *holds. If u is L-harmonic in D and $x \in D$, then*

$$\sup_{x\in U\subset\subset D} \mathbb{E}^x|u(X_{\tau_U})|^p = |u(x)|^p + \int_D G_D(x,y) \int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y. \tag{6.3.12}$$

If u is regular L-harmonic in D, then the left-hand side can be replaced with $\mathbb{E}^x|u(X_{\tau_D})|^p$.

Proof. As we have noted in Remark 4.4.4, $\{u(X_{\tau_U}), U \subset D\}$ is a martingale ordered by the inclusion of open subsets of D. By domain monotonicity of the Green function, cf. the last paragraph of Subsection 2.2.2, and the nonnegativity of F_p, both sides of (6.3.4) increase if U increases. Since every open set $U \subset\subset D$ is included in an open Lipschitz set $U \subset\subset D$, the supremum in (6.3.12) may be taken over open Lipschitz sets $U \subset\subset D$. The first part of the statement follows from the monotone convergence theorem.

If additionally u is regular harmonic, then

$$\mathbb{E}^x|u(X_{\tau_D})|^p = |u(x)|^p + \int_D G_D(x,y) \int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y. \tag{6.3.13}$$

This is delicate. Indeed, by Remark 4.4.4, the martingale $\{u(X_{\tau_U}), U \subset\subset D\}$ is closed by the integrable random variable $u(X_{\tau_D})$. Therefore Lévy's Martingale Convergence Theorem yields that $u(X_{\tau_U})$ converges almost surely and in L^1 to a random variable Z, as $U \uparrow D$, and we have $Z = \mathbb{E}^x[u(X_{\tau_D})|\sigma(\bigcup_{U\subset\subset D}\mathcal{F}_{\tau_U})]$, see, e.g., Dellacherie and Meyer [54, Theorem 31 a,b, p. 26]. We claim that the σ-algebra $\sigma(\bigcup_{U\subset\subset D}\mathcal{F}_{\tau_U})$ is equal to $\mathcal{F}_{\tau_D}$. Indeed, by Proposition 25.20 (i),(ii), and Proposition 25.19 (i),(ii) in Kallenberg [100, p. 501], the filtration of (X_t) is quasi-left continuous. Therefore τ_U increases to τ_D as U increases to D (cf. [16, proof of Lemma 17]), and our claim follows from Dellacherie and Meyer [53, Theorem 83, p. 136]. Consequently, $Z = u(X_{\tau_D})$. Now, if $\sup_{x\in U\subset\subset D}\mathbb{E}^x|u(X_{\tau_U})|^p < \infty$, then [54, Theorem 31 c, p. 26] yields (6.3.13). Else, if the supremum is infinite, then $\mathbb{E}^x|u(X_{\tau_D})|^p = \infty$ by Jensen's inequality, hence (6.3.13) holds, too. □

6.4 The Douglas identity

We now present our main theorem. It is a counterpart of Theorem 4.2.1 with square increments of the function replaced by 'increments' measured in terms F_p or H_p.

Theorem 6.4.1 (Douglas identity). *Let $p > 1$. Assume that the Lévy measure ν satisfies* **A1** *and* **A2***, $D \subset \mathbb{R}^d$ is open, D^c satisfies VDC, $|\partial D| = 0$ and $\mathbb{P}^x(\tau_D < \infty) < 1$ for all $x \in \mathbb{R}^d$.*

(i) Let $g\colon D^c \to \mathbb{R}$ be such that $\mathcal{H}_D^{(p)}[g] < \infty$. Then $P_D[g]$ is well-defined and satisfies

$$\mathcal{H}_D^{(p)}[g] = \mathcal{E}_D^{(p)}[P_D[g]]. \tag{6.4.1}$$

(ii) Furthermore, if $u\colon \mathbb{R}^d \to \mathbb{R}$ satisfies $\mathcal{E}_D^{(p)}[u] < \infty$, then $\mathcal{H}_D^{(p)}[u|_{D^c}] < \infty$.

Here, as usual, $u|_{D^c}$ is the restriction of u to D^c, but in what follows we will abbreviate:

$$\mathcal{H}_D^{(p)}[u] := \mathcal{H}_D^{(p)}[u|_{D^c}]$$

and

$$P_D[u|_{D^c}] = P_D[u].$$

Remark 6.4.2. We note that (ii) of Theorem 6.4.1 is slightly weaker than the counterpart in Theorem 4.2.1. Indeed, here we do not have $\mathcal{E}_D^{(p)}[u] \geq \mathcal{H}_D^{(p)}[u]$, and as we will see in Example 6.5.5 it is in general impossible to prove such relation for $p \neq 2$. However, the inequality for $p = 2$ is obtained in the proof, which is given at the end of this section, therefore the proof of Theorem 6.4.1 yields Theorem 4.2.1.

Recall the space $\mathcal{V}_D^p$, defined in (6.2.12), which is a natural domain of $\mathcal{E}_D^{(p)}$, and the space $\mathcal{X}_D^p$, defined in (6.2.13), which is a natural domain of $\mathcal{H}_D^{(p)}$. From Theorem 6.4.1 we immediately obtain the following trace and extension result in the nonquadratic setting.

Corollary 6.4.3. *Let* $\operatorname{Ext} g = P_D[g]$*, the Poisson extension, and* $\operatorname{Tr} u = u|_{D^c}$*, the restriction to D^c. Then* $\operatorname{Ext}\colon \mathcal{X}_D^p \to \mathcal{V}_D^p$, $\operatorname{Tr}\colon \mathcal{V}_D^p \to \mathcal{X}_D^p$, *and* $\operatorname{Tr}\operatorname{Ext}$ *is the identity operator on $\mathcal{X}_D^p$.*

We next give the Douglas identity for the Poisson extension on D and the form $\mathcal{E}_{\mathbb{R}^d}^{(p)}$, the nonquadratic analogue of Corollary 4.2.3.

Corollary 6.4.4. *If $P_D[|g|] < \infty$ on D, in particular if $\mathcal{H}_D^{(p)}[g] < \infty$, then*

$$\mathcal{E}_{\mathbb{R}^d}^{(p)}[P_D[g]] = \frac{1}{p}\iint\limits_{D^c\times D^c} F_p(g(z), g(w))(\gamma_D(z,w) + \nu(z,w))\,\mathrm{d}z\mathrm{d}w.$$

The proof of Theorem 6.4.1 uses the following lemma, which asserts that the condition $\mathcal{H}_D^{(p)}[g] < \infty$ implies the finiteness of $P_D[|g|^p]$ and $P_D[|g|]$ on D. The analogue for $p = 2$ was given in Lemma 4.4.6, the present case is slightly more involved.

Lemma 6.4.5. *Suppose that $g\colon D^c \to \mathbb{R}$ satisfies $\mathcal{H}_D^{(p)}[g] < \infty$. Then for every $x \in D$ we have $\int_{D^c} |g(z)|^p P_D(x,z)\,\mathrm{d}z < \infty$. In particular, the Poisson integral of g is well-defined.*

Proof. Denote $I = \int_{D^c} |g(z)|^p P_D(x,z)\,\mathrm{d}z$. If $\mathcal{H}_D^{(p)}[g] < \infty$, then

$$\begin{aligned}
&\iint\limits_{D^c\times D^c} F_p(g(w),g(z))\gamma_D(w,z)\,\mathrm{d}w\mathrm{d}z \\
&= \int_D \int_{D^c}\int_{D^c} F_p(g(w),g(z))\nu(w,x)P_D(x,z)\,\mathrm{d}z\mathrm{d}w\mathrm{d}x < \infty. \qquad (6.4.2)
\end{aligned}$$

Since $\nu > 0$, for almost all (hence for some) pairs $(w,x) \in D^c \times D$ we get

$$\int_{D^c} F_p(g(w),g(z))P_D(x,z)\,\mathrm{d}z < \infty. \qquad (6.4.3)$$

For the remainder of the proof, we only consider pairs (w,x) satisfying the above condition.

We will use different approaches for $p \ge 2$ and $p \in (1,2)$. Let $p \ge 2$. From (6.2.2) we obtain

$$A := \int_{D^c} (g(z)-g(w))^2|g(z)|^{p-2}P_D(x,z)\,\mathrm{d}z < \infty.$$

For $z \in D^c$, let $g_n(z) = -n \vee g(z) \wedge n$. Clearly $|g_n(z)| \le |g(z)|$ and $|g_n(z)| \nearrow |g(z)|$ when $n \to \infty$. Since $|g_n(z)| \le n$, the integral $I_n := \int_{D^c} |g_n(z)|^p P_D(x,z)\,\mathrm{d}z$ is finite. It is also true that the increments of g_n do not exceed those of g, that is $|g_n(z) - g_n(w)| \le |g(z) - g(w)|$. Consequently,

$$\begin{aligned}
I_n &= \int_{D^c} g_n(z)^2|g_n(z)|^{p-2}P_D(x,z)\,\mathrm{d}z \\
&\le 2\int_{D^c}(g_n(z)-g_n(w))^2|g_n(z)|^{p-2}P_D(x,z) + 2g_n(w)^2\int_{D^c}|g_n(z)|^{p-2}P_D(x,z)\,\mathrm{d}z \\
&\le A + 2g(w)^2\left(\int_{D^c}|g_n(z)|^pP_D(x,z)\,\mathrm{d}z\right)^{\frac{p-2}{p}}.
\end{aligned}$$

The last inequality is obvious for $p = 2$ and follows from Jensen's inequality if $p > 2$. Thus,

$$I_n \le A + 2g(w)^2(I_n)^{1-\frac{2}{p}}, \qquad (6.4.4)$$

hence the sequence (I_n) is bounded. By the monotone convergence theorem, $I < \infty$. By Jensen's inequality we also get $\int_{D^c} |g(z)|P_D(x,z)\,\mathrm{d}z < \infty$. By the Harnack inequality, the finiteness of the Poisson integral of $|g|$ or $|g|^p$ at any point $x \in D$ guarantees its finiteness at every point of D, cf. Lemma 4.4.6, therefore the proof is finished for $p \ge 2$.

Now let $p \in (1,2)$. If $g \equiv 0$ a.e. on D^c, then the statement is trivial. Otherwise, pick $w \in D^c$ such that $0 < |g(w)| < \infty$. Let $B = \{z \in D^c : |g(z)| > |g(w)|\}$. We have

$$\int_{D^c\setminus B} |g(z)|^pP_D(x,z)\,\mathrm{d}z \le |g(w)|^p < \infty.$$

Using (6.2.1) and (6.4.3) we get

$$\int_B |g(z)|^pP_D(x,z)\,\mathrm{d}z = \int_B g(z)^2|g(z)|^{p-2}P_D(x,z)\,\mathrm{d}z$$

$$\leq 2\int_B (g(z)-g(w))^2|g(z)|^{p-2}P_D(x,z)\,\mathrm{d}z + 2g(w)^2\int_B |g(z)|^{p-2}P_D(x,z)\,\mathrm{d}z$$
$$\approx \int_B F_p(g(w),g(z))P_D(x,z)\,\mathrm{d}z + 2|g(w)|^p < \infty.$$

Thus, $P_D[|g|^p](x) < \infty$. The rest of the proof is the same as in the case $p \geq 2$. □

Proof of Theorem 6.4.1. To prove (i) we let $\mathcal{H}_D^{(p)}[g] < \infty$ and we have (6.4.2). Let $u = P_D[g]$. By Lemma 6.4.5, u is well-defined and it is regular L-harmonic in D, that is $\mathbb{E}^x[u(X_{\tau_D})] = u(x)$ for $x \in D$, cf. Definition 4.4.1 and Corollary 2.2.3. In particular, we have $\mathbb{E}^x|u(X_{\tau_D})| < \infty$.

