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Reduction of integration domain in Triebel–Lizorkin spaces

by

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Abstract. We investigate the comparability of generalized Triebel–Lizorkin and Sobolev seminorms on uniform and nonuniform domains when the integration domain is truncated according to the distance from the boundary. We provide numerous examples of kernels and domains in which the comparability does and does not hold.

1. Introduction. Let $\Omega \subset \mathbb{R}^d$ be a domain, $d \geq 1$, and let $p, q \in (1, \infty)$. Let $K\colon \mathbb{R}^d \times \mathbb{R}^d \to (0, \infty]$ be a homogeneous, radial kernel, i.e. $K(x, y) = k(|x-y|)$, satisfying $\int_{\mathbb{R}^d}(1 \wedge |y|^q)K(0, y)\,dy < \infty$. We define the (generalized) Triebel–Lizorkin space on Ω as

$$(1.1)\qquad F_{p,q}(\Omega) := \Big\{ f \in L^p(\Omega) : \int_\Omega \Big(\int_\Omega |f(x) - f(y)|^q K(x, y)\,dy \Big)^{p/q} dx < \infty \Big\}.$$

The space $F_{p,q}(\Omega)$ obviously depends on K, but we skip it in the notation for simplicity.

$F_{p,q}(\Omega)$ is endowed with the norm

$$\|f\|_{F_{p,q}(\Omega)} = \|f\|_{L^p(\Omega)} + \Big(\int_\Omega \Big(\int_\Omega |f(x) - f(y)|^q K(x, y)\,dy \Big)^{p/q} dx \Big)^{1/p}.$$

We are interested in the Gagliardo-type seminorm

$$(1.2)\qquad \Big(\int_\Omega \Big(\int_\Omega |f(x) - f(y)|^q K(x, y)\,dy \Big)^{p/q} dx \Big)^{1/p},$$

which will be called the *full seminorm*. Let $\theta \in (0, 1]$ and let $\delta(x) = d(x, \partial\Omega)$.

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Our main goal is to establish the comparability of the full seminorm and the *truncated seminorm*

$$\Big(\int_{\Omega}\Big(\int_{B(x,\theta\delta(x))}|f(x)-f(y)|^q K(x,y)\,dy\Big)^{p/q}dx\Big)^{1/p} \tag{1.3}$$

for sufficiently regular K and Ω. Later on, such occurrence will be called a *comparability result.*

Here is our first comparability result.

THEOREM 1.1. *Assume that Ω is a uniform domain, and that K satisfies* **A1**–**A3** *formulated in Subsection* 2.1 *below. Assume that* $1 < q \le p < \infty$. *Then for every* $0 < \theta \le 1$,

$$\Big(\int_{\Omega}\Big(\int_{\Omega}|f(x)-f(y)|^q K(x,y)\,dy\Big)^{p/q}dx\Big)^{1/p} \approx \Big(\int_{\Omega}\Big(\int_{B(x,\theta\delta(x))}|f(x)-f(y)|^q K(x,y)\,dy\Big)^{p/q}dx\Big)^{1/p}.$$

The comparability constant depends on p, q, θ, Ω, and the constants in assumptions **A2**, **A3**.

This is a generalization of the result of Prats and Saksman [21, Theorem 1.6] who proved it for the kernels of the form $K(x,y) = |x-y|^{-d-qs}$ for $s \in (0,1)$. Recently, we were informed that this classical case can also be resolved using the much earlier result of Seeger [24, Corollary 2].

Another result of this flavor was established by Dyda [11, (13)] and was used to obtain Hardy inequalities for nonlocal operators. More recent results on reduction of integration domain in fractional Sobolev spaces include those of Bux, Kassmann, and Schulze [7] who consider certain cones with apex at x instead of $B(x,\delta(x))$, and Chaker and Silvestre [8].

Here we mention that, independently of our work, Kassmann and Wagner [18] have also proved comparability results which extend the ones from [21], allowing for kernels with scaling conditions for $p = q = 2$. However, their overall aim and scope are different than ours.

In Theorem 1.1 we adapt the method of proof from [21] to a wide class of kernels of the form $K(x,y) = |x-y|^{-d}\phi(|x-y|)^{-q}$. The most technical assumption **A2** is tailored for the key Lemmas 2.2, 2.3, however in Subsection 4.2 we argue that it amounts to at least power-type decay of ϕ at 0, and for unbounded Ω to at least power-type growth at ∞.

Notably, we go beyond the uniform domains, where the methods used by Prats and Saksman are no longer available. Namely, we prove that the comparability may hold for fractional Sobolev spaces in strip domains.

THEOREM 1.2. *Assume that* $p = q = 2$. *Let* $\Omega = \mathbb{R}^k \times (0,1)^l \subseteq \mathbb{R}^{k+l}$ *with* $k, l > 0$. *For* $d = k + l$ *let* $K(x,y) = |x-y|^{-d-\alpha}$ *with* $\alpha \in (0,2)$. *If* $k - l - \alpha < -1$, *then the seminorms* (1.2) *and* (1.3) *are comparable.*

We also construct a counterexample for $\alpha < 1$ and $k = l = 1$. This shows an intriguing interplay between the kernel and the width of the domain. Heuristically, it can be seen both in Theorem 1.2 and in Subsection 4.2 that the comparability holds if the stochastic process corresponding to the jump kernel $K \cdot \mathbf{1}_{\Omega\times\Omega}$ and the shape of the domain Ω favor small jumps over large jumps. We remark that the connection between the jump kernel and the stochastic process is a delicate matter. In Section 7 we present a short discussion of this subject and we place our comparability results in this context.

Another object of our studies is the 0-order kernel $K(x,y) \approx |x-y|^{-d}$. We provide examples showing that in this case the comparability does not hold. In an attempt to repeat the proof of Theorem 1.1 we obtain an estimate of (1.2) by a truncated seminorm with a slightly more singular kernel (see Theorem 5.1 below).

The classical Triebel–Lizorkin spaces were introduced independently by Lizorkin [20] and Triebel [26]. The original definition is formulated using Paley–Littlewood theory and is widely used in analysis and applications (see e.g. [1, 6, 15]). For various cases of p, q, d and Ω the classical definition was proved to be equivalent to (1.1) with $K(x,y) = |x-y|^{-d-sq}$, where $s \in (0,1)$ (see [21, 25, 27]).

The definition (1.1) seems more natural if the starting point is $p = q = 2$, e.g., in the case of fractional Sobolev spaces in nonlocal PDEs [9, 12, 23], or Dirichlet forms for Hunt processes [4, 14]. It is also a suitable definition for kernels K more general than $|x-y|^{-d-sq}$, which is also of interest in the field of nonlocal operators and stochastic processes. In this paper we will not attempt to characterize the definition (1.1) in the spirit of classical definitions by Triebel and Lizorkin in full generality. However, we use Fourier methods in Section 5, where we compare spaces with kernels which are only slightly different from each other.

As we argue further in the article, the comparability results can be used to study a class of stochastic processes whose jumps from x are restricted to the ball $B(x, \theta\delta(x))$. The truncated seminorms may also prove useful in peridynamics, as $B(x, \theta\delta(x))$ may be understood as the *variable horizon*; see e.g. [3, 19], and in particular Du and Tian [10] where horizons depending on the distance from the boundary are studied.

The paper is organized as follows. Section 2 contains the notions, assumptions, and basic facts used further in our work. Section 3 is devoted to proving Theorem 1.1. In Section 4 we present positive and negative examples of kernels as regards the comparability results. Section 5 contains the analysis of 0-order kernels. In Section 6 we consider strip domains, in particular we prove Theorem 1.2. Section 7 presents the connection of our development with the theory of Hunt processes.

2. Preliminaries and assumptions

2.1. Assumptions on the kernel. We will consider exponents $1 < q \le p < \infty$ and our assumptions depend on them. As usual, $p' = \frac{p}{p-1}$ is the Hölder conjugate of p. We also let

$$N(r) = \inf\{k \in \mathbb{N} : 2^k r > \operatorname{diam}(\Omega)\}, \quad r > 0.$$

We have $N(r) = \infty$ for every $r > 0$ if and only if Ω is unbounded.

We assume that the kernel K is of the form $K(x,y) = |x-y|^{-d}\phi(|x-y|)^{-q}$, where $\phi\colon (0,\infty) \to (0,\infty)$ satisfies

A1. $(1 \wedge |y|^q)|y|^{-d}\phi(|y|)^{-q} \in L^1(\mathbb{R}^d)$,

A2. ϕ is increasing and there exists $C_2 > 0$ such that for $t_1 = \min(q, p-p/q)$, $t_2 = \frac{1}{q-1}$, and for every $0 < r < \operatorname{diam}(\Omega)$, we have

$$\sum_{k=1}^{N(r)} \frac{\phi(r)^{t_1}}{\phi(2^k r)^{t_1}} \le C_2, \qquad \sum_{k=1}^{\infty} \frac{\phi(2^{-k} r)^{t_2}}{\phi(r)^{t_2}} \le C_2.$$

A3. There exists $C_3 \ge 1$ such that $\phi(2r) \le C_3\phi(r)$ for $0 < r < 3\operatorname{diam}(\Omega)$.

In particular, we allow unbounded domains in which the scaling conditions **A2**, **A3** become global. Note that **A1** is a Lévy-measure-type condition, which ensures the finiteness of (1.2) for smooth, compactly supported f. If $q = 2$ and $\phi(r) = r^s$, $s \in (0,1)$, then K corresponds to the fractional Laplacian of order s and all the assumptions are satisfied. The conditions **A2** and **A3** imply a certain scaling for K; see Subsection 4.2 for the details. The exponents t_1 and t_2 in **A2** stem from the five instances of usage of Lemmas 2.2 and 2.3 in the proof of Theorem 1.1. Since ϕ is increasing, the bounds in **A2** hold for all larger exponents in place of t_1 and t_2. We note the following consequence of **A3** and the monotonicity of ϕ, frequently used below: if $x \lesssim y$, then $\phi(y)^{-1} \lesssim \phi(x)^{-1}$.