For $x \in D$ consider the integral $\int_{D^c} F_p(u(w),u(z))P_D(x,z)\,\mathrm{d}z$. Since $P_D(x,z)$ is the density of the distribution of X_{τ_D} under $\mathbb{P}^x$, we get

$$\int_{D^c} F_p(u(w),u(z))P_D(x,z)\,\mathrm{d}z = \mathbb{E}^x[F_p(u(w),u(X_{\tau_D}))].$$

By Lemma 6.2.1 (ii) applied to $a = u(w)$, $X = u(X_{\tau_D})$ and $\mathbb{E} = \mathbb{E}^x$, the above expression is equal to

$$F_p(u(w),\mathbb{E}^x u(X_{\tau_D})) + \mathbb{E}^x F_p(u(x),u(X_{\tau_D})) = F_p(u(w),u(x)) + \mathbb{E}^x F_p(u(x),u(X_{\tau_D})). \quad (6.4.5)$$

By integrating the first term on the right-hand side of (6.4.5) against $\nu(x,w)\,\mathrm{d}x\mathrm{d}w$ we obtain

$$\iint\limits_{D^c\times D} F_p(u(w),u(x))\nu(x,w)\,\mathrm{d}x\mathrm{d}w. \quad (6.4.6)$$

For the second term in (6.4.5) we use Lemma 6.2.1 (i) and Proposition 6.3.3:

$$\begin{aligned}\mathbb{E}^x F_p(u(x),u(X_{\tau_D})) &= \mathbb{E}^x|u(X_{\tau_D})|^p - |u(x)|^p\\ &= \int_D G_D(x,y)\int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y.\end{aligned}$$

We integrate the latter expression against $\nu(x,w)\,\mathrm{d}x\mathrm{d}w$. By Fubini–Tonelli, the Ikeda–Watanabe formula (2.2.12) and Corollary 2.2.3,

$$\begin{aligned}&\int_{D^c}\int_D\int_D\int_{\mathbb{R}^d} G_D(x,y)F_p(u(y),u(z))\nu(y,z)\nu(x,w)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x\mathrm{d}w\\ &= \int_D\int_{\mathbb{R}^d} F_p(u(y),u(z))\Big(\int_{D^c}\Big(\int_D G_D(x,y)\nu(x,w)\,\mathrm{d}x\Big)\mathrm{d}w\Big)\nu(y,z)\,\mathrm{d}z\mathrm{d}y\\ &= \int_D\int_{\mathbb{R}^d} F_p(u(y),u(z))\Big(\int_{D^c} P_D(y,w)\,\mathrm{d}w\Big)\nu(y,z)\,\mathrm{d}z\mathrm{d}y\\ &= \int_D\int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y. \quad (6.4.7)\end{aligned}$$

Since the sum of (6.4.6) and (6.4.7) equals $p\mathcal{E}_D^{(p)}[u]$, we obtain the Douglas identity.

We now prove (ii) starting with $p = 2$. Note that we cannot use Theorem 4.4.14, because it hinges on the extension and trace theorem which we are proving at the moment. Suppose first that D is bounded. Assume that $u \in \mathcal{V}_D$, i.e., $\mathcal{E}_D[u] < \infty$. Let $g = u|_{D^c}$. This means that g has an extension in $\mathcal{V}_D$, so by Theorem 3.1.1 there exists a function which is weakly harmonic in the sense of Definition 4.4.11 and equal to g on D^c. Furthermore, by Lemma 3.2.5, this weakly harmonic extension minimizes the form $\mathcal{E}_D$ among the functions equal to g on D^c.

Therefore, we may assume that u is weakly harmonic, i.e., $\mathcal{E}_D(u,\phi)=0$ for every $\phi\in\widetilde{\mathcal{V}}^0_D$, cf. (2.3.14). By Lemmas 4.3.4, 4.4.13 we may apply [81, Theorem 1.1] in order to show that, after a modification on a set of Lebesgue measure zero, u is L-harmonic in D. In particular for every Lipschitz $U\subset\subset D$ we have $P_U[|u|]<\infty$. Given that fact, the chain of identities from the proof of part (i) can be reversed with D replaced by U, note that Lemma 6.2.1 holds true. Thus we obtain $\mathcal{E}_U[u]=\mathcal{H}_U[u]$ and then we let $U\uparrow D$. Clearly, $\mathcal{E}_U[u]\uparrow\mathcal{E}_D[u]<\infty$. By Fatou's lemma,

$$\begin{aligned}\infty>2\mathcal{E}_D[u]&=\lim_{U\uparrow D}2\mathcal{E}_U[u]=\lim_{U\uparrow D}2\mathcal{H}_U[u]\\&\geq\iint_{\mathbb{R}^d\times\mathbb{R}^d}(u(z)-u(w))^2\liminf_{U\uparrow D}\left(\gamma_U(z,w)\mathbf{1}_{U\times U}(z,w)\right)\mathrm{d}z\mathrm{d}w\\&\geq\iint_{D\times D}(u(z)-u(w))^2\int_{\mathbb{R}^d}\nu(z,x)\liminf_{U\uparrow D}\left(\int_{\mathbb{R}^d}G_U(x,y)\nu(y,w)\,\mathrm{d}y\right)\mathrm{d}x\mathrm{d}z\mathrm{d}w.\end{aligned}$$

By the quasi-left continuity and the assumption $\mathbb{P}^x(\tau_D<\infty)=1$, $\mathbb{P}^x$–almost surely we have $\tau_U\uparrow\tau_D$, cf. [16, proof of Lemma 17] and [139, Theorem 40.12]. By the monotone convergence theorem,

$$\begin{aligned}\liminf_{U\uparrow D}\int_{\mathbb{R}^d}G_U(x,y)\nu(y,w)\,\mathrm{d}y&=\liminf_{U\uparrow D}\mathbb{E}^x\int_0^{\tau_U}\nu(X_t,w)\,\mathrm{d}t\\&=\mathbb{E}^x\int_0^{\tau_D}\nu(X_t,w)\,\mathrm{d}t=\int_{\mathbb{R}^d}G_D(x,y)\nu(y,w)\,\mathrm{d}y.\end{aligned}$$

Thus, $\mathcal{H}_D[u]\leq\mathcal{E}_D[u]<\infty$, which completes the proof for bounded sets D. Furthermore, since u was taken to be weakly harmonic and $\mathcal{E}_D[u]\geq\mathcal{H}_D[u]=\mathcal{E}_D[P_D[u]]$, we obtain that $u=P_D[u]$ by Lemma 3.2.5.

For unbounded D we consider the nonempty intersections $D\cap B(0,R)$ for $R>0$. We have $\mathcal{E}_{D\cap B(0,R)}[u]<\infty$, so there exists a weakly harmonic u_R such that $u_R=u$ a.e. on D^c. By Lemma 3.2.5 and the above considerations, $\mathcal{E}_D[u]\geq\mathcal{E}_{D\cap B(0,R)}[u]\geq\mathcal{E}_{D\cap B(0,R)}[u_R]=\mathcal{H}_{D\cap B(0,R)}[u]$ for all R. We let $R\to\infty$ and, as above, by the monotone convergence theorem we get $\mathcal{E}_D[u]\geq\mathcal{H}_D[u]$, which ends the case $p=2$ (recall that $\tau_D<\infty$ almost surely).

In order to obtain (ii) for $p\neq 2$, we note that by Lemma 6.2.3, $\mathcal{E}^{(p)}_D[u]<\infty$ is equivalent to $\mathcal{E}_D[u^{\langle p/2\rangle}]<\infty$. By the trace theorem for $p=2$ obtained above, $\mathcal{H}_D[u^{\langle p/2\rangle}]<\infty$. We finish the proof by using Lemma 6.2.3. □

6.5 Douglas and Hardy–Stein identities with remainders

Throughout this section we assume that D is bounded. In the (quadratic) case $p=2$ we know that the Poisson integral $P_D[g]$ is the minimizer of the form $\mathcal{E}_D$ among all Borel functions with a fixed exterior condition $g\in\mathcal{X}_D$, see Theorem 4.4.14. This may not be the case when $p\neq 2$. At the end of this section we give an example of D and $g\in\mathcal{X}^p_D$ for which $P_D[g]$ does not minimize $\mathcal{E}^{(p)}_D$ among functions in $\mathcal{V}^p_D$ equal to g on D^c. However, $P_D[g]$ is always a *quasiminimizer*, if we adopt the following definition:

Definition 6.5.1. Let $K\geq 1$. Function $u\in\mathcal{V}^p_D$ is a K-quasiminimizer of $\mathcal{E}^{(p)}_D$, if $\mathcal{E}^{(p)}_U[u]\leq K\mathcal{E}^{(p)}_U[v]$ for every nonempty open Lipschitz set $U\subset\subset D$ and every $v\in\mathcal{V}^p_U$ equal to u on U^c. We say that u is a quasiminimizer if it is a K-quasiminimizer for some $K\in[1,\infty)$.

The definition is inspired by the classical one given by Giaquinta and Giusti [75, (5.26)]. To avoid technical complications and to make this digression short we require Lipschitz test sets U above, even though we could use the sets as general as permitted in Theorem 6.4.1. However, to be prudent we note that the choice of admissible sets U may affect the definition of quasiminimizers and should be carefully considered, cf. Giusti [77, Example 6.5]. In the classical PDEs, quasiminimizers display many regularity properties similar to minimizers, see, e.g., Adamowicz and Toivanen [1], DiBenedetto and Trudinger [56] and Ziemer [164]. The main motivation for studying quasiminimizers is the fact that the solution of a complicated variational problem may be a quasiminimizer of a better understood functional see, e.g., [75, Theorem 2.1]. Since the harmonic functions of L are already known to be very regular, the following fact may be interesting, but it is unlikely that we will find any use for it.

Proposition 6.5.2. *Suppose that the assumptions of Theorem 6.4.1 are satisfied, D is bounded and let $g \in \mathcal{X}_D^p$. Then $P_D[g]$ is a quasiminimizer of $\mathcal{E}_D^{(p)}$.*

Proof. Fix a Lipschitz subset $U \subset\subset D$ and let $v \in \mathcal{V}_U^p$ be equal to $u := P_D[g]$ on U^c. According to (6.2.15) we have $v^{\langle p/2\rangle} \in \mathcal{V}_U$ and

$$\mathcal{E}_U^{(p)}[v] \approx \mathcal{E}_U[v^{\langle p/2\rangle}],$$

with constants independent of U and v. Note that $v^{\langle p/2\rangle}$ agrees with $u^{\langle p/2\rangle}$ on U^c, U^c satisfies VDC and $|\partial U| = 0$ (since U is Lipschitz), hence by Theorem 4.4.14,

$$\mathcal{E}_U[v^{\langle p/2\rangle}] \geq \mathcal{E}_U[P_U[u^{\langle p/2\rangle}]]. \tag{6.5.1}$$

By applying the Douglas identity for the set U, first with exponent 2, then with exponent p, and by (6.2.16) we get that the right-hand side of (6.5.1) is equal to

$$\mathcal{H}_U[u^{\langle p/2\rangle}] \approx \mathcal{H}_U^{(p)}[u] = \mathcal{E}_U^{(p)}[P_U[u]] = \mathcal{E}_U^{(p)}[u].$$

In the last equality we use the identity $P_U[u] = u$, see Lemma 4.4.2. The proof is complete. □

To prove that Poisson integrals need not be minimizers, we first extend the Hardy–Stein and Douglas identities to functions that are not harmonic. The results are new even for $p = 2$ and $\Delta^{\alpha/2}$.

Recall that D is bounded, hence $\mathbb{E}^x\tau_D$ is bounded. In what follows by $\lim\limits_{U\uparrow D}$ we denote the limit over an arbitrary ascending sequence of Lipschitz open sets $U_n \subset\subset D$ such that $\bigcup_n U_n = D$. Here is an extended version of the Hardy–Stein formula.

Proposition 6.5.3. *Let $p > 1$ and assume that ν satisfies* **A1** *and* **A2**. *Let $u\colon \mathbb{R}^d \to \mathbb{R}$. If $u \in C^2(D)$ and u and Lu are bounded in D, then for every $x \in D$,*

$$\lim_{U\uparrow D} \mathbb{E}^x|u(X_{\tau_U})|^p = |u(x)|^p + \int_D G_D(x,y)\int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y \tag{6.5.2}$$

$$- p\int_D G_D(x,y)u(y)^{\langle p-1\rangle}Lu(y)\,\mathrm{d}y. \tag{6.5.3}$$

If in addition D^c satisfies VDC and $|\partial D| = 0$, then $\lim\limits_{U\uparrow D} \mathbb{E}^x|u(X_{\tau_U})|^p = \mathbb{E}^x|u(X_{\tau_D})|^p$.

Proof. Let $x \in D$. Since u, Lu and $\mathbb{E}^x\tau_D$ are bounded on D, by (2.2.9) we get that the integral in (6.5.3) is finite. Therefore, using the arguments from the proof of Proposition 6.3.3, in what follows we may and do assume that $\int_{\mathbb{R}^d} |u(x)|^p(1 \wedge \nu(x))\,\mathrm{d}x < \infty$, because otherwise both sides of (6.5.2) are infinite. With this in mind we first consider open Lipschitz $U \subset\subset D$ so large that $x \in U$.

Let $p \geq 2$. Since $u \in C^2(D)$, we get that $L|u|^p(x)$ and $\mathbb{E}^x|u(X_{\tau_U})|^p$ are finite for $x \in U$, and (6.3.6) holds. Furthermore, since Lu is finite in D, the following manipulations are justified for $y \in D$:

$$\begin{aligned} L|u|^p(y) &= L|u|^p(y) - pu(y)^{\langle p-1\rangle}Lu(y) + pu(y)^{\langle p-1\rangle}Lu(y) \\ &= -\lim_{\epsilon\to 0^+}\int_{|z-y|>\epsilon} (|u(z)|^p - |u(y)|^p - pu(y)^{\langle p-1\rangle}(u(z)-u(y)))\nu(z,y)\,\mathrm{d}z \\ &\quad + pu(y)^{\langle p-1\rangle}Lu(y) \\ &= -\int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z + pu(y)^{\langle p-1\rangle}Lu(y). \end{aligned} \tag{6.5.4}$$

Consequently, (6.3.6) takes on the form

$$\mathbb{E}^x|u(X_{\tau_U})|^p = |u(x)|^p + \int_U G_U(x,y)\int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y \tag{6.5.5}$$
$$- \int_U G_U(x,y)u(y)^{\langle p-1\rangle}Lu(y)\,\mathrm{d}y. \tag{6.5.6}$$

For clarity we note that the left-hand side of (6.5.5) is finite and the integral in (6.5.6) is absolutely convergent, so the integral in (6.5.5) is finite as well.

For $p \in (1,2)$ we proceed as in the proof of Proposition 6.3.2, that is, instead of $|u(x)|^p$ we consider $\varepsilon > 0$ and the function $x \mapsto (u(x)^2+\varepsilon^2)^{p/2}$. We obtain (cf. (6.3.10) and (6.5.4)),

$$\mathbb{E}^x(u(X_{\tau_U})^2+\varepsilon^2)^{p/2} = (u(x)^2+\varepsilon^2)^{p/2} + \int_U G_U(x,y)\int_{\mathbb{R}^d} F_p^{(\varepsilon)}(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y \tag{6.5.7}$$
$$- p\int_U G_U(x,y)u(y)(u(y)^2+\varepsilon^2)^{(p-2)/2}Lu(y)\,\mathrm{d}y. \tag{6.5.8}$$

As in the proof of Proposition 6.3.3, the left-hand side tends to $\mathbb{E}^x|u(X_{\tau_U})|^p$ as $\varepsilon \to 0^+$. Furthermore, since Lu and u are bounded in D, the integral in (6.5.8) converges to that in (6.5.6). Then we apply Fatou's lemma and the dominated convergence theorem to the integral on the right-hand side of (6.5.7) and we obtain (6.5.5) for $p \in (1,2)$, too.

We let $U \uparrow D$ in (6.5.5). By the boundedness of u and Lu in D, the integral in (6.5.6) tends to the one in (6.5.3), which is absolutely convergent. The integral on the right-hand side of (6.5.5) converges to the one on the right-hand side of (6.5.2) by the domain monotonicity and the monotone convergence theorem. Since the limit on the right-hand side of (6.5.2) exists, the limit on the left-hand side must exist as well. This proves (6.5.2).

If D^c satisfies VDC and $|\partial D| = 0$, then Corollary 2.2.3 holds true. Furthermore, we have

$$\mathbb{E}^x|u(X_{\tau_U})|^p = \mathbb{E}^x[|u(X_{\tau_U})|^p;\, \tau_U \neq \tau_D] + \mathbb{E}^x[|u(X_{\tau_D})|^p;\, \tau_U = \tau_D].$$

The first term on the right converges to 0 by the boundedness of u on D and the fact that $\mathbb{P}^x(\tau_U \neq \tau_D)$ decreases to 0 as $U \uparrow D$, see Corollary 2.2.3. The second term converges to $\mathbb{E}^x|u(X_{\tau_D})|^p$ by the monotone convergence theorem. Thus the left-hand side of (6.5.5) tends to $\mathbb{E}^x|u(X_{\tau_D})|^p$. □

We next provide a Douglas-type identity for a class of non-harmonic functions:

Theorem 6.5.4. *Suppose that the assumptions of Theorem 6.4.1 hold with the addition that D is bounded. Let $u\colon \mathbb{R}^d \to \mathbb{R}$ be bounded, $u \in C^2(D)$ and Lu be bounded in D. Then*

$$\mathcal{E}_D^{(p)}[P_D[u]] = \mathcal{E}_D^{(p)}[u] + A_D(u), \tag{6.5.9}$$

where

$$A_D(u) = -\int_D u(x)^{\langle p-1\rangle} Lu(x)\,\mathrm{d}x + \int_D\int_{D^c} u(w)^{\langle p-1\rangle}(u(x) - P_D[u](x))\,\nu(w,x)\,\mathrm{d}w\mathrm{d}x.$$

Proof. Since u is bounded on $\mathbb{R}^d$, we have $\int_{\mathbb{R}^d} |u(x)|(1\wedge\nu(x))\,\mathrm{d}x < \infty$.

Assume first that $\mathcal{H}_D^{(p)}[u] < \infty$. From Theorem 6.4.1 we have

$$\mathcal{E}_D^{(p)}[P_D[u]] = \mathcal{H}_D^{(p)}[u].$$

By the definition of γ_D, see (2.2.20), and Fubini–Tonelli,

$$p\mathcal{H}_D^{(p)}[u] = \int_D\int_{D^c}\int_{D^c} F_p(u(w),u(z))P_D(x,z)\nu(x,w)\,\mathrm{d}z\mathrm{d}w\mathrm{d}x.$$

We apply Lemma 6.2.1 (iii) to $a = u(w)$, $b = u(x)$, with $w \in D^c$, $x \in D$, $X = u(X_{\tau_D})$ and $\mathbb{E} = \mathbb{E}^x$. Note that $\mathbb{E}X = P_D[u](x)$. This yields:

$$\begin{aligned}&\int_{D^c} F_p(u(w),u(z))P_D(x,z)\,\mathrm{d}z\\ &= \int_{D^c} F_p(u(x),u(z))P_D(x,z)\,\mathrm{d}z + F_p(u(w),u(x)) + (pu(w)^{\langle p-1\rangle} - pu(x)^{\langle p-1\rangle})(u(x) - P_D[u](x)).\end{aligned}$$

After integration, we obtain

$$\begin{aligned}p\mathcal{H}_D^{(p)}[u] =& \int_D\int_{D^c}\int_{D^c} F_p(u(x),u(z))P_D(x,z)\nu(x,w)\,\mathrm{d}z\mathrm{d}w\mathrm{d}x\\ &+ \int_D\int_{D^c} F_p(u(w),u(x))\nu(x,w)\,\mathrm{d}w\mathrm{d}x\\ &+ \int_D\int_{D^c} (pu(w)^{\langle p-1\rangle} - pu(x)^{\langle p-1\rangle})(u(x) - P_D[u](x))\,\nu(x,w)\,\mathrm{d}w\mathrm{d}x\\ =:& A_1(u) + A_2(u) + A_3(u).\end{aligned}$$

Note that every term above is finite. Indeed, by the boundedness of u,

$$|A_3(u)| \lesssim \int_D\int_{D^c} |u(x) - P_D[u](x)|\nu(x,w)\,\mathrm{d}w\mathrm{d}x.$$

To prove that this is finite, let $v = u - P_D[u]$. We have $Lv = Lu = f \in L^\infty(D)$ and $v = 0$ on D^c. Note that $v \in C^2(D)$ and $\int_{\mathbb{R}^d} |v(x)|(1\wedge\nu(x))\,\mathrm{d}x < \infty$, cf. Lemma 4.3.4. Let $U \subset\subset D$. By Lemma 6.3.1,

$$\mathbb{E}^x v(X_{\tau_U}) - v(x) = -\int_U G_U(x,y)f(y)\,\mathrm{d}y, \quad x \in U.$$

Since u is bounded on $\mathbb{R}^d$, we have $\mathbb{E}^x u(X_{\tau_U}) \to \mathbb{E}^x u(X_{\tau_D}) = P_D[u](x)$ as $U \uparrow D$, cf. the last part of the proof of Proposition 6.5.3. Hence, the boundedness of f, the domain monotonicity and the dominated convergence theorem yield

$$v(x) = \int_D G_D(x,y)f(y)\,\mathrm{d}y, \quad x \in D.$$

This allows us to further estimate A_3:

$$|A_3(u)| \lesssim \int_D \int_{D^c} \int_D G_D(x,y)\nu(w,x)\,\mathrm{d}y\mathrm{d}w\mathrm{d}x = \int_D \int_{D^c} P_D(y,w)\,\mathrm{d}w\mathrm{d}y = |D| < \infty.$$

Since $A_1(u)$ and $A_2(u)$ are nonnegative, they must be finite as well, because $\mathcal{H}_D^{(p)}[u] < \infty$. We then have

$$\begin{aligned}\int_{D^c} F_p(u(x),u(z))P_D(x,z)\,\mathrm{d}z &= \mathbb{E}^x F_p(u(x),u(X_{\tau_D}))\\ &= \mathbb{E}^x|u(X_{\tau_D})|^p - |u(x)|^p - pu(x)^{\langle p-1\rangle}(P_D[u](x)-u(x)).\end{aligned}$$

Thus, by Proposition 6.5.3 we obtain

$$\begin{aligned}A_1(u) = A_4(u) &- p\int_D\int_{D^c}\int_D G_D(x,y)u(y)^{\langle p-1\rangle}Lu(y)\nu(x,w)\,\mathrm{d}y\mathrm{d}w\mathrm{d}x \\ &- p\int_D\int_{D^c} u(x)^{\langle p-1\rangle}(P_D[u](x)-u(x))\nu(x,w)\,\mathrm{d}w\mathrm{d}x,\end{aligned} \tag{6.5.10}$$

where $A_4(u)$ is the integral in (6.4.7). Note that $A_2(u) + A_4(u) = p\mathcal{E}_D^{(p)}[u]$. Also, all the expressions in (6.5.10) are finite, see the discussion of $A_3(u)$. To finish the proof of (6.5.9) in the case $\mathcal{H}_D^{(p)}[u] < \infty$, we simply note that $pA_D(u) = A_1(u) - A_4(u) + A_3(u)$.

The situation $\mathcal{H}_D^{(p)}[u] = \infty$ remains to be considered. Since $P_D[u]$ is bounded in D, by arguments similar to those in the estimates of $A_3(u)$ above, we prove that $A_D(u)$ is finite. Therefore by Theorem 6.4.1 the identity (6.5.9) holds with both sides infinite. □

Knowing the form of the remainder $A_D(u)$ in the Douglas identity (6.5.9), we may provide an example which shows that Poisson integral need not be a minimizer of $\mathcal{E}_D^{(p)}$ for $p \neq 2$; it is only a quasiminimizer by Proposition 6.5.2.

Example 6.5.5 (The Poisson extension need not be a minimizer for $p \neq 2$)**.** Let $p > 2$ and consider $0 < R < R_1$ such that $D \subset\subset B_R$. Define

$$g_n(z) = (((|z|-R)^{-1/(p-1)} \wedge n)\mathbf{1}_{B_{R_1}\setminus B_R}(z).$$

Since each g_n is bounded with support separated from D, we have $g_n \in \mathcal{X}_D^p \cap \mathcal{X}_D$; see the discussion following Example 4.2.4. By (2.2.17) there exists $c > 0$ such that

$$P_D(x,z) \le c, \quad x \in D,\ z \in B_{R_1} \setminus B_R. \tag{6.5.11}$$

Furthermore, for every $U \subset\subset D$ there is $\epsilon > 0$ such that

$$P_D(x,z) \ge \epsilon, \quad x \in U,\ z \in B_{R_1} \setminus B_R. \tag{6.5.12}$$

For $x \in D$ we let

$$u_n(x) = G_D[1](x) + P_D[g_n](x).$$

Obviously u_n are bounded on $\mathbb{R}^d$. We will verify that $G_D[1] \in C^2(D)$. For this purpose we let f be a smooth, compactly supported, nonnegative function equal to 1 on D. By the Hunt's formula and Fubini–Tonelli we get

$$G_D[f](x) = G_D[1](x) = \int_{\mathbb{R}^d} G(x-y)f(y)\,\mathrm{d}y - \mathbb{E}^x\int_{\mathbb{R}^d} G(X_{\tau_D},y)f(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d. \tag{6.5.13}$$

Here G is either the potential kernel or the compensated potential kernel of (X_t); see [81, Appendix A] for details. In particular, by [81, Corollary A.3] and [139, Theorem 35.4] G is locally integrable, thus the first term in (6.5.13) is finite and smooth in D. Since the latter term in (6.5.13) is a harmonic function, we get that $G_D[1] \in C^2(D)$. In particular, by Lemma 4.4.10 and Dynkin [64, Lemma 5.7] we have $Lu_n = 1$ in D. We are now in a position to apply Theorem 6.5.4. Fix open $U \subset\subset D$. We get

$$\begin{aligned} A_D(u_n) &= -\int_D u_n(x)^{p-1}\,\mathrm{d}x + \int_D\int_{D^c} u_n(w)^{p-1}G_D[1](x)\nu(x,w)\,\mathrm{d}w\mathrm{d}x \\ &= \int_D (\mathbb{E}^x u_n(X_{\tau_D})^{p-1} - (\mathbb{E}^x u_n(X_{\tau_D}) + G_D[1](x))^{p-1})\,\mathrm{d}x = \int_U + \int_{D\setminus U}. \end{aligned} \tag{6.5.14}$$

We claim that $A_D(u_n) > 0$ for large n. Indeed, recall that $G_D[1](x) = \mathbb{E}^x\tau_D$ is bounded. Since the integrals $\int_{D^c} g_n(x)\,\mathrm{d}x$ are bounded, by (6.5.11) there is $M > 0$ such that $\mathbb{E}^x u_n(X_{\tau_D}) < M$ for every $x \in D$ and $n \in \mathbb{N}$. Therefore the integral $\int_{D\setminus U}$ in (6.5.14) is bounded from below, independently of n. Note that $\int_{D^c} g_n(x)^{p-1}\mathrm{d}x \to \infty$ as $n \to \infty$. Thus, by (6.5.12) we obtain that $\int_U \to \infty$ in (6.5.14) as $n \to \infty$. Hence, for sufficiently large n we get that $A_D(u_n) > 0$, which proves that $\mathcal{E}_D^{(p)}[P_D[u_n]] > \mathcal{E}_D^{(p)}[u_n]$ for some n, as needed. The case $p \in (1,2)$ may be handled similarly, by using $g_n(z) = ((|z| - R)^{-1} \wedge n)\mathbf{1}_{B_{R_1}\setminus B_R}(z)$ and $u_n = P_D[g_n] - G_D[1]$.