2.2. Whitney decomposition and uniform domains. For cubes Q, R in $\mathbb{R}^d$ we consider $l(R)$, the length of the side of R, and the *long distance* between Q and R: $D(Q,R) = l(Q) + d(Q,R) + l(R)$, where d is the Euclidean distance. The scaling of the cube is done from its center x_Q.

We say that a family of (closed) dyadic cubes $\mathcal{W}$ is a *Whitney decomposition* of Ω if for every $Q, S \in \mathcal{W}$,

- if $Q \ne S$, then $\operatorname{int}(Q) \cap \operatorname{int}(S) = \emptyset$;
- if $Q \cap S \ne \emptyset$, then $l(Q) \le 2l(S)$;

- if $Q \subseteq 5S$, then $l(S) \leq 2l(Q)$;
- there is a constant $C_{\mathcal{W}}$ such that $C_{\mathcal{W}} l(Q) \leq d(Q, \partial\Omega) \leq 4C_{\mathcal{W}} l(Q)$.

A sequence $(Q, R_1, \ldots, R_n, S)$ of cubes is a *chain* connecting Q and S, if every cube is a neighbor of its successor and predecessor (if any), by which we mean that their boundaries have nonempty intersection. We will denote the chain by $[Q, S]$ and the sum of the side lengths of its cubes by $l([Q, S])$. We let $[Q, S) = [Q, S] \setminus \{S\}$.

The Whitney decomposition is *admissible* if there exists $\varepsilon > 0$ such that for every pair of cubes Q, S, there exists an *ε-admissible chain* $[Q, S] = (Q_1, \ldots, Q_n)$, i.e.

- $l([Q, S]) \leq \frac{1}{\varepsilon} D(Q, S)$,
- there exists $j_0 \in \{1, \ldots, n\}$ such that $l(Q_j) \geq \varepsilon D(Q, Q_j)$ for every $1 \leq j \leq j_0$, and $l(Q_j) \geq \varepsilon D(Q_j, S)$ for every $j_0 \leq j \leq n$; Q_{j_0} will be denoted as Q_S and called the *central cube* of the chain $[Q, S]$.

A domain which has an admissible Whitney decomposition is called a *uniform domain.* Unless otherwise stated, $[Q, S]$ is an arbitrary (ε-)admissible chain connecting Q and S.

The *shadow* of a cube is $\mathbf{Sh}_\rho(Q) = \{S \in \mathcal{W} : S \subseteq B(x_Q, \rho l(Q))\}$, $\rho > 0$. We also denote $\mathbf{SH}_\rho(Q) = \bigcup \mathbf{Sh}_\rho(Q)$. Note that we can take a sufficiently large ρ_ε so that

- for every ε-admissible chain $[Q, S]$, and every $P \in [Q, Q_S]$, we have $Q \in \mathbf{Sh}_{\rho_\varepsilon}(P)$,
- if $[Q, S]$ is ε-admissible, then every cube from it belongs to $\mathbf{Sh}_{\rho_\varepsilon}(Q_S)$,
- for every $Q \in \mathcal{W}$, $5Q \subseteq \mathbf{SH}_{\rho_\varepsilon}(Q)$.

From now on we fix ρ_ε and write $\mathbf{Sh}(Q) = \mathbf{Sh}_{\rho_\varepsilon}(Q)$ and $\mathbf{SH}(Q) = \mathbf{SH}_{\rho_\varepsilon}(Q)$.

REMARK 2.1. The proofs throughout the paper involve numerous '$\lesssim$' and '$\gtrsim$' signs. We stress that any comparability for ϕ stems from **A2** and **A3**. In particular, for fixed p, q the constants can be chosen to depend only on the geometry of Ω (including the dimension) and on the constants in **A2** and **A3** wherever ϕ is used.

The next lemma provides some inequalities for the noncentered Hardy–Littlewood maximal operator (denoted by M) with connection to the kernel K. It is inspired by the results of [21, Section 2] and Prats and Tolsa [22, Section 3].

LEMMA 2.2. *Let Ω be a domain with Whitney covering $\mathcal{W}$ and let ϕ satisfy* **A1**–**A3**. *Assume $g \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ is nonnegative and $0 < r < 3\,\mathrm{diam}(\Omega)$.*

For every $\eta \geq \min(q, p - p/q)$, $Q \in \mathcal{W}$ *and* $x \in \Omega$, *we have*

$$\int\limits_{\Omega\cap\{|x-y|>r\}} \frac{g(y)\,dy}{|x-y|^d \phi(|x-y|)^\eta} \lesssim \frac{Mg(x)}{\phi(r)^\eta}, \tag{2.1}$$

$$\sum_{S: D(Q,S)>r} \frac{\int_S g(y)\,dy}{D(Q,S)^d \phi(D(Q,S))^\eta} \lesssim \frac{\inf_{x\in Q} Mg(x)}{\phi(r)^\eta}, \tag{2.2}$$

$$\sum_{S\in\mathcal{W}} \frac{l(S)^d}{D(Q,S)^d \phi(D(Q,S))^\eta} \lesssim \frac{1}{\phi(l(Q))^\eta}. \tag{2.3}$$

Proof. Let us look at (2.1). For clarity, assume that $\Omega \ni x = 0$. Since $1/\phi$ is decreasing, we get

$$\begin{aligned}\int\limits_{\Omega\cap\{|y|>r\}} \frac{\phi(r)^\eta g(y)\,dy}{|y|^d\phi(|y|)^\eta} &\leq \sum_{k=1}^{N(r)} \int\limits_{2^{k-1}r<|y|<2^k r} \frac{g(y)}{|y|^d}\frac{\phi(r)^\eta}{\phi(|y|)^\eta}\,dy \\ &\lesssim \sum_{k=1}^{N(r)} \frac{\phi(r)^\eta}{\phi(2^{k-1}r)^\eta}\frac{1}{|B_{2^k r}|} \int\limits_{2^{k-1}r<|y|<2^k r} g(y)\,dy \leq \sum_{k=1}^{N(r)} \frac{\phi(r)^\eta}{\phi(2^{k-1}r)^\eta} Mg(0).\end{aligned}$$

The sum is bounded with respect to r thanks to **A2**.

In order to prove (2.2) note that if $D(Q,S) > r$, then for all $x \in Q$ and $y \in S$, we have $|x-y| + r \lesssim D(Q,S)$. Therefore, by **A3** and the fact that ϕ is increasing, for every $x \in Q$ we have

$$\begin{aligned}\sum_{S: D(Q,S)>r} \frac{\phi(r)^\eta \int_S g(y)\,dy}{D(Q,S)^d\phi(D(Q,S))^\eta} &\lesssim \int\limits_{\Omega} \frac{\phi(r)^\eta g(y)\,dy}{(|x-y|+r)^d\phi(|x-y|+r)^\eta} \\ &\leq \int\limits_{\Omega\cap\{|x-y|>r\}} \frac{\phi(r)^\eta g(y)\,dy}{|x-y|^d\phi(|x-y|)^\eta} + \int\limits_{|x-y|<r} \frac{\phi(r)^\eta g(y)\,dy}{r^d\phi(r)^\eta} \\ &\lesssim \int\limits_{\Omega\cap\{|x-y|>r\}} \frac{\phi(r)^\eta g(y)\,dy}{|x-y|^d\phi(|x-y|)^\eta} + \frac{1}{|B_r|}\int\limits_{|x-y|<r} g(y)\,dy.\end{aligned}$$

The claim follows from this estimate. Since the implied constants do not depend on x, the same holds for the infimum.

Inequality (2.3) can be obtained by taking $g \equiv 1$ and $r = l(Q)$ in (2.2). In that case $D(Q,S) > r$ for every S, including Q. ∎

The following lemma is an extension of [21, (2.7), (2.8)].

Lemma 2.3. *Let* $\eta \geq \min(q, p-p/q)$, $\kappa \geq \frac{1}{q-1}$, *assume that* **A2** *and* **A3** *hold, and assume that* $\mathcal{W}$ *is admissible. Then*

$$\sum_{R:\, P\in\mathbf{Sh}_\rho(R)} \phi(l(R))^{-\eta} \lesssim \phi(l(P))^{-\eta}. \tag{2.4}$$

Furthermore, if $S \in \mathbf{Sh}_\rho(R)$, *then*

$$\sum_{P\in[S,R]} \phi(l(P))^\kappa \lesssim \phi(l(R))^\kappa. \tag{2.5}$$

Proof. Since the cubes are dyadic, we may and do assume in (2.4) that $l(P) = 2^{p_0}$ for some $p_0 \in \mathbb{Z}$. Every R which satisfies $P \in \mathbf{Sh}_\rho(R)$ must be at a distance from P smaller than a multiple of $l(R)$, therefore there can only be a bounded number K of such cubes R with a given side length. Furthermore, the cubes considered must be sufficiently large to contain P in its shadow, that is, $l(R) \geq 2^{p_0-l_0}$ with $l_0 \in \mathbb{N}_0$ independent of p_0. We also obviously have $l(R) < \operatorname{diam}(\Omega)$. Thus, the sum in the first assertion can be bounded from above as follows:

$$\begin{aligned}\sum_{R:P\in\mathbf{Sh}_\rho(R)} \phi(l(R))^{-\eta} &\leq K \sum_{k=p_0-l_0}^{p_0+N(2^{p_0})} \phi(2^k)^{-\eta}\\ &= K \sum_{k=p_0-l_0}^{p_0} \phi(2^k)^{-\eta} + K \sum_{k=p_0+1}^{p_0+N(2^{p_0})} \phi(2^k)^{-\eta}.\end{aligned}$$

The sums are estimated by a multiple of $\phi(2^{p_0})^{-\eta}$ using **A3** and **A2** respectively, which proves (2.4).

As in the proof of [21, (2.8)] we may deduce that if $S \in \mathbf{Sh}_\rho(R)$, then there are a bounded number L of cubes $P \in [S, R]$ of a given side length. Furthermore, for every $P \in [S, R]$ we have $l(P) \leq 2^{r_0+l_0}$, where $l(R) = 2^{r_0}$ and l_0 is a fixed natural number independent of S and R. Therefore we estimate (2.5) as follows:

$$\sum_{P\in[S,R]} \phi(l(P))^\kappa \leq L \sum_{k=-\infty}^{r_0+l_0} \phi(2^k)^\kappa = L \sum_{k=-\infty}^{r_0} \phi(2^k)^\kappa + L \sum_{k=r_0+1}^{r_0+l_0} \phi(2^k)^\kappa.$$

The first sum is bounded from above by a multiple of $\phi(2^{r_0})^\kappa$ because of the second assertion of **A2**, and the second is handled by using **A3**. This ends the proof. ∎

3. Proof of Theorem 1.1. Obviously it suffices to show that the truncated seminorm dominates the full one up to a multiplicative constant.