6.6 Further discussion

6.6.1 Extension theorem for spaces $\mathcal{W}_D^p$

As usual, D is a nonempty open set in $\mathbb{R}^d$. We define yet another type of function spaces generalizing $\mathcal{V}_D$. They are based on the forms reminiscent of those considered, e.g., by Dyda [60] but with more general Lévy kernels.

$$\mathcal{W}_D^p = \Big\{u\colon \mathbb{R}^d \to \mathbb{R} \ \Big|\ \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} |u(x) - u(y)|^p\nu(x,y)\,\mathrm{d}x\mathrm{d}y < \infty\Big\}. \tag{6.6.1}$$

We will show that the functions from the following space have an extension in $\mathcal{W}_D^p$.

$$\mathcal{Y}_D^p = \Big\{g\colon D^c \to \mathbb{R} \ \Big|\ \iint\limits_{D^c\times D^c} |g(w) - g(z)|^p\gamma_D(w,z)\,\mathrm{d}w\mathrm{d}z < \infty\Big\}.$$

Proposition 6.6.1. *If $p \geq 2$ then* (6.1.5) *holds true under the assumptions on D and ν from Theorem 6.4.1, and the Poisson extension acts from $\mathcal{Y}_D^p$ to $\mathcal{W}_D^p$.*

Proof. Assume that $g \in \mathcal{Y}_D^p$, i.e., the right-hand side of (6.1.5) is finite. By a simple modification of the proof of Lemma 4.4.6 we get that $g \in L^p(D^c, P_D(x,z)\,\mathrm{d}z)$ for every $x \in D$, in particular the Poisson integral $P_D[g](x)$ converges absolutely. By the definition of γ_D, see (2.2.20), the right-hand side of (6.1.5) equals

$$\int_{D^c}\int_{D^c}\int_D |g(w) - g(z)|^p\nu(w,x)P_D(x,z)\,\mathrm{d}x\mathrm{d}w\mathrm{d}z.$$

We use Fubini–Tonelli and consider the integral

$$\int_{D^c} |g(w) - g(z)|^p P_D(x,z)\,\mathrm{d}z = \mathbb{E}^x\,|u(X_{\tau_D}) - g(w)|^p\,.$$

By Lemma 6.2.2 we get that for $x \in D$ and $w \in D^c$,

$$\mathbb{E}^x |u(X_{\tau_D}) - g(w)|^p \approx \mathbb{E}^x|u(X_{\tau_D}) - u(x)|^p + |u(x) - g(w)|^p \geq \mathbb{E}^x|u(X_{\tau_D}) - u(x)|^p.$$

We apply Proposition 6.3.3, to $\widetilde{u}(z) := u(z) - u(x)$. It is L-harmonic on D and $\widetilde{u}(x) = 0$, therefore

$$\mathbb{E}^x |u(X_{\tau_D}) - u(x)|^p = \int_D G_D(x,y) \int_{\mathbb{R}^d} F_p(\widetilde{u}(y), \widetilde{u}(z))\nu(z,y)\,\mathrm{d}z\mathrm{d}y.$$

For $p \neq 2$ it is not true that $F_p(a+t, b+t)$ is comparable with $F_p(a,b)$, but since $p \geq 2$, by Lemma 6.2.4 we have $F_p(a+t, b+t) \geq c|a+t-b-t|^p = c|a-b|^p$. It follows that

$$F_p(\widetilde{u}(y), \widetilde{u}(z)) \gtrsim |u(y) - u(z)|^p,$$

and thus

$$\mathbb{E}^x |u(X_{\tau_D}) - g(w)|^p \gtrsim \int_D G_D(x,y) \int_{\mathbb{R}^d} |u(y) - u(z)|^p \nu(z,y)\,\mathrm{d}z\mathrm{d}y.$$

We integrate the inequality on $D^c \times D$ against $\nu(w,x)\,\mathrm{d}w\mathrm{d}x$ as in (6.4.7), and the right-hand side is

$$\int_D \int_{\mathbb{R}^d} |u(x) - u(y)|^p \nu(x,y)\,\mathrm{d}x\mathrm{d}y \geq \tfrac{1}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus(D^c\times D^c)} |u(x) - u(y)|^p \nu(x,y)\,\mathrm{d}x\mathrm{d}y.$$

The result follows. □

We remark that in general (6.1.5) fails for $p \in (1,2)$; see Lemma 6.6.4 and Example 6.6.5.

6.6.2 Inclusion of smooth functions

Below we discuss whether the spaces $\mathcal{V}^p_D$ and $\mathcal{W}^p_D$ contain the smooth functions. In most cases we get a positive answer.

Lemma 6.6.2. *For every $p > 1$ we have $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{V}^p_{\mathbb{R}^d} \subseteq \mathcal{V}^p_D$.*

Proof. The inclusion $\mathcal{V}^p_{\mathbb{R}^d} \subseteq \mathcal{V}^p_D$ follows from the definition. The remaining arguments are rather standard, cf. Proposition 2.3.2, but due to the numerous characterizations of $\mathcal{V}^p_{\mathbb{R}^d}$ it is not obvious which one we should work with. For example, it is rather unclear how to proceed with the version using $(b-a)^2(|b| \vee |a|)^{p-2}$, cf. Lemma 2.3.5. Here is an efficient approach.

To prove that $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{V}^p_{\mathbb{R}^d}$, we let $\phi \in C_c^\infty(\mathbb{R}^d)$. We have

$$|2\phi(x) - \phi(x+z) - \phi(x-z)| \leq M(1 \wedge |z|^2), \quad x, z \in \mathbb{R}^d.$$

It follows that $L\phi$ is bounded on $\mathbb{R}^d$, cf. (2.3.1). Thus,

$$\int_{\mathbb{R}^d} |\phi(x)|^{p-1} |L\phi(x)|\,\mathrm{d}x < \infty. \tag{6.6.2}$$

Furthermore, by the dominated convergence theorem and the symmetry of ν,

$$\begin{aligned}\int_{\mathbb{R}^d} \phi(x)^{\langle p-1\rangle} L\phi(x)\,\mathrm{d}x &= \tfrac{1}{2} \int_{\mathbb{R}^d} \phi(x)^{\langle p-1\rangle} \lim_{\epsilon\to0^+} \int_{|z|>\epsilon} (2\phi(x) - \phi(x+z) - \phi(x-z))\nu(z)\,\mathrm{d}z\mathrm{d}x \\ &= \lim_{\epsilon\to0^+} \int_{\mathbb{R}^d} \int_{|z|>\epsilon} \phi(x)^{\langle p-1\rangle} (\phi(x) - \phi(x+z))\nu(z)\,\mathrm{d}z\mathrm{d}x.\end{aligned}$$

By Fubini's theorem, the substitutions $z \to -z$ and $x \to x+z$, and the symmetry of ν,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{|z|>\epsilon} \phi(x)^{\langle p-1\rangle}(\phi(x)-\phi(x+z))\nu(z)\,\mathrm{d}z\mathrm{d}x \\
=&\int_{\mathbb{R}^d}\int_{|z|>\epsilon} \phi(x+z)^{\langle p-1\rangle}(\phi(x+z)-\phi(x))\nu(z)\,\mathrm{d}z\mathrm{d}x \\
=&\tfrac{1}{2}\int_{|z|>\epsilon}\int_{\mathbb{R}^d}(\phi(x+z)^{\langle p-1\rangle}-\phi(x)^{\langle p-1\rangle})(\phi(x+z)-\phi(x))\,\mathrm{d}x\,\nu(z)\,\mathrm{d}z
\end{aligned}$$

for every $\epsilon > 0$. By (6.2.10), the monotone convergence theorem and the above,

$$\begin{aligned}
\mathcal{E}^{(p)}_{\mathbb{R}^d}[\phi] &= \tfrac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\phi(x+z)^{\langle p-1\rangle}-\phi(x)^{\langle p-1\rangle})(\phi(x+z)-\phi(x))\nu(z)\,\mathrm{d}x\mathrm{d}z \\
&= \int_{\mathbb{R}^d}\phi(x)^{\langle p-1\rangle}L\phi(x)\,\mathrm{d}x. \qquad (6.6.3)
\end{aligned}$$

The result follows from (6.6.2) and (6.2.12). □

The inclusion $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{V}^p_D$ indicates that the Sobolev–Bregman spaces will be useful in variational problems posed in L^p.

The situation with the spaces $\mathcal{W}^p_D$ is more complicated. While for $p \ge 2$ we have a result similar to that of Lemma 6.6.2, for $p \in (1,2)$ it is not so. More precisely, we have the following two lemmas:

Lemma 6.6.3. *For $p \ge 2$ we have $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{W}^p_{\mathbb{R}^d} \subseteq \mathcal{W}^p_D$.*

Proof. For $\phi \in C_c^\infty(\mathbb{R}^d)$ let $K = \operatorname{supp}\phi$. Then we have $|\phi(x)-\phi(y)| = 0$ on $K^c \times K^c$ and

$$|\phi(x)-\phi(y)|^p \lesssim 1 \wedge |x-y|^p \le 1 \wedge |x-y|^2, \quad x,y \in \mathbb{R}^d \times \mathbb{R}^d \setminus K^c \times K^c.$$

It follows that $\phi \in \mathcal{W}^p_{\mathbb{R}^d}$, cf. Lemma 2.3.5. The inclusion $\mathcal{W}^p_{\mathbb{R}^d} \subseteq \mathcal{W}^p_D$ is clear from the definition of the spaces. □

Lemma 6.6.4. *Let $p \in (1,2)$ and assume that for some $r > 0$ we have $\nu(y) \gtrsim |y|^{-d-p}$ for $|y| < r$. If $u \in \mathcal{W}^p_D$ has compact support in $\mathbb{R}^d$ and vanishes on D^c, then $u \equiv 0$.*

Results of this type are well-known for the spaces with integration over $D \times D$, where D is connected. Brezis [31, Proposition 2] shows that any measurable function must be constant in this case; a simpler proof of this fact was given by De Marco, Mariconda and Solimini [50, Theorem 4.1]. Lemma 6.6.4 follows by taking $\Omega = \mathbb{R}^d$ in the aforementioned results, but we present a different proof. Such facts also hold true in the context of metric spaces, see, e.g., Pietruska-Pałuba [126]. We will see in the proof of Lemma 6.6.4 that the result reduces to that with $D = \mathbb{R}^d$.