We will work with dual norms, namely

$$\sup_{\substack{g\geq 0\\ \|g\|_{L^{p'}(L^{q'}(\Omega))}\leq 1}} \int_\Omega\int_\Omega |f(x) - f(y)|\, |x-y|^{-d/q}\phi(|x-y|)^{-1} g(x,y)\, dy\, dx. \tag{3.1}$$

From now on, g will be as in (3.1).

First let us take care of the case when x and y are close to each other. By Hölder's inequality, we get

$$\sum_{Q\in\mathcal{W}}\int_Q\int_{2Q}\frac{|f(x)-f(y)|g(x,y)}{|x-y|^{d/q}\phi(|x-y|)}\,dy\,dx$$
$$\le\sum_{Q\in\mathcal{W}}\int_Q\Big(\int_{2Q}\frac{|f(x)-f(y)|^q}{|x-y|^d\phi(|x-y|)^q}\,dy\Big)^{1/q}\Big(\int_{2Q}g(x,y)^{q'}\,dy\Big)^{1/q'}\,dx$$
$$\le\Big(\sum_{Q\in\mathcal{W}}\int_Q\Big(\int_{2Q}\frac{|f(x)-f(y)|^q}{|x-y|^d\phi(|x-y|)^q}\,dy\Big)^{p/q}\,dx\Big)^{1/p}.$$

What remains is the integral over $(\Omega\times\Omega)\setminus\bigcup_{Q\in\mathcal{W}}Q\times 2Q=\bigcup_{Q\in\mathcal{W}}Q\times(\Omega\setminus 2Q)=\bigcup_{Q,S\in\mathcal{W}}Q\times(S\setminus 2Q)$. We claim that in this case $|x-y|\approx D(Q,S)$. Indeed, since $y\notin 2Q$, we immediately get $l(Q)\le|x-y|$. Furthermore, if $l(S)\ge l(Q)$ and $|x-y|\le 2l(S)$, then $Q\subseteq 5S$, and by the definition of the Whitney decomposition $l(Q)\ge\frac{1}{2}l(S)$, which proves the claim. Therefore, by **A3** we get

$$\sum_{Q,S}\int_Q\int_{S\setminus 2Q}\frac{|f(x)-f(y)|g(x,y)}{|x-y|^{d/q}\phi(|x-y|)}\,dy\,dx \tag{3.2}$$
$$\lesssim\sum_{Q,S}\int_Q\int_S\frac{|f(x)-f(y)|g(x,y)}{D(Q,S)^{d/q}\phi(D(Q,S))}\,dy\,dx.$$

Let $f_Q=\frac{1}{|Q|}\int_Q f(x)\,dx$. By the triangle inequality, (3.2) does not exceed $(\mathbf{A})+(\mathbf{B})+(\mathbf{C})$ where

$$(\mathbf{A})=\sum_{Q,S}\int_Q\int_S\frac{|f(x)-f_Q|g(x,y)}{D(Q,S)^{d/q}\phi(D(Q,S))}\,dy\,dx,$$
$$(\mathbf{B})=\sum_{Q,S}\int_Q\int_S\frac{|f_Q-f_{Q_S}|g(x,y)}{D(Q,S)^{d/q}\phi(D(Q,S))}\,dy\,dx,$$
$$(\mathbf{C})=\sum_{Q,S}\int_Q\int_S\frac{|f_{Q_S}-f(y)|g(x,y)}{D(Q,S)^{d/q}\phi(D(Q,S))}\,dy\,dx.$$

Using Hölder's inequality and (2.3) we can estimate $(\mathbf{A})$ from above by

(3.3)
$$\sum_Q\int_Q|f(x)-f_Q|\Big(\int_\Omega g(x,y)^{q'}\,dy\Big)^{1/q'}\Big(\sum_S\frac{l(S)^d}{D(Q,S)^d\phi(D(Q,S))^q}\Big)^{1/q}\,dx$$
$$\lesssim\sum_Q\int_Q|f(x)-f_Q|\Big(\int_\Omega g(x,y)^{q'}\,dy\Big)^{1/q'}\frac{1}{\phi(l(Q))}\,dx$$
$$\lesssim\Big(\sum_Q\int_Q\Big(\frac{|f(x)-f_Q|}{\phi(l(Q))}\Big)^p\,dx\Big)^{1/p}.$$

Now, by the definition of f_Q, Jensen's inequality, and **A3** we get

$$(\mathbf{A}) \lesssim \bigg(\sum_Q \int_Q \bigg(\int_Q \frac{|f(x)-f(y)|^q}{l(Q)^d\phi(l(Q))^q}\,dy\bigg)^{p/q}\,dx\bigg)^{1/p}$$
$$\lesssim \bigg(\sum_Q \int_Q \bigg(\int_Q \frac{|f(x)-f(y)|^q}{|x-y|^d\phi(|x-y|)^q}\,dy\bigg)^{p/q}\,dx\bigg)^{1/p}.$$

Let us consider $(\mathbf{B})$. If we denote by $\mathcal{N}(P)$ the successor of P in a chain $[Q,S)$, then by the triangle inequality,

$$(\mathbf{B}) \le \sum_{Q,S} \bigg(\int_Q\int_S \frac{g(x,y)}{D(Q,S)^{d/q}\phi(D(Q,S))}\,dy\,dx \sum_{P\in[Q,Q_S)} |f_P - f_{\mathcal{N}(P)}|\bigg).$$

Recall that $\mathcal{N}(P)\subseteq 5P$, and for every $P\in[Q,Q_S]$, $Q\in\mathbf{Sh}(P)$. For such P it is also true that $D(P,S)\approx D(Q,S)$ (see [21, (2.6)]). Therefore, by **A3** we estimate $(\mathbf{B})$ from above by a multiple of

$$\sum_P \int_P\int_{5P} \frac{|f(\xi)-f(\zeta)|}{|P|\,|5P|}\,d\xi\,d\zeta \sum_{Q\in\mathbf{Sh}(P)} \int_Q \sum_S \int_S \frac{g(x,y)}{D(P,S)^{d/q}\phi(D(P,S))}\,dy\,dx.$$

By Hölder's inequality and (2.3) this expression does not exceed (up to a constant)

$$\sum_P \int_P\int_{5P} \frac{|f(\xi)-f(\zeta)|}{|P|\,|5P|}\,d\xi\,d\zeta \int_{\mathbf{SH}(P)} \bigg(\int_\Omega g(x,y)^{q'}\,dy\bigg)^{1/q'} \frac{1}{\phi(l(P))}\,dx. \tag{3.4}$$

Let $G(x)=\big(\int_\Omega g(x,y)^{q'}\,dy\big)^{1/q'}$. By [21, Lemma 2.7] we have $\int_{\mathbf{SH}(P)} G(x)\,dx \lesssim \inf_{y\in P} MG(y)l(P)^d$. Using this, Jensen's inequality, Hölder's inequality, and the fact that the maximal operator is continuous in $L^{p'}$, $p'>1$, we get

$$(\mathbf{B}) \lesssim \sum_P \frac{1}{|P|\,|5P|}\frac{l(P)^d}{\phi(l(P))}\int_P\int_{5P} |f(\xi)-f(\zeta)|MG(\zeta)\,d\xi\,d\zeta$$
$$\lesssim \sum_P \int_P \frac{MG(\zeta)}{l(P)^{d/q}\phi(l(P))}\bigg(\int_{5P} |f(\xi)-f(\zeta)|^q\,d\xi\bigg)^{1/q}\,d\zeta$$
$$\lesssim \bigg(\sum_P \int_P \bigg(\int_{5P} \frac{|f(\xi)-f(\zeta)|^q}{l(P)^d\phi(l(P))^q}\,d\xi\bigg)^{p/q}\,d\zeta\bigg)^{1/p}.$$

Since $|\xi-\zeta|\le 5l(P)$, $(\mathbf{B})$ is estimated.

Now we will deal with $(\mathbf{C})$. Since $D(Q,S)\approx l(Q_S)$, by **A3** we obtain

$$(\mathbf{C}) \lesssim \sum_{Q,S}\int_Q\int_S \frac{|f_{Q_S}-f(y)|g(x,y)}{l(Q_S)^{d/q}\phi(l(Q_S))}\,dy\,dx.$$

Furthermore, for every admissible chain we have $Q,S\in\mathbf{Sh}(Q_S)$, and so for all $Q,S\in\mathcal{W}$,

$$(Q_S,Q,S) \in \bigcup_{R\in\mathcal{W}} \{(R,P,P') : P,P'\in\mathbf{Sh}(R)\}.$$

Consequently,

$$(\mathbf{C}) \lesssim \sum_{R\in\mathcal{W}} \sum_{Q\in\mathbf{Sh}(R)} \sum_{S\in\mathbf{Sh}(R)} \int_Q\int_S \frac{|f_R - f(y)|g(x,y)}{l(R)^{d/q}\phi(l(R))}\,dy\,dx. \tag{3.5}$$

By Hölder's inequality the above expression does not exceed

$$\sum_{R\in\mathcal{W}} \frac{(\int_{\mathbf{SH}(R)} |f_R - f(y)|^q\,dy)^{1/q}}{l(R)^{d/q}\phi(l(R))} \int_{\mathbf{SH}(R)} \Big(\int_{\mathbf{SH}(R)} g(x,y)^{q'}\,dy\Big)^{1/q'}dx$$
$$\le \sum_{R\in\mathcal{W}} \frac{(\int_{\mathbf{SH}(R)} |f_R - f(y)|^q\,dy)^{1/q}}{l(R)^{d/q}\phi(l(R))} \int_{\mathbf{SH}(R)} G(x)\,dx.$$

By the last estimate of [21, Lemma 2.7], the fact that $\inf_R MG \le \frac{1}{l(R)^d}\int_R MG$, and Hölder's inequality we get

$$\begin{aligned}(\mathbf{C}) &\lesssim \sum_{R\in\mathcal{W}} \frac{1}{l(R)^{d/q}\phi(l(R))}\Big(\int_{\mathbf{SH}(R)} |f_R - f(y)|^q\,dy\Big)^{1/q} \int_R MG(\xi)\,d\xi \\ &\le \Big(\sum_{R\in\mathcal{W}} \int_R \frac{1}{l(R)^{dp/q}\phi(l(R))^p}\Big(\int_{\mathbf{SH}(R)} |f_R - f(y)|^q\,dy\Big)^{p/q}d\xi\Big)^{1/p}\|MG\|_{L^{p'}(\Omega)} \\ &\le \Big(\sum_{R\in\mathcal{W}} \frac{l(R)^d}{l(R)^{dp/q}\phi(l(R))^p}\Big(\sum_{S\in\mathbf{Sh}(R)}\int_S |f_R - f(y)|^q\,dy\Big)^{p/q}\Big)^{1/p}.\end{aligned}$$

Let $[S,R]$ be an admissible chain between S and R. Then, after using the inequality $|f_R - f(y)|^q \lesssim |f_R - f_S|^q + |f_S - f(y)|^q$, we get

$$\begin{aligned}(\mathbf{C})^p \lesssim{}& \sum_{R\in\mathcal{W}} \frac{l(R)^d}{l(R)^{dp/q}\phi(l(R))^p}\Big(\sum_{S\in\mathbf{Sh}(R)}\Big|\sum_{P\in[S,R)} f_P - f_{\mathcal{N}(P)}\Big|^q l(S)^d\Big)^{p/q} \\ &+ \sum_{R\in\mathcal{W}} \frac{l(R)^d}{l(R)^{dp/q}\phi(l(R))^p}\Big(\sum_{S\in\mathbf{Sh}(R)}\int_S |f_S - f(y)|^q\,dy\Big)^{p/q} = (\mathbf{C1}) + (\mathbf{C2}).\end{aligned}$$

If we write $f_P - f_{\mathcal{N}(P)} = (f_P - f_{\mathcal{N}(P)})\frac{\phi(l(P))^{1/q}}{\phi(l(P))^{1/q}}$, then by Hölder's inequality we estimate $(\mathbf{C1})$ from above by

$$\sum_{R\in\mathcal{W}} \frac{l(R)^d}{l(R)^{dp/q}\phi(l(R))^p}$$
$$\times\Big(\sum_{S\in\mathbf{Sh}(R)}\sum_{P\in[S,R)} \frac{|f_P - f_{\mathcal{N}(P)}|^q l(S)^d}{\phi(l(P))}\Big(\sum_{P\in[S,R)} \phi(l(P))^{q'/q}\Big)^{q/q'}\Big)^{p/q}.$$

By Lemma 2.3,

$$(\mathbf{C1}) \lesssim \sum_{R\in\mathcal{W}} \frac{l(R)^d}{l(R)^{dp/q}} \phi(l(R))^{p/q-p} \Big(\sum_{S\in\mathbf{Sh}(R)} \sum_{P\in[S,R)} \frac{|f_P - f_{\mathcal{N}(P)}|^q}{\phi(l(P))} l(S)^d \Big)^{p/q}.$$

Let us take ρ_2 large enough for $S \in \mathbf{Sh}^2(P) := \mathbf{Sh}_{\rho_2}(P)$ and $P \in \mathbf{Sh}^2(R)$ to hold. Then $\sum_{S\in\mathbf{Sh}(R)} \sum_{P\in[S,R)} \lesssim \sum_{P\in\mathbf{Sh}^2(R)} \sum_{S\in\mathbf{Sh}^2(P)}$. We denote by U_P the sum of the neighbors of P. Since $\sum_{S\in\mathbf{Sh}^2(P)} l(S)^d \lesssim l(P)^d$, we find that, up to a multiplicative constant, $(\mathbf{C1})$ does not exceed

$$\sum_{R\in\mathcal{W}} \frac{l(R)^d}{l(R)^{dp/q}} \phi(l(R))^{p/q-p} \Big(\sum_{P\in\mathbf{Sh}^2(R)} \frac{(l(P)^{-d} \int_{U_P} |f_P - f(\xi)|\, d\xi)^q}{\phi(l(P))} l(P)^d \Big)^{p/q}.$$

Since $p \geq q$, we can use Hölder's inequality with exponent p/q to estimate this expression from above by

$$\begin{aligned} &\sum_{R\in\mathcal{W}} \frac{l(R)^d}{l(R)^{dp/q}} \phi(l(R))^{p/q-p} \Big(\sum_{P\in\mathbf{Sh}^2(R)} \frac{(l(P)^{-d} \int_{U_P} |f_P - f(\xi)|\, d\xi)^p}{\phi(l(P))^{p/q}} l(P)^d \Big) \\ &\qquad\qquad \times \Big(\sum_{P\in\mathbf{Sh}^2(R)} l(P)^d \Big)^{(1-q/p)p/q} \\ &\qquad \lesssim \sum_{R\in\mathcal{W}} \sum_{P\in\mathbf{Sh}^2(R)} \phi(l(R))^{p/q-p} \frac{(l(P)^{-d} \int_{U_P} |f_P - f(\xi)|\, d\xi)^p l(P)^d}{\phi(l(P))^{p/q}} \\ &\qquad \lesssim \sum_{P\in\mathcal{W}} \frac{(l(P)^{-d} \int_{U_P} |f_P - f(\xi)|\, d\xi)^p l(P)^d}{\phi(l(P))^{p/q}} \sum_{R: P\in\mathbf{Sh}^2(R)} \phi(l(R))^{p/q-p}. \end{aligned}$$

Furthermore, Lemma 2.3 and Jensen's inequality give

$$\begin{aligned} (\mathbf{C1}) &\lesssim \sum_{P\in\mathcal{W}} \frac{(l(P)^{-d} \int_{U_P} |f_P - f(\xi)|\, d\xi)^p l(P)^d}{\phi(l(P))^p} \\ &\lesssim \sum_{P\in\mathcal{W}} \int_{U_P} \frac{|f_P - f(\xi)|^p}{\phi(l(P))^p}\, d\xi \\ &\leq \sum_{P\in\mathcal{W}} \int_{U_P} \Big(\int_P \frac{|f(\zeta) - f(\xi)|^q}{l(P)^d \phi(l(P))^q}\, d\zeta \Big)^{p/q} d\xi. \end{aligned} \tag{3.6}$$

Since $U_P \subseteq 5P$, we have finished estimating $(\mathbf{C1})$.

Now we proceed with $(\mathbf{C2})$. By Hölder's inequality,

$$(\mathbf{C2}) = \sum_{R\in\mathcal{W}} \frac{l(R)^{d(1-p/q)}}{\phi(l(R))^p} \Big(\sum_{S\in\mathbf{Sh}(R)} \int_S |f_S - f(\xi)|^q\, d\xi\, \frac{l(S)^{d(1-q/p)}}{l(S)^{d(1-q/p)}} \Big)^{p/q}$$

$$\le \sum_{R\in\mathcal{W}} \frac{l(R)^{d(1-p/q)}}{\phi(l(R))^p}\Big(\sum_{S\in\mathbf{Sh}(R)} l(S)^d\Big)^{p/q-1} \sum_{S\in\mathbf{Sh}(R)} \frac{(\int_S |f_S-f(\xi)|^q\,d\xi)^{p/q}}{l(S)^{d(p/q-1)}}$$
$$\lesssim \sum_{R\in\mathcal{W}} \sum_{S\in\mathbf{Sh}(R)} \frac{(\int_S |f_S-f(\xi)|^q\,d\xi)^{p/q}}{l(S)^{d(p/q-1)}\phi(l(R))^p}.$$

By rearranging and using Lemma 2.3 we obtain

$$(\mathbf{C2}) \lesssim \sum_{S\in\mathcal{W}} \frac{(\int_S |f_S-f(\xi)|^q\,d\xi)^{p/q}}{l(S)^{d(p/q-1)}} \sum_{R:S\in\mathbf{Sh}(R)} \phi(l(R))^{-p}$$
$$\lesssim \sum_{S\in\mathcal{W}} \Big(\int_S \frac{|f_S-f(\xi)|^q}{l(S)^d}\,d\xi\Big)^{p/q} \frac{l(S)^d}{\phi(l(S))^p}.$$

Hence, by Jensen's inequality,

$$(\mathbf{C2}) \lesssim \sum_{S\in\mathcal{W}} \frac{l(S)^d}{\phi(l(S))^p} \int_S \frac{|f_S-f(\xi)|^p}{l(S)^d}\,d\xi = \sum_{S\in\mathcal{W}} \int_S \frac{|f_S-f(\xi)|^p}{\phi(l(S))^p}\,d\xi.$$

Thus we have arrived at the same situation as in (3.6) and the proof is finished (we may need to enlarge the constant $C_{\mathcal{W}}$, which can be done by diminishing the cubes in the Whitney decomposition). ■

4. Examples of ϕ

4.1. Positive examples. We will present some examples of kernels which satisfy **A2** and **A3**.

EXAMPLE 4.1. Stable scaling is more than enough for **A2** to hold. Indeed, if we assume that there exist $\beta_1, \beta_2 \in (0,1)$ for which we have

$$\lambda^{\beta_1} \lesssim \frac{\phi(\lambda r)}{\phi(r)} \lesssim \lambda^{\beta_2}, \qquad r>0,\ \lambda\le 1,$$

then by the first inequality we get **A3** and by the second inequality the series in **A2** are geometric and independent of r.

Let us examine the constant C_2 in **A2** for $p=q=2$, $\alpha\in(0,2)$, and for kernels of the form $K(x,y)=(2-\alpha)|x-y|^{-d-\alpha}$, i.e. $\phi(t)=(2-\alpha)t^{\alpha/2}$. In this case $\frac{1}{q-1}=\min(q,p-p/q)=1$ and for every $r>0$ we have

$$\sum_{k=1}^{\infty} \frac{\phi(r)}{\phi(2^k r)} = \sum_{k=1}^{\infty} \frac{\phi(2^{-k}r)}{\phi(r)} = \sum_{k=1}^{\infty} \frac{1}{(2^{\alpha/2})^k} = \frac{1}{2^{\alpha/2}-1}.$$

This quantity is bounded as $\alpha\to 2^-$. Since the constant in **A3** is also bounded in this case, we conclude that the comparability in Theorem 1.1 is uniform for $\alpha\in(\varepsilon,2)$ for every $\varepsilon>0$.