Proof of Lemma 6.6.4. We may assume that u is bounded, because the p-increments of $(0\vee u)\wedge 1$ do not exceed those of u. Thus, since u is compactly supported, we get that $u \in L^p(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. Let

$$\widehat{u}(\xi) = \int_{\mathbb{R}^d} u(x)e^{-2\pi i\xi x}\,\mathrm{d}x, \quad \xi \in \mathbb{R}^d.$$

The Hausdorff–Young inequality asserts that for $u \in L^p(\mathbb{R}^d)$ we have

$$\|u\|_p \ge \|\widehat{u}\|_{p'}, \qquad (6.6.4)$$

where $p' = \frac{p}{p-1}$, see, e.g., Grafakos [78, Proposition 2.2.16]. We estimate the left-hand side of (6.1.5) by using (6.6.4):

$$\begin{aligned}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} |u(x)-u(y)|^p\nu(x,y)\,\mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |u(x)-u(x+y)|^p\nu(y)\,\mathrm{d}x\mathrm{d}y \\ &\geq \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |(u(\cdot)-u(\cdot+y))^\wedge(\xi)|^{p'}\,\mathrm{d}\xi\right)^{\frac{p}{p'}}\nu(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |1-e^{-2\pi i\xi y}|^{p'}|\widehat{u}(\xi)|^{p'}\,\mathrm{d}\xi\right)^{\frac{p}{p'}}\nu(y)\,\mathrm{d}y.\end{aligned}$$

By (6.6.4), $|\widehat{u}(\xi)|^{p'}\,\mathrm{d}\xi$ is a finite measure on $\mathbb{R}^d$. As we have $p/p' < 1$, by Jensen and Fubini–Tonelli,

$$\begin{aligned}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |1-e^{-2\pi i\xi y}|^{p'}|\widehat{u}(\xi)|^{p'}\,\mathrm{d}\xi\right)^{\frac{p}{p'}}\nu(y)\,\mathrm{d}y &\gtrsim \int_{\mathbb{R}^d}\nu(y)\int_{\mathbb{R}^d}|1-e^{-2\pi i\xi y}|^p|\widehat{u}(\xi)|^{p'}\,\mathrm{d}\xi\mathrm{d}y \\ &= \int_{\mathbb{R}^d}|\widehat{u}(\xi)|^{p'}\int_{\mathbb{R}^d}|1-e^{2\pi i\xi y}|^p\nu(y)\,\mathrm{d}y\mathrm{d}\xi.\end{aligned}$$

Since $|1-e^{2\pi i\xi y}| \geq |\sin 2\pi\xi y|$ and $\nu(y) \gtrsim |y|^{-d-p}$ for small $|y|$, the integral is infinite, unless $u = 0$ a.e. in $\mathbb{R}^d$. □

As a comment to Lemmas 6.6.2 and 6.6.4 we recall that $\mathcal{V}^p_D$ is defined in terms of F_p. When a is close to b then, regardless of $p > 1$, the Bregman divergence $F_p(a,b)$ is of order $(b-a)^2$ rather than $|b-a|^p$. Thus $\mathcal{V}^p_D$ agrees with the Lévy measure integrability condition better than $\mathcal{W}^p_D$ does.

The following example indicates that the scale of linear spaces $\mathcal{W}^p_D$ may not be suitable for analysis of harmonic functions when $p \leq 2$:

Example 6.6.5. Let ν and p be as in Lemma 6.6.4. Let $B = B(0,1)$ and assume that D is bounded and $d(D,B) > 0$. Then there is $g \in \mathcal{Y}^p_D$ such that $u := P_D[g] \notin \mathcal{W}^p_D$, i.e.,

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} |u(x)-u(y)|^p\nu(x,y)\,\mathrm{d}x\mathrm{d}y = \infty. \tag{6.6.5}$$

Let $g(z) = \mathbf{1}_B(z)$ for $z \in D^c$. Then $g \in \mathcal{Y}^p_D$, cf. the arguments following Example 4.2.4. Clearly, u is bounded in D. By the positivity of P_D, see (2.2.17), we have $u(x) > 0$ for every $x \in D$.

The illustration in Figure 2.5 may prove useful for the following argument. Note that B, $D^c \setminus B = B^c \setminus D$ and D form a partition of $\mathbb{R}^d$. Therefore their Cartesian products partition $\mathbb{R}^d \times \mathbb{R}^d$; in fact also $B^c \times B^c$ and $\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c$ (see below). Since u vanishes on $D^c \setminus B$, $u(x)-u(y)$ vanishes on $(D^c \setminus B) \times (D^c \setminus B)$. It follows that

$$\int_{B^c}\int_{B^c} |u(x)-u(y)|^p\nu(x,y)\,\mathrm{d}x\mathrm{d}y \leq \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} |u(x)-u(y)|^p\nu(x,y)\,\mathrm{d}x\mathrm{d}y. \tag{6.6.6}$$

Define $\widetilde{u} = u$ on B^c and $\widetilde{u} = 0$ on B. Then, $\widetilde{u} = u$ on D, $\widetilde{u} = 0$ on D^c, and

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} |\widetilde{u}(x)-\widetilde{u}(y)|^p\nu(x,y)\,\mathrm{d}x\mathrm{d}y = \int_D\int_D + \int_D\int_{D^c\setminus B} + \int_{D^c\setminus B}\int_D + \int_B\int_D + \int_D\int_B$$

$$= \int_{B^c}\int_{B^c} |u(x)-u(y)|^p \nu(x,y)\,\mathrm{d}x\mathrm{d}y + 2\int_D |u(y)|^p \int_B \nu(x,y)\,\mathrm{d}x\mathrm{d}y.$$

By the boundedness of u, the boundedness of D and the separation of D and B, the last integral is finite. Furthermore, since $\widetilde{u}$ is not constant and vanishes on D^c, the left-hand side is infinite by Lemma 6.6.4. Therefore the left-hand side of (6.6.6) is infinite, which yields (6.6.5).

6.6.3 Comparison of $\mathcal{V}^p_D$, $\mathcal{W}^p_D$ and the fractional Sobolev-type spaces for $p > 2$

The extension and trace theorem of Dyda and Kassmann [62] reaches beyond the case $p = 2$. The spaces considered there are similar to the fractional Sobolev spaces $W^{s,p}$ from Example 2.3.9 in terms of the integrand, and similar to the Sobolev–Bregman spaces $\mathcal{V}^p_D$ in terms of the integration domain. Namely, for $p > 1$ and $s \in (0,1)$ we let:

$$\mathcal{V}^{s,p}(D) = \Big\{ u\colon \mathbb{R}^d \to \mathbb{R} \;\Big|\; \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\mathrm{d}y < \infty \Big\}.$$

Contrary to the authors of [62], we disregard the L^p norm, because our goal below is to gain deeper understanding on how different methods of measuring the increments affect the smoothness of functions, cf. Remark 2.3.6. For $s = \alpha/2$, $p = 2$, and the fractional Laplacian $L = (-\Delta)^{\alpha/2}$, the spaces $\mathcal{V}^{s,p}(D)$, $\mathcal{V}^p_D$ and $\mathcal{W}^p_D$ coincide, cf. (6.2.12) and (6.6.1). For $p > 2$ this is no longer true in general. We will prove that in certain range of p and s, these three spaces are pairwise different.

Proposition 6.6.6. *Let $p > 2$. Then we have*

(i) $\mathcal{V}^p_D \subseteq \mathcal{W}^p_D$.

If furthermore $\nu(x,y) = c|x-y|^{d+2s}$ with $s \in (0,1)$, then the following statements hold.

(ii) *If $2s(1-2/p) > 1$, then the above inclusion is proper, that is, there exists $v\colon \mathbb{R}^d \to \mathbb{R}$ such that $v \in \mathcal{W}^p_D \setminus \mathcal{V}^p_D$.*

(iii) *If $2s - 2/p > 1$, then the set $\mathcal{V}^p_D \setminus \mathcal{V}^{s,p}(D)$ is nonempty. Consequently, $\mathcal{W}^p_D \setminus \mathcal{V}^{s,p}(D)$ is nonempty as well.*

(iv) *If D is bounded, then for every $s \in (0,1)$ both $\mathcal{V}^{s,p}(D) \setminus \mathcal{V}^p_D$ and $\mathcal{V}^{s,p}(D) \setminus \mathcal{W}^p_D$ are nonempty.*

Proof. (i) The inclusion follows immediately from Lemma 6.2.4.

In the proof of (ii) and (iii), we assume for clarity that $B(0,1/2) \subseteq D$. Recall the notation $\mathbb{R}^d \ni x = (x', x_d)$, where $x' \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$, and consider $\mathbb{R}^d \ni x \mapsto (x_d)_+ := x_d \vee 0$.

(ii) Take $q \in (0,1)$ and let $v_q(x) := ((x_d)_+^q + 1)\varphi(x)$, where $0 \le \varphi \le 1$ is a smooth function compactly supported in $B(0,1/2)$, equal to 1 on $B(0,1/4)$. Since $1 \le v_q \le 2$ in $B(0,1/4)$, by (6.2.1) we have $F_p(v_q(x), v_q(y)) \approx (v_q(x) - v_q(y))^2$ for $x, y \in B(0,1/4)$. Thus,

$$\begin{aligned}
\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} \frac{F_p(v_q(x), v_q(y))}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y &\gtrsim \int_{B(0,1/4)}\int_{B(0,1/4)} \frac{(v_q(x)-v_q(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y \\
&\ge \int_{B(0,1/8)\cap\{x_d>0\}}\int_{B(0,1/4)\cap\{y_d<0\}} \frac{x_d^{2q}}{|x-y|^{d+2s}}\,\mathrm{d}y\mathrm{d}x \\
&\gtrsim \int_{B(0,1/8)\cap\{x_d>0\}} x_d^{2q-2s}\,\mathrm{d}x,
\end{aligned}$$

which is infinite if $q \le (2s-1)/2$, i.e., $v_q \notin \mathcal{V}_D^p$ for q within that range.

On the other hand, we have $|v_q(x) - v_q(y)| \lesssim 1 \wedge |x-y|^q$, and since $\operatorname{supp} v_q \subset B(0,1/2)$,

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} \frac{|v_q(x)-v_q(y)|^p}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y \lesssim \int_{B(0,1/2)}\int_{B(0,3/4)} \frac{|x-y|^{pq}}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y + \int_{B(0,1/2)}\int_{B(0,3/4)^c} \frac{1}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y.$$

The latter integral is finite for any $s \in (0,1)$, and the former is finite if and only if $q > 2s/p$, which means that $v_q \in \mathcal{W}_D^p$ for this set of parameters. To summarize, we get that $v_q \notin \mathcal{V}_D^p$ and $v_q \in \mathcal{W}_D^p$ provided that $s(1-2/p) > 1/2$ and $q \in (2s/p, (2s-1)/2]$.

(iii) Now take $u_q(x) = (x_d)_+^q \varphi(x)$ with φ as above and $q \in (0,1)$. We first determine sufficient conditions for $u_q \notin \mathcal{V}^{s,p}(D)$. Similarly as above we write

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} \frac{|u_q(x)-u_q(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}x\mathrm{d}y \gtrsim \int_{B(0,1/8)\cap\{x_d>0\}}\int_{B(0,1/4)\cap\{y_d<0\}} \frac{x_d^{pq}}{|x-y|^{d+ps}}\,\mathrm{d}y\mathrm{d}x \gtrsim \int_{B(0,1/8)\cap\{x_d>0\}} x_d^{pq-ps}\,\mathrm{d}x,$$

which is infinite if $q \le s - \frac{1}{p}$.

We now analyze the integral defining $\mathcal{V}_D^p$. By using (6.2.2) and the boundedness of u_q, we get

$$\begin{aligned}&\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} \frac{F_p(u_q(x),u_q(y))}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y\\ &\lesssim \int_{B(0,3/4)}\int_{B(0,3/4)} \frac{((x_d)_+^q-(y_d)_+^q)^2((x_d)_+^{q(p-2)}+(y_d)_+^{q(p-2)})}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y\\ &\quad+\int_{B(0,1/2)}\int_{B(0,3/4)^c}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{d+2s}}.\end{aligned}$$

The latter integral is finite for every $s \in (0,1)$, while the former is bounded from above by a multiple of

$$\int_{B(0,3/4)\cap\{y_d>0\}}\int_{B(0,3/4)\cap\{x_d>y_d\}} \frac{(x_d^q-y_d^q)^2 x_d^{p(q-2)}}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y + \int_{B(0,3/4)\cap\{y_d<0\}}\int_{B(0,3/4)\cap\{x_d>0\}} \frac{x_d^{pq}}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y =: I_1 + I_2.$$

To estimate I_1, we make use of the following observation: if $q \in (0,1)$ and $a \ge b > 0$, then we have

$$a^q - b^q \le q b^{q-1}(a-b). \tag{6.6.7}$$

Therefore, we have

$$I_1 \lesssim \int_{B(0,3/4)\cap\{y_d>0\}}\int_{B(0,3/4)\cap\{x_d>y_d\}} \frac{y_d^{2q-2} x_d^{p(q-2)}}{|x-y|^{d+2s-2}}\,\mathrm{d}x\mathrm{d}y.$$

Since $2s-2<0$, the integrals over x are bounded and thus $q > 1/2$ suffices for I_1 to be finite.

Next, we have

$$I_2 \le \int_{B(0,3/4)\cap\{x_d>0\}} x_d^{pq} \int_{B(0,3/2)\setminus B(0,x_d)} \frac{\mathrm{d}y}{|y|^{d+2s}}\,\mathrm{d}x \approx \int_{B(0,3/4)\cap\{x_d>0\}} x_d^{pq-2s}\,\mathrm{d}x.$$

Thus, I_2 is finite provided that $p > (2s-1)/q$, which is not really a restriction, because with $q > 1/2$ we have $(2s-1)/q \le 4s-2 < 2$ and $p > 2$ is the standing assumption in this result. Therefore we get that if p, q, s satisfy $1/2 < q \le s - \frac{1}{p}$, then $v_q \in \mathcal{V}_D^p$, but $v_q \notin \mathcal{V}^{s,p}(D)$. Also, by (i), $v_q \in \mathcal{W}_D^p \setminus \mathcal{V}^{s,p}(D)$.