EXAMPLE 4.2. Assume that Ω is bounded. Let $\gamma \in (0,1)$, $\phi(r) = [\log(1+r)]^\gamma$ and $R = \operatorname{diam}(\Omega)$. Note that for $r > 0$ we have

$$1 \le \frac{\log(1+2r)}{\log(1+r)} \le 2.$$

Indeed, by looking at the derivative we see that the ratio is decreasing, the inequalities result from its limits at 0^+ and at ∞. Therefore, ϕ satisfies **A3**. Furthermore, for $r < R$ the lower bound can be replaced with a constant $C = C(R) > 1$, hence both series in **A2** become geometric and so this condition is satisfied.

4.2. O-regularly varying functions

DEFINITION 4.3. We say that ϕ is *O-regularly varying* at infinity if there exist $a, b \in \mathbb{R}$ and $A, B, R > 0$ such that

$$A\left(\frac{r_2}{r_1}\right)^a \le \frac{\phi(r_2)}{\phi(r_1)} \le B\left(\frac{r_2}{r_1}\right)^b \tag{4.1}$$

whenever $R < r_1 < r_2$. Analogously, ϕ is O-regularly varying at zero if (4.1) holds for $0 < r_1 < r_2 < R$. The supremum of a and the infimum of b for which (4.1) is satisfied are called the *lower*, respectively *upper*, *Matuszewska indices* (or lower/upper indices).

A nice short review of the O-regularly varying functions can be found in the work of Grzywny and Kwaśnicki [16, Appendix A]; for further reading we refer to the book by Bingham, Goldie, and Teugels [2].

Assume **A2** and **A3**. We will show that these assumptions enforce O-regular variation with positive lower index at 0 and, for unbounded Ω, at infinity by using [16, Proposition A.1]. Note that by **A3** for $r > 0$, $k \in \mathbb{Z}$, and $z \in [2^{k-1}r, 2^k r]$ we have $\phi(z) \approx \phi(2^k r)$.

We first consider the regular variation at zero using [16, Proposition A.1(c)]. Let $R = \operatorname{diam}(\Omega)$ and $t_2 = \frac{1}{q-1}$. Then for every $r \in (0, R)$ and $\eta \in \mathbb{R}$ we have

$$\int_0^r z^{-\eta}\phi(z)^{t_2}\,\frac{dz}{z} \approx \sum_{k=1}^{\infty} \phi(2^{-k}r)^{t_2}(2^{-k}r)^{-\eta} = r^{-\eta}\phi(r)^{t_2}\sum_{k=1}^{\infty}\frac{\phi(2^{-k}r)^{t_2}}{\phi(r)^{t_2}}2^{k\eta}.$$

By **A2** the latter sum is finite for $\eta \le 0$, it is also bounded away from 0 because of **A3**. Therefore ϕ^{t_2} (and thus also ϕ) has to be O-regularly varying at 0 with some lower index $a_0 > 0$, that is,

$$\frac{\phi(r_2)}{\phi(r_1)} \gtrsim \left(\frac{r_2}{r_1}\right)^{a_0/t_2}, \quad 0 < r_1 \le r_2 \le R.$$

The above condition yields power-type decay of ϕ at 0. This could also be obtained using the other summation condition in **A2** by applying [16, Proposition A.1(d)].

The behavior of ϕ at infinity only comes into play when Ω is unbounded, so we assume that $\operatorname{diam}(\Omega) = \infty$ for the remainder of this subsection. Let $r > 0$, $\eta \in \mathbb{R}$, and $t_1 = \min(q, p - p/q)$. We have

$$\int_r^\infty z^{-\eta}\phi(z)^{-t_1}\,\frac{dz}{z} \approx \sum_{k=1}^\infty \phi(2^k r)^{-t_1}(2^k r)^{-\eta} = r^{-\eta}\phi(r)^{-t_1}\sum_{k=1}^\infty \frac{\phi(r)^{t_1}}{\phi(2^k r)^{t_1}}2^{-k\eta}.$$

By **A2** and **A3** the sum is finite and bounded away from 0 if $\eta \geq 0$. Thus ϕ^{-t_1} is O-regularly varying at infinity with upper index $-a_\infty < 0$, which is equivalent to the O-regular variation with lower index a_∞ for ϕ^{t_1}:

$$\frac{\phi(r_2)}{\phi(r_1)} \gtrsim \left(\frac{r_2}{r_1}\right)^{a_\infty/t_1}, \qquad R < r_1 \leq r_2 < \infty.$$

4.3. Negative examples. We will show some examples for which the seminorms (1.2) and (1.3) are not comparable. Assume for clarity that $p = q = 2$.

Example 4.4. Let $\Omega = (0,1) \subset \mathbb{R}$ and let $K(x,y) \equiv 1$. Consider the function $f(x) = x^{-\gamma}$ with $\gamma \in (0, 1/2)$. A direct calculation shows that

$$\int_0^1\int_0^1 (f(x) - f(y))^2\,dy\,dx = 2\left(\frac{1}{1-2\gamma} - \frac{1}{(1-\gamma)^2}\right). \tag{4.2}$$

In particular, f belongs to the corresponding Sobolev space (actually the "Sobolev space" is $L^2(\Omega)$ in this case). Let $\varepsilon \in (0,1)$. We have

$$\begin{aligned}\int_0^1 \int_{x-\varepsilon\delta(x)}^{x+\varepsilon\delta(x)} (f(x) - f(y))^2\,dy\,dx &\leq \int_0^1 \int_{x(1-\varepsilon)}^{x(1+\varepsilon)} (f(x) - f(y))^2\,dy\,dx \\ &= \frac{\varepsilon}{1-\gamma} - \frac{(1+\varepsilon)^{1-\gamma} - (1-\varepsilon)^{1-\gamma}}{(1-\gamma)^2} + \frac{(1+\varepsilon)^{1-2\gamma} - (1-\varepsilon)^{1-2\gamma}}{(1-2\gamma)(2-2\gamma)}.\end{aligned} \tag{4.3}$$

As $\gamma \to (1/2)^-$ the ratio of the right hand sides of (4.2) and (4.3) goes to infinity, which shows that in this case the result of Theorem 1.1 does not hold.

Example 4.5. The preceding example gives an idea on how to show an analogous fact for any nonzero K such that $K(0,\cdot) \in L^1([0,1])$. On the restricted domain of integration we have $x \approx y$. Therefore $|1/x^\gamma - 1/y^\gamma| \lesssim 1/x^\gamma$, hence

$$\int_0^1 \int_{B(x,\varepsilon\delta(x))} \left(\frac{1}{x^\gamma} - \frac{1}{y^\gamma}\right)^2 K(x,y)\,dy\,dx \lesssim \int_0^1 \frac{1}{x^{2\gamma}} \int_{B(x,\varepsilon\delta(x))} K(x,y)\,dy\,dx. \tag{4.4}$$

On the other hand, since K is nontrivial, there exists $\eta > 0$ such that for every $x \in (0,\eta)$ we have $\int_\eta^1 K(x,y)\,dy \geq C > 0$. Therefore

$$\int_0^1\!\!\int_0^1 \left(\frac{1}{x^\gamma} - \frac{1}{y^\gamma}\right)^2 K(x,y)\,dy\,dx \geq \int_0^{\eta/2}\!\!\int_\eta^1 \left(\frac{1}{x^\gamma} - \frac{1}{\eta^\gamma}\right)^2 K(x,y)\,dy\,dx \tag{4.5}$$
$$\gtrsim \int_0^{\eta/2} \frac{1}{x^{2\gamma}} \int_\eta^1 K(x,y)\,dy\,dx \gtrsim \int_0^{\eta/2} \frac{1}{x^{2\gamma}}\,dx.$$

Note that (4.4) is of the form $\int_0^1 \frac{f(x)}{x^{2\gamma}}\,dx$ with $f(x)$ bounded and $\lim_{x\to 0^+} f(x) = 0$. Let us fix an arbitrarily small $\xi > 0$, and let ρ be so small that $f(x) \leq \xi$ for $x \in (0,\rho)$. If we decompose $\int_0^1 = \int_0^\rho + \int_\rho^1$, then we see that the ratio of (4.4) and (4.5) tends to 0 as $\gamma \to 1/2$.

Remark 4.6. In previous examples the kernel was integrable. This means that

$$\int_\Omega\int_\Omega (f(x)-f(y))^2 K(x,y)\,dy\,dx \leq 2\int_\Omega\int_\Omega f(x)^2 K(x,y)\,dy\,dx$$
$$\leq 2\|f\|^2_{L^2(\Omega)} \|K(0,\cdot)\|_{L^1(\mathbb{R}^d)}.$$

Therefore, even though the quadratic forms (1.2) and (1.3) are incomparable, the Triebel–Lizorkin norm $\|\cdot\|_{F_{p,q}(\Omega)}$ is comparable when we replace the full seminorm with the truncated one.

Example 4.7. For $K(x,y) = |x-y|^{-1}$ on $\Omega = (0,1)$ the seminorms also fail to be comparable. Consider the functions $f_n(x) = n \wedge \frac{1}{x}$. Since

$$\int_0^1\!\!\int_0^x (f(x)-f(y))^2 K(x,y)\,dy\,dx = \frac{1}{2}\int_0^1\!\!\int_0^1 (f(x)-f(y))^2 K(x,y)\,dy\,dx,$$

we will assume that $y < x$, and work only with the integral on the left hand side. Note that for f_n, the integral over $(0,1/n)^2$ vanishes. We split the integral as follows:

$$\int_0^1\!\!\int_0^x (f_n(x)-f_n(y))^2 K(x,y)\,dy\,dx$$
$$= \int_{1/n}^1 \int_{1/n}^x \left(\frac{1}{x} - \frac{1}{y}\right)^2 K(x,y)\,dy\,dx + \int_{1/n}^1 \int_0^{1/n} \left(n - \frac{1}{x}\right)^2 K(x,y)\,dy\,dx = \mathrm{I} + \mathrm{II}.$$

We first compute I. Note that the integrand is equal to $\frac{(x-y)^2}{y^2x^2} \cdot \frac{1}{x-y} = \frac{x-y}{y^2x^2}$.