(iv) Now consider the functions $u_q(x) = |x|^q \mathbf{1}_{B_R^c}(x)$, $q > 0$ with R so large that $u = 0$ in the neighborhood of D. Both $u_q \in \mathcal{V}_D^p$ and $u_q \in \mathcal{W}_D^p$ are equivalent to $q < \frac{2}{p}s$, while $u_q \in \mathcal{V}^{s,p}(D)$ is equivalent to $q < s$. Therefore, for $q = \frac{2}{p}s$ we have $u_q \in \mathcal{V}^{s,p}(D)$, while $u_q \notin \mathcal{V}_D^p$, $u_q \notin \mathcal{W}_D^p$. □

Appendix A

Apart from Subsection A.1.3, the contents of this Appendix appeared in [21] up to some modifications and explanations.

A.1 Further results from potential theory

We recollect the definitions of the concentration functions for the convenience of the reader.

$$K(r) = \int_{|z|\le r} \frac{|z|^2}{r^2}\nu(z)\,\mathrm{d}z, \qquad h(r) = K(r) + \nu(B_r^c) = \int_{\mathbb{R}^d} \left(\frac{|z|^2}{r^2} \wedge 1\right)\nu(z)\,\mathrm{d}z,$$
$$V(r) = \frac{1}{\sqrt{h(r)}}.$$

A.1.1 Hitting the boundary

Assume that ν satisfies **A2**. Then for every $R \in (0,\infty)$,

$$\nu(\lambda r) \le c\lambda^{-d-\beta}\nu(r), \quad 0 < \lambda \le 1,\ 0 < r \le R. \tag{A.1.1}$$

Indeed, for $r \in (0,1]$ we can take $c = C_2$, and if $1 < r \le R$, then

$$\nu(\lambda r) \le \nu(\lambda 1) \le C_2\lambda^{-d-\alpha}\nu(1) \le C_2\frac{\nu(1)}{\nu(R)}\ \lambda^{-d-\alpha}\nu(r).$$

We have $K > 0$ and $h > 0$. Furthermore, h is strictly decreasing, but $r^2h(r)$ is increasing, which is seen directly from the definition. Thus for $a \ge 1$ and $r > 0$,

$$h(r) \ge h(ar) = (ar)^2h(ar)/(ar)^2 \ge r^2h(r)/(ar)^2 = h(r)/a^2. \tag{A.1.2}$$

Recall that $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in $\mathbb{R}^d$. We obtain

$$K(r) = r^{-2}\int_0^r \omega_d s^{d-1+2}\nu(s)\mathrm{d}s \ge \nu(r)r^d\omega_d/(d+2), \quad r > 0.$$

By (A.1.1), for every $R < \infty$ we get

$$\begin{aligned} K(r) &= r^{-2}\int_0^r \omega_d s^{d+1}\nu(s)\mathrm{d}s \le cr^{-2}\int_0^r \omega_d s^{d+1}\nu(r)(s/r)^{-d-\beta}\mathrm{d}s \\ &= \nu(r)r^d c\omega_d/(2-\beta), \quad r \le R. \end{aligned}$$

Therefore, for every $R \in (0,\infty)$,

$$\nu(r) \approx \frac{K(r)}{r^d}, \quad 0 < r \le R. \tag{A.1.3}$$

It is known that

$$h'(r) = -2K(r)/r, \tag{A.1.4}$$

see Bogdan, Grzywny and Ryznar [26, (3.5)]. If $0 < r \le R/2$, then by (A.1.3) and (A.1.4) we have

$$\begin{aligned}\nu(B_r^c) &= \int_r^\infty \omega_d s^{d-1}\nu(s)\,\mathrm{d}s \ge c\int_r^R s^{d-1}\frac{K(s)}{s^d}\,\mathrm{d}s\\ &= -c\int_r^R h'(s)\,\mathrm{d}s = c(h(r)-h(R)) \ge c\left(1-\frac{h(R)}{h(R/2)}\right)h(r).\end{aligned}$$

Thus, by the definition of h, for every $R \in (0,\infty)$,

$$\nu(B_r^c) \approx h(r), \quad 0 < r \le R/2. \tag{A.1.5}$$

The below proof follows the argument given for the fractional Laplacian by Wu [161, Theorem 1].

Proof of Lemma 2.2.2. The trajectories of X are càdlàg, so locally bounded, therefore

$$\mathbb{P}^x(X_{\tau_D} \in \partial D) = \mathbb{P}^x(\tau_D < \infty, X_{\tau_D} \in \partial D) = \lim_{R\to\infty} \mathbb{P}^x(\tau_D < \tau_{B_R}, X_{\tau_D} \in \partial D).$$

We have $\mathbb{P}^x(\tau_D < \tau_{B_R}, X_{\tau_D} \in \partial D) \le \mathbb{P}^x(X_{\tau_{D\cap B_R}} \in \partial(D\cap B_R))$ for every $R > 0$. Indeed, if $\tau_D < \tau_{B_R}$, then $X_{\tau_D} \in B_R$ and it suffices to note that $\partial D \cap B_R \subset \partial(D\cap B_R)$. Therefore in what follows we may assume that D is bounded and that VDC (2.1.1) holds. Let $a = \max\{(2|B_1|/C_{\mathrm{vdc}})^{1/d}, 2\}$, where C_{vdc} is the constant from (2.1.1). By (A.1.5),

$$\nu(B_r^c) \approx h(r), \quad r \le a^2\mathrm{diam}(D).$$

Here, as usual,

$$\mathrm{diam}(D) = \sup\{|x-y| : x, y \in D\}.$$

For $x \in D$ we let $r_x = \delta_D(x)/2$ and $B_x = B(x, r_x)$. If $Q \in \partial D$ is such that $|x-Q| = \delta_D(x)$, then by (2.1.1),

$$|D^c \cap (B(Q,ar)\setminus B(Q,r))| \ge |B(Q,r)|, \quad r > 0. \tag{A.1.6}$$

By unimodality of ν, (A.1.6), (A.1.5) and then (A.1.2) we get

$$\begin{aligned}\nu(x,D^c) \ge& \sum_{k\ge1}\nu\left(D^c\cap\left(B(Q,a^k r_x)\setminus B(Q,a^{k-1}r_x)\right)-x\right)\\ \ge& \sum_{k\ge1}\nu(a^k r_x + 2r_x)|B(Q,a^{k-1}r_x)|\\ \ge& \sum_{k\ge1}\nu(a^{k+1}r_x)|B(Q,a^{k+2}r_x)\setminus B(Q,a^{k+1}r_x)|\\ =& \sum_{k\ge1}\nu(a^{k+1}r_x)|B(0,a^{k+2}r_x)\setminus B(0,a^{k+1}r_x)|\\ \ge& a^{-3d}\nu(B^c_{a^2r_x}) \approx h(a^2r_x) \approx h(r_x).\end{aligned}$$

The estimates for Poisson kernel for the ball [82, Lemma 2.2] give

$$P_{B_r}(0,z) \gtrsim \frac{\nu(z)}{h(r)}, \quad |z| > r > 0.$$

By (2.2.10) we have $\omega_{B_x}(x,A) := \mathbb{P}^x(X_{\tau_{B_x}} \in A) = \int_A P_{B_{r_x}}(0, z-x)\,\mathrm{d}z$, if $\operatorname{dist}(D,A) > 0$, hence

$$\mathbb{P}^x(X_{\tau_{B_x}} \in D^c) \gtrsim \frac{\nu(x,D^c)}{h(r_x)} \geq c, \tag{A.1.7}$$

where $c > 0$ does not depend on x. Following [161], we write

$$\mathbb{P}^x(X_{\tau_D} \in \partial D) = \mathbb{P}^x(X_{\tau_{B_x}} \in \partial D) + \mathbb{P}^x(X_{\tau_{B_x}} \in D, X_{\tau_D} \in \partial D)$$

The first term vanishes because $|\partial D| = 0$. By the strong Markov property and (A.1.7), the second term is equal to

$$\begin{aligned}\mathbb{E}^x[\mathbb{P}^{X_{\tau_{B_x}}}(X_{\tau_D} \in \partial D)\mathbf{1}_{D\setminus B_x}(X_{\tau_{B_x}})] &\leq \sup_{y\in D} \mathbb{P}^y(X_{\tau_D} \in \partial D)\mathbb{P}^x(X_{\tau_{B_x}} \in D \setminus B_x) \\ &\leq (1-c)\sup_{y\in D} \mathbb{P}^y(X_{\tau_D} \in \partial D).\end{aligned}$$

Thus, for every $x \in D$ we have

$$\mathbb{P}^x(X_{\tau_D} \in \partial D) \leq (1-c)\sup_{y\in D} \mathbb{P}^y(X_{\tau_D} \in \partial D).$$

This implies that $\sup_{x\in D} \mathbb{P}^x(X_{\tau_D} \in \partial D) = 0$.

□

A.1.2 Estimates of the interaction kernel

In the proof of the result for the half-space we often use the following global scalings.

A4 There exist constants $\alpha, \beta \in (0,2)$ and $c > 0$ such that

$$\nu(\lambda r) \leq c\lambda^{-d-\beta}\nu(r), \qquad 0 < \lambda \leq 1,\ r > 0. \tag{A.1.8}$$

$$\nu(\lambda r) \geq c\lambda^{-d-\alpha}\nu(r), \qquad 0 < \lambda \leq 1,\ r > 0. \tag{A.1.9}$$

Note that (A.1.8) is but a global version of (4.2.1), equivalent to $r^{d+\beta}\nu(r)$ being almost increasing on $(0,\infty)$: $p^{d+\beta}\nu(p) \leq cr^{d+\beta}\nu(r)$ if $0 < p < r < \infty$, cf. Bogdan, Grzywny and Ryznar [24, Section 3]. Clearly, **A4** holds true if $L = \Delta^{\alpha/2}$.

We start with some further observations about the functions h, K and V, see the beginning of this appendix, under the global scalings **A4**. Note that V is increasing. If (A.1.8) holds, then by a similar procedure as in (A.1.3) we obtain its global version

$$\nu(s) \approx \frac{K(s)}{s^d}, \quad s > 0. \tag{A.1.10}$$

For $a \in (0,2]$ we denote

$$U_a(s) = \frac{K(s)}{h(s)^a s^d}, \quad s > 0.$$

Due to [82, Theorem 1.2], unimodality of ν and (A.1.10), U_a is almost decreasing, i.e., there is a constant $c_a > 0$ such that for all $0 < s_1 < s_2$ we have $U_a(s_1) \geq c_a U_a(s_2)$, in short $U_a(s_1) \gtrsim U_a(s_2)$. A direct calculation gives

$$-\Big(\frac{1}{V(s)}\Big)' = \frac{V(s)K(s)}{s} \approx V(s)\nu(s)s^{d-1}, \qquad [V^2]'(s) = 2s^{d-1}U_2(s). \tag{A.1.11}$$

The factor s^{d-1} will be useful for integrations in polar coordinates. It is also easy to verify that $s^2h(s)$ is nondecreasing, hence $V(s)/s$ is nonincreasing and for every $\lambda \in (0,1)$ we have

$$V(s) \geq V(\lambda s) \geq \lambda V(s), \quad s > 0. \tag{A.1.12}$$

Recall that for an open $D \subset \mathbb{R}^d$, $r(z,w) = |z-w| + \delta_D(z) + \delta_D(w)$ and $\delta_D(w) = d(w, \partial D)$. Here is our main result for the half-space

$$H = \{x \in \mathbb{R}^d : x_d > 0\}.$$

Theorem A.1.1. *Let $d \geq 3$ and assume that* (A.1.8) *holds true. Then,*

$$\gamma_H(z,w) \lesssim \frac{V^2(r(z,w))\nu(r(z,w))}{V(\delta_H(z))V(\delta_H(w))}.$$

If we additionally assume (A.1.9)*, then*

$$\gamma_H(z,w) \approx \frac{V^2(r(z,w))\nu(r(z,w))}{V(\delta_H(z))V(\delta_H(w))}.$$

The proof of Theorem A.1.1 given below uses the following estimates of the Poisson kernel of the half-space. Below we will use the 'profile' notation for Lévy measure, that is $\nu(|x-y|)$ rather than $\nu(x,y)$. This should help to keep track of the usages of various functions and their properties given above.

Lemma A.1.2. *Let $d \geq 3$. Assume that* (A.1.8) *holds true. Then,*

$$P_H(x,z) \approx \frac{V(\delta_H(x))}{V(\delta_H(z))} V^2(|x-z|)\nu(|x-z|), \quad x \in H,\ z \in \overline{H}^c.$$

Proof. By [82, Theorem 1.13],

$$G_H(x,y) \approx \frac{V(\delta_H(x))}{V(\delta_H(x) + |x-y|)} \frac{V(\delta_H(y))}{V(\delta_H(y) + |x-y|)} U_2(|x-y|), \quad x,y \in H. \tag{A.1.13}$$

From (A.1.8), the Ikeda–Watanabe formula (2.2.12) and the monotonicity properties of V, U_a, ν,

$$\begin{aligned}
\frac{P_H(x,z)}{V(\delta_H(x))} &\lesssim \int_{H\cap\{|x-z|\leq 2|x-y|\}} \frac{V(\delta_H(y))}{V^2(|x-z|/2)} U_2(|x-z|/2)\nu(|y-z|)\,\mathrm{d}y \\
&\quad + \int_{|x-z|>2|x-y|} \frac{1}{V(|x-y|)} U_2(|x-y|)\nu(|x-z|/2)\,\mathrm{d}y \\
&\leq U_1(|x-z|/2) \int_{\overline{B}^c(z,\delta_H(z))} V(|y-z|)\nu(|y-z|)\,\mathrm{d}y \\
&\quad + \nu(|z-x|) \int_{B_{|x-z|/2}} U_{3/2}(|y|)\,\mathrm{d}y
\end{aligned}$$

$$\le U_1(|x-z|/2)\int_{\overline{B}^c_{\delta_H(z)}} V(|y|)\nu(|y|)\mathrm{d}y + \nu(|z-x|)\int_{B_{|x-z|/2}} U_{3/2}(|y|)\mathrm{d}y$$
$$\lesssim \frac{U_1(|x-z|)}{V(\delta_H(z))} + U_{1/2}(|x-z|) = 2\frac{U_1(|x-z|)}{V(\delta_H(z))}.$$

In the last inequality we use (A.1.10) and the formula $h'(r) = -2K(r)/r$, see (A.1.4), which result in

$$\int_r^\infty \frac{K(s)}{h^{1/2}(s)s}\,\mathrm{d}s = h^{1/2}(r) \quad \text{and} \quad \int_0^r \frac{K(s)}{sh^{3/2}(s)}\,\mathrm{d}s = \frac{1}{h^{1/2}(r)}.$$

We next prove a matching lower estimate. Using repeatedly the monotonicity properties of U_a, V, the inequality (A.1.12) and the scaling of ν we see that up to a multiplicative constant, $P_H(x,z)/V(\delta_H(x))$ is not less than

$$\int_{H\cap\{|y-z|\le 2|x-z|\}} \frac{V(\delta_H(y))}{V^2(5|x-z|)} U_2(3|x-z|)\nu(|y-z|)\,\mathrm{d}y$$
$$+\int_{H\cap\{|y-x|\le 2|x-z|\}} \frac{V(\delta_H(y))}{V(\delta_H(y)+|x-y|)V(3|x-z|)} U_2(|x-y|)\nu(3|x-z|)\,\mathrm{d}y$$
$$\gtrsim U_1(5|x-z|)\mathrm{I} + \frac{\nu(|x-z|)}{V(3|x-z|)}\mathrm{I\!I} \gtrsim U_1(|x-z|)\left(\mathrm{I} + \frac{1}{V^3(2|x-z|)}\mathrm{I\!I}\right),$$

where

$$\mathrm{I} = \int_{H\cap\{|y-z|\le 2|x-z|\}} V(\delta_H(y))\nu(|y-z|)\,\mathrm{d}y,$$
$$\mathrm{I\!I} = \int_{H\cap\{|y-x|\le 2|x-z|\}} \frac{V(\delta_H(y))}{V(\delta_H(y)+|x-y|)} U_2(|x-y|)\,\mathrm{d}y.$$

First we estimate the integral I. Without loss of generality we may and do assume that $z = (0,\ldots,0,z_d)$ with $z_d < 0$. Consider the cone $\Gamma = \{(\tilde{y}, y_d) : |\tilde{y}| < y_d\}$. For $y \in \Gamma$ we have $2\delta_H(y) \ge |y-z| - \delta_H(z)$. Hence, by the rotational invariance of ν, (A.1.11) and (A.1.12) we obtain

$$\mathrm{I} \ \ge \int_{\Gamma\cap\{|y-z|\le 2|x-z|\}} V((|y-z|-\delta_H(z))/2)\nu(|y-z|)\,\mathrm{d}y$$
$$\ge c(d)\int_{3\delta_H(z)/2\le|y-z|\le 2|x-z|} V(|y-z|-\delta_H(z))\nu(|y-z|)\,\mathrm{d}y$$
$$\gtrsim \int_{3\delta_H(z)/2}^{2|x-z|} V(s)\nu(s)s^{d-1}\mathrm{d}s \approx \frac{1}{V(3\delta_H(z)/2)} - \frac{1}{V(2|x-z|)}.$$

Similarly,

$$\mathrm{I\!I} \ \ge \int_{|y-x|\le 2(|x-z|\wedge y_d)} \frac{V(\delta_H(y))}{V(3\delta_H(y))} U_2(|x-y|)\,\mathrm{d}y$$
$$\gtrsim \int_{|y-x|\le 2|x-z|} U_2(|x-y|)\,\mathrm{d}y \approx V^2(2|x-z|),$$

where in the second inequality we use the isotropy of U_2 and the inclusion

$$\{y : |y-x| \le 2y_d\} \supset \{y : |y-x| \le 2(y_d - x_d)_+\} \supset x + \Gamma.$$

Hence, up to a multiplicative constant, $P_H(x,z)/V(\delta_H(x))$ is not less than

$$U_1(|x-z|)\left(\frac{1}{V(3\delta_H(z)/2)}-\frac{1}{V(2|x-z|)}+\frac{1}{V(2|x-z|)}\right)\geq\frac{U_1(|x-z|)}{V(\delta_H(z))}.$$

Since $U_1(s)\approx\nu(s)V^2(s)$, the proof is complete. $\square$

Proof of Theorem A.1.1. By the definition of γ_D, see (2.2.20), and Lemma A.1.2 we have

$$\gamma_H(z,w)\approx\frac{1}{V(\delta_H(z))}\int_H V(\delta_H(x))V^2(|x-z|)\nu(|z-x|)\nu(|w-x|)\,\mathrm{d}x. \tag{A.1.14}$$

Let $\tilde{z}\in H$ be the reflection of $z\in H^c$ in the hyperplane $\{x_d=0\}$. Then $|w-\tilde{z}|\approx r(z,w)$ and for $x\in H$ we have $|x-\tilde{z}|<|x-z|$, and $\delta_H(\tilde{z}),\delta_H(x)\leq|x-z|$. Consequently, by (A.1.14), the estimates of the Green function (A.1.13) and Lemma A.1.2 we get

$$\gamma_H(z,w)V^2(\delta_H(\tilde{z}))\approx\int_H\frac{V(\delta_H(x))V(\delta_H(\tilde{z}))}{V^2(\delta_H(\tilde{z})+|x-z|)}V^4(|x-z|)\nu(|z-x|)\nu(|w-x|)\,\mathrm{d}x \tag{A.1.15}$$

$$\lesssim\ P_H(\tilde{z},w)\approx\frac{V(\delta_H(\tilde{z}))}{V(\delta_H(w))}V^2(|\tilde{z}-w|)\nu(|\tilde{z}-w|). \tag{A.1.16}$$

We next assume (A.1.9) and prove the matching lower bound. It suffices to replace z with $\tilde{z}$ in the right-hand side of (A.1.15) because then we have approximation $\approx$ instead of inequality $\lesssim$ in (A.1.16). Thus, we are going to prove that the integral with $\tilde{z}$ in place of z is comparable to the original one. We once again use (A.1.12) and obtain

$$\int_{B(\tilde{z},\delta_H(z)/2)}\frac{V(\delta_H(x))V(\delta_H(\tilde{z}))}{V^2(\delta_H(\tilde{z})+|x-z|)}V^4(|x-z|)\nu(|z-x|)\nu(|w-x|)\,\mathrm{d}x$$
$$\approx V^4(\delta_H(z))\nu(\delta_H(z))\nu(|w-\tilde{z}|)\delta_H(z)^d. \tag{A.1.17}$$

For the integrand with $\tilde{z}$ we have

$$\int_{B(\tilde{z},\delta_H(z)/2)}\frac{V(\delta_H(x))V(\delta_H(\tilde{z}))}{V^2(\delta_H(\tilde{z})+|x-\tilde{z}|)}V^4(|x-\tilde{z}|)\nu(\tilde{z}-x)\nu(|w-x|)\,\mathrm{d}x$$
$$\approx\nu(|w-\tilde{z}|)\int_{B_{\delta_H(z)/2}}V^4(|x|)\nu(|x|)\,\mathrm{d}x\approx V^2(\delta_H(z))\nu(r(z,w)). \tag{A.1.18}$$

The last comparison follows from $V^4(s)\nu(s)s^{d-1}\approx[V^2]'(s)$, cf. (A.1.11). Since (A.1.9) gives $\nu(r)r^dV^2(r)\approx 1$, cf. (A.1.10) and the definitions of V and h, the right-hand sides of (A.1.17) and (A.1.18) are comparable. We have $|x-\tilde{z}|\approx|x-z|$, for $x\in H$ such that $|x-\tilde{z}|\geq\delta_H(z)/2$. Therefore we can replace z by $\tilde{z}$ in the integrand in (A.1.15), and so

$$\gamma_H(z,w)\approx\frac{P_H(\tilde{z},w)}{V^2(\delta_H(\tilde{z}))}\approx\frac{V^2(r(z,w))}{V(\delta_H(z))V(\delta_H(w))}\nu(r(z,w)).$$

$\square$

The result for the bounded $C^{1,1}$ open sets has a similar proof, so we will be brief.

Proof of Theorem 4.2.5. Let D be $C^{1,1}$ at scale $2R > 0$, cf. Lemma 2.1.4 and recall that we assume **A2** and **A3** given in Section 4.2. Obviously, $R \le \operatorname{diam}(D)$.
(i) First we let $\delta_D(z), \delta_D(w) \ge R$. By using (2.2.17) and the unimodality of ν, we obtain

$$\nu(\delta_D(w) + \operatorname{diam}(D))\mathbb{E}^x\tau_D \le P_D(x,w) \le \nu(\delta_D(w))\mathbb{E}^x\tau_D.$$

By (4.2.2),

$$\nu\left(\delta_D(w) + \operatorname{diam}(D)\right) \approx \nu(\delta_D(w)).$$

These imply

$$\gamma_D(z,w) \approx \nu(\delta_D(z))\nu(\delta_D(w)) \int_D \mathbb{E}^x\tau_D \,\mathrm{d}x,$$

which ends the proof in the first case.
(ii) We next assume that $\delta_D(z) \le R \le \delta_D(w)$. We get

$$\gamma_D(z,w) \approx \nu(\delta_D(w)) \int_D \mathbb{E}^x\tau_D\nu(|z-x|)\,\mathrm{d}x.$$

Let $A = B(z, 2\operatorname{diam}(D)) \setminus \overline{B(z,\delta_D(z))} \supseteq D$. Note that for $x \in A$ we have $\delta_A(x) \le |x-z|$. By [26, Theorem 4.6, Lemma 7.2, Proposition 5.2] and (A.1.5), we have

$$\begin{aligned}\int_D \mathbb{E}^x\tau_D\nu(|z-x|)\,\mathrm{d}x &\le \int_A \mathbb{E}^x\tau_A\nu(|z-x|)\,\mathrm{d}x \lesssim \int_A V(\delta_A(x))\nu(|z-x|)\,\mathrm{d}x \\ &\le \int_{\overline{B(0,\delta_D(z))}^c} V(|y|)\nu(y)\,\mathrm{d}y.\end{aligned}$$

Using [26, Lemma 3.5] we obtain

$$\gamma_D(z,w) \le c\nu(\delta_D(w))\frac{1}{V(\delta_D(z))}.$$

Before we proceed with the lower bound and the remaining case, we recall that by [25, Proposition 4.4 and Theorem 4.5] the Dirichlet heat kernel of D (2.2.5) satisfies

$$p_t^D(x,y) \approx e^{-\lambda(D)t}\left(\frac{V(\delta_D(x))}{\sqrt{t}\wedge V(r)} \wedge 1\right)\left(\frac{V(\delta_D(x))}{\sqrt{t}\wedge V(r)} \wedge 1\right) p_{t\wedge V^2(r)}(x,y), \quad t>0,\, x,y \in D,$$

where $\lambda(D) \approx 1/V^2(R)$. By integrating against time we get rather standard estimates of the Green function, cf. Chen, Kim and Song [43, proof of Theorem 7.3]. For instance if $d \ge 2$, then

$$G_D(x,y) \approx U_2(|x-y|)\Big(\frac{V(\delta_D(x))V(\delta_D(y))}{V^2(|x-y|)} \wedge 1\Big), \quad x,y \in D.$$

The Ikeda–Watanabe formula yields estimates for the Poisson kernel (cf. Kang and Kim [101, Theorem 2.6] and [26, Proposition 2.4]),

$$P_D(x,z) \approx \frac{V(\delta_D(x))}{V(\delta_D(z))}\frac{1}{|x-z|^d}, \quad x \in D,\, \delta_D(z) < R. \tag{A.1.19}$$

Since D is $C^{1,1}$, there is $x_0 \in D$ such that $B = B(x_0,R) \subset D$. Thus, by using (A.1.19) we obtain

$$\int_D \mathbb{E}^x\tau_D\nu(|z-x|)\,\mathrm{d}x = \int_D P_D(x,z)\,\mathrm{d}x \gtrsim \int_{B(x_0,R/2)} \frac{V(\delta_D(x))}{V(\delta_D(z))}\frac{1}{|x-z|^d}\,\mathrm{d}x \gtrsim \frac{1}{V(\delta_D(z))}.$$

Therefore,

$$\gamma_D(z,w) \approx \nu(\delta_D(w)) \frac{1}{V(\delta_D(z))}.$$

(iii) The case $\delta_D(z), \delta_D(w) < R$ requires calculations which are almost identical to those from the proof of Theorem A.1.1. The only slight difference is in the geometrical arguments, therefore we briefly discuss them. First of all, the reflection through the boundary is well-defined for the considered z, and if $\tilde{z}$ is the reflection of z, then $\delta_D(z) = \delta_D(\tilde{z})$, cf. the discussion preceding Definition 4.5.1.