Hence

$$\mathrm{I} = \int_{1/n}^{1}\int_{1/n}^{x} \frac{x-y}{y^2x^2}\,dy\,dx = \int_{1/n}^{1}\int_{1/n}^{x} \frac{1}{y^2x}\,dy\,dx - \int_{1/n}^{1}\int_{1/n}^{x} \frac{1}{yx^2}\,dy\,dx$$
$$= n\log n - 2n + \log n + 2.$$

For II we only show the asymptotics:

$$\begin{aligned}\mathrm{II} &= \int_{1/n}^{1}\int_{0}^{1/n} \left(n-\frac{1}{x}\right)^2 K(x,y)\,dy\,dx \\ &= \int_{1/n}^{1} \left(n-\frac{1}{x}\right)^2 \left(\log x - \log\left(x-\frac{1}{n}\right)\right) dx \\ &= -n^2 \int_{1/n}^{1} \left(1-\frac{1}{nx}\right)^2 \log\left(1-\frac{1}{nx}\right) dx \\ &= -n \int_{0}^{1-1/n} \frac{t^2}{(1-t)^2}\log t\,dt.\end{aligned}$$

For $n>2$ we split the last integral: $\int_0^{1-1/n} = \int_0^{1/2} + \int_{1/2}^{1-1/n}$. The first one converges, i.e. it is a (negative) constant. In the second one $t^2 \approx 1$ and $\frac{\log t}{1-t} \approx -1$, therefore

$$-n\int_0^{1-1/n} \frac{t^2}{(1-t)^2}\log t\,dt \approx n\left(1+\int_{1/2}^{1-1/n}\frac{dt}{1-t}\right) = n(1+\log n - \log 2). \tag{4.6}$$

Thus we get the asymptotics

$$\int_0^1\int_0^1 (f_n(x)-f_n(y))^2 K(x,y)\,dy\,dx \approx n\log n. \tag{4.7}$$

Now consider the truncated case. For clarity, assume that $\epsilon = \frac{1}{2}$. Then

$$\begin{aligned}\int_0^1\int_{x/2}^{x} (f_n(x)-f_n(y))^2 K(x,y)\,dy\,dx &= \int_{2/n}^{1}\int_{x/2}^{x} \left(\frac{1}{x}-\frac{1}{y}\right)^2 K(x,y)\,dy\,dx \\ &\quad+ \int_{1/n}^{2/n}\int_{1/n}^{x} \left(\frac{1}{x}-\frac{1}{y}\right)^2 K(x,y)\,dy\,dx \\ &\quad+ \int_{1/n}^{2/n}\int_{x/2}^{1/n} \left(n-\frac{1}{x}\right)^2 K(x,y)\,dy\,dx \\ &= \mathrm{III} + \mathrm{IV} + \mathrm{V}.\end{aligned}$$

We note that

$$\mathrm{III} = \int_{2/n}^{1} \int_{x/2}^{x} \left(\frac{1}{x} - \frac{1}{y}\right)^2 K(x,y)\,dy\,dx \le \int_{2/n}^{1} \int_{x/2}^{x} \frac{1}{y^2 x}\,dy\,dx = \frac{n}{2} - 1,$$

and

$$\mathrm{IV} = \int_{1/n}^{2/n} \int_{1/n}^{x} \left(\frac{1}{x} - \frac{1}{y}\right)^2 K(x,y)\,dy\,dx \le \int_{1/n}^{2/n} \int_{1/n}^{x} \frac{1}{y^2 x}\,dy\,dx = n\log 2 - \frac{n}{2}.$$

The last integral V is estimated as follows:

$$\begin{aligned}
\mathrm{V} &= \int_{1/n}^{2/n} \int_{x/2}^{1/n} \left(n - \frac{1}{x}\right)^2 K(x,y)\,dy\,dx \\
&= \int_{1/n}^{2/n} \left(n - \frac{1}{x}\right)^2 \left(\log\frac{x}{2} - \log\left(x - \frac{1}{n}\right)\right) dx \\
&= -\,n^2 \int_{1/n}^{2/n} \left(1 - \frac{1}{nx}\right)^2 \left(\log\left(1 - \frac{1}{nx}\right) + \log 2\right) dx \\
&\le -\,n \int_{0}^{1/2} \frac{t^2}{(1-t)^2} \log t\,dt \approx n.
\end{aligned}$$

To conclude, we get

$$\int_0^1 \int_{B(x,\delta(x)/2)} (f_n(x) - f_n(y))^2 K(x,y)\,dy\,dx \lesssim n. \tag{4.8}$$

Since the ratio of the right hand sides of (4.7) and (4.8) diverges as $n \to \infty$, our claim is proved.

5. The 0-order kernel

THEOREM 5.1. *Let Ω be a bounded uniform domain. If $1 < q \le p < \infty$, then for every $0 < \theta \le 1$,*

$$\begin{aligned}
&\left(\int_\Omega \left(\int_\Omega \frac{|f(x)-f(y)|^q}{|x-y|^d}\,dy\right)^{p/q} dx\right)^{1/p} \\
&\quad\lesssim \left(\int_\Omega \left(\int_{B(x,\theta\delta(x))} \frac{|f(x)-f(y)|^q}{|x-y|^d}\,(|\log|x-y|| \vee 1)^q\,dy\right)^{p/q} dx\right)^{1/p}.
\end{aligned} \tag{5.1}$$

The implied constant depends only on p, q, θ, Ω.

In order to obtain this result we first prove an analogue of Lemma 2.2 for $K(x,y)=|x-y|^{-d}$, i.e. $\phi\equiv 1$. For now every integral is restricted to Ω by default.

LEMMA 5.2. *Let Ω be a bounded domain with Whitney covering $\mathcal{W}$. Assume that $g\in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $0<r<\operatorname{diam}(\Omega)$. Then for every $Q\in\mathcal{W}$ and $x\in\Omega$, we have*

$$\int\limits_{|y-x|>r} \frac{g(y)\,dy}{|y-x|^d} \lesssim Mg(x)(|\log r|\vee 1), \tag{5.2}$$

$$\sum_{S\colon D(Q,S)>r} \frac{\int_S g(y)\,dy}{D(Q,S)^d} \lesssim \inf_{x\in Q} Mg(x)(|\log r|\vee 1), \tag{5.3}$$

$$\sum_{S\in\mathcal{W}} \frac{l(S)^d}{D(Q,S)^d} \lesssim |\log l(Q)|\vee 1. \tag{5.4}$$

Proof. Let $x\in\Omega$. If we take $R=\operatorname{diam}(\Omega)$, then proceeding as in Lemma 2.2 we get

$$\begin{aligned}\int\limits_{|y-x|>r} \frac{g(y)\,dy}{|y-x|^d} &\le \sum_{k=1}^{\lceil\log_2(R/r)\rceil} \int\limits_{2^{k-1}r\le|y-x|<2^k r} \frac{g(y)\,dy}{|x-y|^d}\\ &\lesssim Mg(x)\lceil\log_2(R/r)\rceil \lesssim Mg(x)(|\log r|\vee 1).\end{aligned}$$

As in the proof of Lemma 2.2, in order to prove (5.3) we use (5.2), and we are left with

$$\int\limits_{|x-y|<r} \frac{g(y)\,dy}{(|x-y|+r)^d} \lesssim \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} g(y)\,dy \le Mg(x)(|\log r|\vee 1).$$

Finally, (5.4) is obtained by taking $r=l(Q)$ and $g\equiv 1$. ∎

We will also use the following result similar to Lemma 2.3.

LEMMA 5.3. *Let Ω be a bounded uniform domain with admissible Whitney decomposition $\mathcal{W}$ and let $\rho>0$ and $\eta>1$. Then, for every $S\in\mathcal{W}$,*

$$\sum_{R\colon S\in\mathbf{Sh}_\rho(R)} 1 \lesssim |\log l(S)|\vee 1. \tag{5.5}$$

If $S\in\mathbf{Sh}_\rho(R)$, then

$$\sum_{P\in[S,R)} (|\log l(P)|\vee 1)^{-\eta} \lesssim (|\log l(R)|\vee 1)^{1-\eta}. \tag{5.6}$$

Furthermore, for every $P\in\mathcal{W}$,

$$\sum_{R:P\in\mathbf{Sh}_\rho(R)} (|\log l(R)|\vee 1)^{-\eta} \lesssim 1. \tag{5.7}$$

Proof. Throughout the proof we let $l(S) = 2^{s_0}$, $l(R) = 2^{r_0}$, $l(P) = 2^{p_0}$, whenever the cubes are fixed.

Arguing as in the proof of Lemma 2.3 we deduce that there are a limited number of cubes of a given side length contributing to the sum in (5.5) and the smallest of these cubes must have side length at least $2^{s_0-l_0}$ for some fixed natural number $l_0 \geq 0$. Therefore, if we let 2^{m_0} be the side length of the largest cube in $\mathcal{W}$, then

$$\sum_{R:\, S\in \mathbf{Sh}_\rho(R)} 1 \lesssim \sum_{k=s_0-l_0}^{m_0} 1 = m_0 - s_0 + l_0 + 1 \approx |\log l(S)| \vee 1.$$

As in Lemma 2.3, in (5.6) we have a limited number of cubes of the same side length, and the length cannot be larger than $2^{r_0+l_0}$ and smaller than $2^{s_0-l_0}$ (l_0 may be different than above, but it does not depend on S and R). Therefore we estimate the sum in (5.6) as follows:

$$\sum_{P\in[S,R)} (|\log l(P)| \vee 1)^{-\eta} \lesssim \sum_{k=s_0-l_0}^{r_0+l_0} (|k| \vee 1)^{-\eta} \leq \sum_{k=-\infty}^{r_0+l_0} (|k| \vee 1)^{-\eta}.$$

Since $\eta > 1$, the latter series is finite and it is of order $(|r_0| \vee 1)^{1-\eta}$, which proves (5.6).

In order to prove (5.7) we argue as above in terms of the numbers of the cubes, and because of $\eta > 1$ we find that there is a constant C such that

$$\sum_{R:\, P\in \mathbf{Sh}_\rho(R)} (|\log l(R)| \vee 1)^{-\eta} \lesssim \sum_{k=p_0-l_0}^{m_0} (|k| \vee 1)^{-\eta} \leq \sum_{k=-\infty}^{m_0} (|k| \vee 1)^{-\eta} = C. \quad \blacksquare$$

Proof of Theorem 5.1. We proceed as in Theorem 1.1 starting with 1 in place of ϕ. The integrals over $Q \times 2Q$ are trivially estimated, because the kernel on the right hand side of (5.1) is larger than the one on the left hand side.

In (**A**) and (**B**) the modification is quite straightforward. Lemma 2.2 is used in (3.3) and (3.4) respectively. Using Lemma 5.2 instead, we get respectively $(|\log l(Q)| \vee 1)^{1/q}$ and $(|\log l(P)| \vee 1)^{1/q}$. The remaining arguments are conducted with $(|\log r| \vee 1)^{-1/q}$ in place of $\phi(r)$. Note that $(|\log r| \vee 1)^{-1/q} \approx (|\log 2r| \vee 1)^{-1/q}$. We remark that this yields estimates for (**A**) and (**B**), which are better than the ones in the statement, in fact both expressions are bounded from above by

$$\left(\int_{\Omega} \left(\int_{B(x,\theta\delta(x))} \frac{|f(x)-f(y)|^q}{|x-y|^d} \big(|\log|x-y|\,| \vee 1\big)\, dy\right)^{p/q} dx\right)^{1/p}. \tag{5.8}$$

Notice the lack of exponent q in the logarithmic term. At this point we distinguish between the cases $p = q$ and $p \neq q$. In the former case the test functions g from (3.1) are defined by the condition $\int_\Omega \int_\Omega g(x,y)^{p'}\,dy\,dx \le 1$, therefore (**C**) can be estimated exactly as (**A**) and (**B**) because we can interchange the roles of Q, S and x, y using Tonelli's theorem. Thus, in this case we in fact obtain an estimate better than postulated, as the whole expression on the left hand side of (5.1) is approximately bounded from above by (5.8).

For the remainder of the proof we assume that $p > q$. The procedure for (**C**) is also similar to the one in the proof of Theorem 1.1, but the computations are slightly different in terms of exponents, therefore we give the details. There are no essential changes up to the moment of splitting into (**C1**) and (**C2**), thus we make it our starting point. As in the proof of Theorem 1.1 we get

$$\begin{aligned}(\mathbf{C2}) &= \sum_{R\in\mathcal{W}} l(R)^{d(1-p/q)}\Big(\sum_{S\in\mathbf{Sh}(R)} \int_S |f_S - f(\xi)|^q\,d\xi \frac{l(S)^{d(1-q/p)}}{l(S)^{d(1-q/p)}}\Big)^{p/q}\\ &\lesssim \sum_{R\in\mathcal{W}} \sum_{S\in\mathbf{Sh}(R)} l(S)^{d(1-p/q)}\Big(\int_S |f_S - f(\xi)|^q\,d\xi\Big)^{p/q}.\end{aligned}$$

We rearrange, use (5.5) and then Jensen's inequality twice to obtain

$$\begin{aligned}(\mathbf{C2}) &\lesssim \sum_{S\in\mathcal{W}} l(S)^{d(1-p/q)}\Big(\int_S |f_S - f(\xi)|^q\,d\xi\Big)^{p/q}\Big(\sum_{R:S\in\mathbf{Sh}(R)} 1\Big)\\ &\lesssim \sum_{S\in\mathcal{W}} l(S)^d(|\log l(S)|\vee 1)\Big(\frac{1}{l(S)^d}\int_S |f_S - f(\xi)|^q\,d\xi\Big)^{p/q}\\ &\le \sum_{S\in\mathcal{W}} (|\log l(S)|\vee 1)\int_S |f_S - f(\xi)|^p\,d\xi\\ &\le \sum_{S\in\mathcal{W}} \int_S\Big(\int_S \frac{|f(\zeta)-f(\xi)|^q}{l(S)^d}(|\log l(S)|\vee 1)^{q/p}\,d\zeta\Big)^{p/q}\,d\xi,\end{aligned}$$

and thus (**C2**) is estimated, since $q/p < 1 < q$.

In order to estimate (**C1**) we write $|f_P - f_{\mathcal{N}(P)}| = |f_P - f_{\mathcal{N}(P)}|\frac{|\log l(P)|\vee 1}{|\log l(P)|\vee 1}$ and we use Hölder's inequality with exponent q and (5.6):

$$\begin{aligned}(\mathbf{C1}) \le \sum_{R\in\mathcal{W}} l(R)^{d(1-p/q)}\Big[&\sum_{S\in\mathbf{Sh}(R)}\Big(\sum_{P\in[S,R)} |f_P - f_{\mathcal{N}(P)}|^q(|\log l(P)|\vee 1)^q l(S)^d\Big)\\ &\times\Big(\sum_{P\in[S,R)}(|\log l(P)|\vee 1)^{-q'}\Big)^{q/q'}\Big]^{p/q}\\ \lesssim \sum_{R\in\mathcal{W}} l(R)^{d(1-p/q)}&(|\log l(R)|\vee 1)^{-p/q}\\ &\times\Big(\sum_{S\in\mathbf{Sh}(R)}\sum_{P\in[S,R)} |f_P - f_{\mathcal{N}(P)}|^q(|\log l(P)|\vee 1)^q l(S)^d\Big)^{p/q}.\end{aligned}$$

By rearranging as in the proof of Theorem 1.1 and by using Hölder's and Jensen's inequalities we further estimate (**C1**) from above by a multiple of

$$\begin{aligned}
&\sum_{R\in\mathcal{W}} l(R)^{d(1-p/q)}(|\log l(R)|\vee 1)^{-p/q}\\
&\qquad\times\Bigg(\sum_{P\in\mathbf{Sh}^2(R)}\sum_{S\in\mathbf{Sh}^2(P)}\Big(\int_{U_P}\frac{|f_P-f(\xi)|}{l(P)^d}\,d\xi\Big)^q(|\log l(P)|\vee 1)^q l(S)^d\Bigg)^{p/q}\\
&\quad\lesssim \sum_{R\in\mathcal{W}} l(R)^{d(1-p/q)}(|\log l(R)|\vee 1)^{-p/q}\\
&\qquad\qquad\times\Bigg(\sum_{P\in\mathbf{Sh}^2(R)}\Big(\int_{U_P}\frac{|f_P-f(\xi)|}{l(P)^d}\,d\xi\Big)^q(|\log l(P)|\vee 1)^q l(P)^d\Bigg)^{p/q}\\
&\quad\le \sum_{R\in\mathcal{W}}\sum_{P\in\mathbf{Sh}^2(R)}(|\log l(R)|\vee 1)^{-p/q}(|\log l(P)|\vee 1)^p\int_{U_P}|f_P-f(\xi)|^p\,d\xi.
\end{aligned}$$

We rearrange once more and use (5.7) (recall that $p>q$) and Jensen's inequality to find that, up to a multiplicative constant, (**C1**) does not exceed

$$\begin{aligned}
&\sum_{P\in\mathcal{W}}(|\log l(P)|\vee 1)^p\int_{U_P}|f_P-f(\xi)|^p\,d\xi\Big(\sum_{R:\,P\in\mathbf{Sh}^2(R)}(|\log l(R)|\vee 1)^{-p/q}\Big)\\
&\qquad\lesssim \sum_{P\in\mathcal{W}}(|\log l(P)|\vee 1)^p\int_{U_P}|f_P-f(\xi)|^p\,d\xi\\
&\qquad\lesssim \sum_{P\in\mathcal{W}}\int_{U_P}\Big(\int_P\frac{|f(\zeta)-f(\xi)|^q}{l(P)^d}(|\log l(P)|\vee 1)^q\,d\zeta\Big)^{p/q}\,d\xi.
\end{aligned}$$

This finishes the proof. ■

Since the kernel in (5.1) is significantly larger than the one in (5.1), it is plausible that the reverse inequality is not true. We will show the existence of a counterexample when $\Omega=(0,1)$, $p=q=2$. For an open interval $I\subseteq\mathbb{R}$ we let

$$\begin{aligned}
F_0(I)&=\Big\{f\in L^2(I):\iint_{I\,I}\frac{(f(x)-f(y))^2}{|x-y|}\,dy\,dx<\infty\Big\},\\
F_{\log}(I)&=\Big\{f\in L^2(I):\iint_{I\,I}\frac{(f(x)-f(y))^2}{|x-y|}\big(|\log|x-y||\vee 1\big)\,dy\,dx<\infty\Big\}.
\end{aligned}$$

We note that in $F_{\log}(I)$ the logarithm is with exponent 1. This suffices for our present purpose, because $q>1$ in Theorem 5.1.

THEOREM 5.4. *For every $\theta \in (0,1]$, there exists $f \in F_0(0,1) \cap L^\infty(0,1)$ such that*

$$\int_0^1 \int_{B(x,\theta\delta(x))} (f(x)-f(y))^2 |x-y|^{-1} \big(\big|\log|x-y|\big| \vee 1\big)\,dy\,dx = \infty. \tag{5.9}$$

Proof. STEP 1. First, note that the finiteness of the left hand side of (5.9) implies that $f \in F_{\log}(\frac{n}{2n+1}, \frac{n+1}{2n+1})$ for a sufficiently large $n \in \mathbb{N}$. Indeed, if $\theta \geq 1/n$ for some natural number $n \geq 2$, then

$$\begin{aligned}\int_0^1 \int_{B(x,\theta\delta(x))} (\ldots) \geq \int_0^1 \int_{B(x,\delta(x)/n)} (\ldots) &\geq \int_{\frac{n}{2n+1}}^{\frac{n+1}{2n+1}} \int_{B\left(x,\frac{1}{2n+1}\right)} (\ldots) \\ &\geq \int_{\frac{n}{2n+1}}^{\frac{n+1}{2n+1}} \int_{\frac{n}{2n+1}}^{\frac{n+1}{2n+1}} (\ldots).\end{aligned} \tag{5.10}$$

We fix a number n for which (5.10) is satisfied.