We will show that $|w-\tilde{z}| \approx r(z,w)$. First, assume that $|w-z| < 3\delta_D(z)$. Then we also have $\delta_D(w) \lesssim \delta_D(z)$ and consequently $|w-\tilde{z}| \geq \delta_D(z) \gtrsim r(z,w)$. For $|w-z| \geq 3\delta_D(z)$ we consider two cases. Assume that $|w-z| > 3\delta_D(w)$. Then $|w-\tilde{z}| \geq |w-z| - 2\delta_D(z) \geq \frac{1}{3}|w-z| \gtrsim r(z,w)$. If on the other hand $|w-z| \leq 3\delta_D(w)$, then $|w-\tilde{z}| \geq \delta_D(w) \gtrsim r(z,w)$.

Obviously, $\delta_D(\tilde{z}), \delta_D(x) \leq |x-z|$. We also have $|x-\tilde{z}| \lesssim |x-z|$. Indeed, it suffices to consider the cases when $|x-\tilde{z}|$ is smaller and greater than $\delta_D(z)$. Furthermore, $|x-\tilde{z}| \gtrsim |x-z|$ provided that $|x-\tilde{z}| \geq \delta_D(z)/2$, because we have $|x-z| \leq |x-\tilde{z}| + 2\delta_D(z)$. □

A.1.3 The Green function

As announced in Subsection 2.2.2, we discuss the finiteness of the Green function G_D. First, recall that the domain monotonicity holds — if $U \subseteq D$, then $G_U \leq G_D$. Therefore in order to prove the finiteness of G_D for proper open $D \subset \mathbb{R}^d$, it suffices to show that either $G_{\{0\}^c}$ or $G_{\mathbb{R}^d}$ is finite. For $d \geq 3$ all nondegenerate Lévy processes are transient, that is, the potential kernel for the whole space $G_{\mathbb{R}^d}$ is finite, see [139, Theorem 35.4 and Theorem 37.8]. For $d = 1,2$, the case of the bounded sets D is resolved by [82, Theorem 1.3], see also [81, Theorem A.4]. Below we will impose certain assumptions on ν in order to obtain the finiteness of G_D also for unbounded D, but we note that the following discussion is only an informative digression irrelevant to the results of this dissertation, cf. the discussion following (2.2.7).

Let $d = 2$. Recall the Chung–Fuchs criterion [139, Corollary 37.6], which says that the isotropic process (X_t) is transient if and only if its Lévy–Khinchine exponent ψ (cf. (2.2.1)) satisfies

$$\int_{B_1} \frac{\mathrm{d}x}{\psi(x)} < \infty. \tag{A.1.20}$$

Obviously, it holds if $\psi(x)$ is bounded from below by a multiple of $|x|^\alpha$ for some $\alpha \in (0,2)$ near the origin, but below we will give a sufficient condition for (A.1.20) in terms of ν. By [26, Proposition 2.4] and the definition of h we have

$$\psi(1/r) \approx h(r) \geq \nu(B_r^c), \tag{A.1.21}$$

see also the first paragraph of [26, Section 3]. Therefore, by introducing the polar coordinates we get

$$\int_{B_1} \frac{\mathrm{d}x}{\psi(x)} \approx \int_0^1 \frac{s}{\psi(s)}\,\mathrm{d}s = \int_1^\infty \frac{\mathrm{d}t}{t^3\psi(1/t)} \leq \int_1^\infty \frac{\mathrm{d}t}{t^3\nu(B_t^c)}. \tag{A.1.22}$$

Thus, if we assume that the scaling (A.1.8) holds, then for $t \gtrsim 1$,

$$\nu(B_t^c) \approx \int_t^\infty s\nu(s)\,\mathrm{d}s = t^2 \int_1^\infty u\nu(tu)\,\mathrm{d}u = t^2 \int_1^\infty u\nu(u)\frac{\nu(tu)}{\nu(u)}\,\mathrm{d}u \gtrsim t^{-\beta}$$

for some $\beta \in (0,2)$. Consequently the integral on the right-hand side of (A.1.22) converges. We remark that the above argument requires the scaling of ν only at infinity.

For $d = 1$ we have the result by Grzywny and Ryznar [83, Proposition 2.3] which gives the finiteness of $G_{\{0\}^c}$, see also the references therein. These facts are obtained under the assumption

$$\int_0^\infty \frac{\mathrm{d}t}{1+\psi(t)} < \infty.$$

We will show that this condition holds provided that **A3** holds with $\alpha > 1$. Note that equivalently we may study the finiteness of $\int_1^\infty$. By (A.1.5) the inequality in (A.1.21) becomes an equivalence for $r < 1$, and thus we have

$$\int_1^\infty \frac{\mathrm{d}t}{1+\psi(t)} \le \int_0^1 \frac{\mathrm{d}t}{t^2\psi(1/t)} \approx \int_0^1 \frac{\mathrm{d}t}{t^2\nu(B_t^c)}. \tag{A.1.23}$$

Assume the scaling condition from **A3** and let $R > 1$ be fixed. Then, for $t \in (0, 1)$,

$$\nu(B_t^c) \ge \nu(B_R \setminus B_t) = \int_t^R \nu(s)\,\mathrm{d}s = \nu(R)\int_t^R \frac{\nu(s)}{\nu(R)}\,\mathrm{d}s \gtrsim \int_t^R s^{-1-\alpha}\,\mathrm{d}s \approx t^{-\alpha}.$$

We see that the right-hand side of (A.1.23) is finite provided that $\alpha > 1$. Thus, with such scaling we get that $G_{\{0\}^c}$ is finite. Another way to obtain the finiteness of the Green function (for a different class of Lévy measures) is by repeating the argument from the case $d = 2$. By doing it, we may show that if (A.1.8) holds with $\beta < 1$, then the condition (A.1.20) holds, and as a consequence $G_{\mathbb{R}}$ is finite.

A.2 The core of the Dirichlet form for the Lévy process

Recall that $\widetilde{\mathcal{V}}_D = \mathcal{V}_D \cap L^2(D)$ for $D \subseteq \mathbb{R}^d$. The following result is well-known, see [139, Theorem 31.5], but we present a short and straightforward proof for the sake of completeness.

Lemma A.2.1. *The class $C_c^\infty(\mathbb{R}^d)$ is a core of the Dirichlet form $(\mathcal{E}, \widetilde{\mathcal{V}}_{\mathbb{R}^d})$ on $L^2(\mathbb{R}^d)$.*

Obviously, the functions from $C_c^\infty(\mathbb{R}^d)$ are dense in the uniform norm in $C_c(\mathbb{R}^d)$. In order to show that a function $u \in \widetilde{\mathcal{V}}_{\mathbb{R}^d}$ can be approximated by the test functions, we do a cut-off and then we convolve it with a smooth, compactly supported mollifier. Below we show that these operations are continuous in an appropriate sense.

Consider a sequence of smooth functions q_j, $j \in \mathbb{N}$ such that $0 \le q_j \le 1$, $q_j = 1$ in B_j, $q_j = 0$ in B_{j+1}^c and such that $|\nabla q_j(x)| < M$, $x \in \mathbb{R}^d$, $j = 1, 2, \ldots$.

Lemma A.2.2 (Cut-off)**.** *For every $u \in \widetilde{\mathcal{V}}_{\mathbb{R}^d}$, $q_j u \to u$ as $j \to \infty$.*

Proof. The convergence in L^2 follows immediately from the dominated convergence theorem. Since $|(q_j u)(x) - (q_j u)(x+y) - u(x) + u(x+y)| \le |(1 - q_j(x))(u(x+y) - u(x))| + |(q_j(x) - q_j(x+y))u(x+y)|$, we get

$$\mathcal{E}[q_j u - u] \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (1 - q_j(x))^2 (u(x) - u(x+y))^2\,\mathrm{d}\nu(y)\mathrm{d}x \tag{A.2.1}$$

$$+ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (q_j(x) - q_j(x+y))^2 u(x+y)^2\,\mathrm{d}\nu(y)\mathrm{d}x. \tag{A.2.2}$$

The integrands in (A.2.1) and (A.2.2) converge to 0 *a.e.* as $j \to \infty$. For (A.2.1) we have $(q_j(x) - 1)^2(u(x) - u(x+y))^2 \le (u(x) - u(x+y))^2$, which is integrable against $\mathrm{d}\nu(y)\mathrm{d}x$ since $u \in \widetilde{\mathcal{V}}_{\mathbb{R}^d}$. For (A.2.2) we use the smoothness of q_j:

$$(q_j(x) - q_j(x+y))^2 u(x+y)^2 \le C(1 \wedge |y|^2)u(x+y)^2,$$

and so

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(1\wedge|y|^2)u(x+y)^2\,\mathrm{d}x\mathrm{d}\nu(y)=\int_{\mathbb{R}^d}(1\wedge|y|^2)\,\mathrm{d}\nu(y)\int_{\mathbb{R}^d}u(x)^2\,\mathrm{d}x<\infty.$$

By the dominated convergence theorem we obtain the desired result. □

Let $\eta\in C_c^\infty(B_1)$ be a nonnegative radial function on $\mathbb{R}^d$ satisfying $\int\eta=1$, and let $\eta_\varepsilon(x)=\varepsilon^{-d}\eta(x/\varepsilon)$ for $\varepsilon>0$, $x\in\mathbb{R}^d$.

Lemma A.2.3 (Mollification)**.** *For every $u\in\mathcal{V}_{\mathbb{R}^d}$, $\mathcal{E}[\eta_\varepsilon*u-u]\to 0$ as $\varepsilon\to 0^+$.*

Proof. It suffices to verify that $\mathrm{I}:=\iint_{\mathbb{R}^d\times\mathbb{R}^d}(u*\eta_\varepsilon(x)-u(x)-u*\eta_\varepsilon(x+y)+u(x+y))^2\,\mathrm{d}\nu(y)\mathrm{d}x\to 0$ as $\varepsilon\to 0^+$. By Fubini–Tonelli theorem and Jensen's inequality,

$$\begin{aligned}\mathrm{I}&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\int_{B_1}(u(x-\varepsilon z)-u(x+y-\varepsilon z)-u(x)+u(x+y))\,\eta(z)\,\mathrm{d}z\right)^2\mathrm{d}\nu(y)\mathrm{d}x\\&\le\int_{B_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big(u(x-\varepsilon z)-u(x+y-\varepsilon z)-u(x)+u(x+y)\Big)^2\mathrm{d}x\mathrm{d}\nu(y)\,\eta(z)\mathrm{d}z.\end{aligned}$$

We will apply the dominated convergence theorem to the integral over $B_1\times\mathbb{R}^d$. By the translation invariance of the Lebesgue measure,

$$\eta(z)\int_{\mathbb{R}^d}(u(x-\varepsilon z)-u(x+y-\varepsilon z)-u(x)+u(x+y))^2\,\mathrm{d}x\le 4\eta(z)\int_{\mathbb{R}^d}(u(x)-u(x+y))^2\,\mathrm{d}x,$$

which is integrable against $\mathrm{d}\nu(y)\mathrm{d}z$. Furthermore, by the continuity of translations in $L^2(\mathbb{R}^d)$ the expression on the left-hand side converges to 0 as $\varepsilon\to 0^+$, for every $z\in B_1$, and $y\in\mathbb{R}^d$. This ends the proof. □

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