STEP 2. In order to construct a counterexample we will use the asymptotics of the Fourier expansions of functions in $F_0(I)$ and $F_{\log}(I)$. We adopt the following convention for the Fourier coefficients of an integrable function f on an interval (a,b):

$$\widehat{f}(m) = \frac{1}{b-a}\int_a^b f(x) e^{-\frac{2\pi i m x}{b-a}}\,dx, \quad m \in \mathbb{Z}.$$

Below, $\widehat{f}(m)$ will mean the Fourier coefficient on $(0,1)$. Let f satisfy $f(x+1) = f(x)$ for $x \in \mathbb{R}$. Let $K(x,y)$ equal $|x-y|^{-1}$ (resp. $|x-y|^{-1}(|\log|x-y||\vee 1)$). We claim that a function $f \in L^\infty(0,1)$ belongs to $F_0(0,1)$ (resp. $F_{\log}(0,1)$) if and only if

$$\int_0^1\int_0^1 (f(x)-f(x-h))^2 K(0,h)\,dh\,dx < \infty.$$

Indeed, we have

$$\begin{aligned}\int_0^1\int_0^1 (f(x)-f(y))^2 K(x,y)\,dy\,dx &= 2\int_0^1\int_0^x (f(x)-f(y))^2 K(x,y)\,dy\,dx \\ &= 2\int_0^1\int_0^x (f(x)-f(x-h))^2 K(0,h)\,dh\,dx.\end{aligned}$$

Therefore, it suffices to verify that $\int_0^1\int_x^1 (f(x)-f(x-h))^2 K(0,h)\,dh\,dx < \infty$ for bounded f. Clearly we can assume that $K(x,y) = |x-y|^{-1}(|\log|x-y||\vee 1)$.

Then

$$\int_0^1\int_x^1 (f(x)-f(x-h))^2 K(0,h)\,dh\,dx \lesssim \int_0^1\int_x^1 \frac{(-\log h)\vee 1}{h}\,dh\,dx$$

$$= \int_0^{1/e}\int_x^{1/e} \frac{-\log h}{h}\,dh\,dx + \int_0^{1/e}\int_{1/e}^{1} \frac{1}{h}\,dh\,dx + \int_{1/e}^{1}\int_x^{1} \frac{1}{h}\,dh\,dx.$$

All the integrals are finite, therefore the claim is proved.

By Parseval's identity and Tonelli's theorem we get

$$\int_0^1 K(0,h)\int_0^1 (f(x)-f(x-h))^2\,dx\,dh$$

$$= \int_0^1 K(0,h)\sum_{m\in\mathbb{Z}} |\widehat{f}(m)|^2 |1-e^{2\pi imh}|^2\,dh$$

$$= \sum_{m\in\mathbb{Z}} |\widehat{f}(m)|^2 \int_0^1 |1-e^{2\pi imh}|^2 K(0,h)\,dh$$

$$= 2\sum_{m\in\mathbb{Z}} |\widehat{f}(m)|^2 \int_0^1 (1-\cos(2\pi mh))K(0,h)\,dh.$$

Now let us inspect the remaining integrals for both cases of K. For $m\neq 0$ we have

$$\int_0^1 \frac{1-\cos(2\pi mh)}{h}\,dh = \int_0^{|m|} \frac{1-\cos(2\pi h)}{h}\,dh \approx \log|m|.$$

In the logarithmic case

$$\int_0^1 \frac{1-\cos(2\pi mh)}{h}(-\log h\vee 1)\,dh = \int_0^{|m|} \frac{1-\cos(2\pi h)}{h}\left(-\log\frac{h}{|m|}\vee 1\right)dh$$

$$\approx \log^2|m|.$$

To summarize, for bounded functions we can characterize $F_0(0,1)$ by

$$\sum_{m\in\mathbb{Z},m\neq 0} |\widehat{f}(m)|^2\log|m| < \infty \tag{5.11}$$

and $F_{\log}(0,1)$ by

$$\sum_{m\in\mathbb{Z},m\neq 0} |\widehat{f}(m)|^2\log^2|m| < \infty. \tag{5.12}$$

The same characterizations hold for $I = (\frac{n}{2n+1},\frac{n+1}{2n+1})$ and the respective Fourier expansion.

STEP 3. We give an example of $f \in F_0(0,1) \cap L^\infty(0,1)$ for which (5.11) is satisfied and (5.12) is not. For $m = (2n+1)2^l$, $l = 1, 2, \ldots$, we put $\widehat{f}(m) = \frac{1}{l^{3/2}}$; for other m we let $\widehat{f}(m) = 0$. Note that f is $\frac{1}{2n+1}$-periodic. Therefore the jth Fourier coefficient of f on $\left(\frac{n}{2n+1}, \frac{n+1}{2n+1}\right)$ is equal to its $(2n+1)j$th Fourier coefficient on $(0,1)$. Since $(\widehat{f}(m))_{m\in\mathbb{Z}}$ is summable, f is bounded. Furthermore, $l^{-3}\log[(2n+1)2^l] = l^{-2}\log 2 + l^{-3}\log(2n+1)$ and $l^{-3}\log^2(2^l) \approx l^{-1}$. Therefore (5.11) is satisfied and (5.12) is not. By (5.10), the proof is finished. ∎

6. Uniformity is not a sharp condition. In this section we examine the strip $\mathbb{R} \times (0,1)$, which is a nonuniform domain. We will show that the comparability fails for fractional Sobolev spaces with $\alpha < 1$. Then we prove that for $\alpha > 1$ and slightly more general kernels the comparability holds. Later, we present a higher-dimensional case in which the comparability may also hold for $\alpha < 1$ in nonuniform domains. For clarity of presentation, we assume that $p = q = 2$.

EXAMPLE 6.1. Let $\Omega = \mathbb{R} \times (0,1)$ and let $K(x,y) = |x-y|^{-2-\alpha}$. Note that Ω is not uniform: if we take two cubes far from each other we will fail to find a sufficiently large central cube in any chain connecting them.

We will show that for $\alpha \in (0,1)$ the comparability does not hold. Consider a sequence of functions (f_n) given by $f_n(x_1, x_2) = \left(1 - \frac{|x_1|}{n}\right) \vee 0$. Since f_n is constant in the second variable, for every $\xi \in (0,1)$ we have

$$\int_\Omega \int_\Omega \frac{(f_n(x) - f_n(y))^2}{|x-y|^{2+\alpha}}\, dy\, dx = \int_{\mathbb{R}} \int_{\mathbb{R}} (f_n(x_1,\xi) - f_n(y_1,\xi))^2 \int_0^1 \int_0^1 |x-y|^{-2-\alpha}\, dy_2\, dx_2\, dy_1\, dx_1.$$

Let the integral over $(0,1)\times(0,1)$ be called $\kappa(x_1,y_1)$. We claim that $\kappa(x_1,y_1)$ is comparable with $|x_1-y_1|^{-2-\alpha}$ if $|x_1-y_1| \ge 1$ and with $|x_1-y_1|^{-1-\alpha}$ otherwise. Indeed, we have $|x-y| \approx |x_1-y_1| + |x_2-y_2|$. If $|x_1-y_1| \ge 1$, then

$$\int_0^1 \int_0^1 |x-y|^{-2-\alpha}\, dy_2\, dx_2 \approx |x_1-y_1|^{-2-\alpha} \int_0^1 \int_0^1 dy_2\, dx_2 = |x_1-y_1|^{-2-\alpha}.$$

For $|x_1-y_1| < 1$ note that for fixed $a > 0$,

$$a^{1+\alpha} \int_0^1 \int_0^1 (a + |x_2-y_2|)^{-2-\alpha}\, dy_2\, dx_2 \approx a^{1+\alpha} \int_0^1 \int_0^{x_2} (a + x_2 - y_2)^{-2-\alpha}\, dy_2\, dx_2$$

$$= \frac{a^{1+\alpha}}{1+\alpha} \int_0^1 (a^{-1-\alpha} - (a+x_2)^{-1-\alpha})\, dx_2$$
$$= \frac{1}{1+\alpha} - \frac{1}{1+\alpha} \int_0^1 \left(1 + \frac{x_2}{a}\right)^{-1-\alpha} dx_2.$$

For $a = |x_1 - y_1| < 1$ we have $x_2/a > x_2$, so the last integral is bounded from above by $C \in (0,1)$. Thus the whole expression is approximately equal to a positive constant, which proves our claim.

The shape of Ω grants that for every $\theta \in (0,1]$ we have

$$\int_\Omega \int_{B(x,\theta\delta(x))} \frac{(f_n(x) - f_n(y))^2}{|x-y|^{2+\alpha}}\, dy\, dx \le \int_{\mathbb{R}} \int_{B(x_1,1)} (f_n(x_1,\xi) - f_n(y_1,\xi))^2 \kappa(x_1,y_1)\, dy_1\, dx_1.$$

To simplify the notation we will write $f_n(x_1) = f_n(x_1,\xi)$ for some fixed $\xi \in (0,1)$, $x \in \mathbb{R}$. Since f_n is Lipschitz with constant $1/n$, we have

$$\int_{\mathbb{R}} \int_{B(x_1,1)} (f_n(x_1) - f_n(y_1))^2 \kappa(x_1,y_1)\, dy_1\, dx_1$$
$$\approx \int_{\mathbb{R}} \int_{B(x_1,1)} (f_n(x_1) - f_n(y_1))^2 |x_1 - y_1|^{-1-\alpha}\, dy_1\, dx_1$$
$$= \int_{-n-1}^{n+1} \int_{B(x_1,1)} (f_n(x_1) - f_n(y_1))^2 |x_1 - y_1|^{-1-\alpha}\, dy_1\, dx_1$$
$$\lesssim \frac{1}{n^2} \int_{-n-1}^{n+1} \int_{B(x_1,1)} |x_1 - y_1|^{1-\alpha}\, dy_1\, dx_1 \approx \frac{1}{n}.$$

Thanks to the fact that $\alpha < 1$, the full seminorm is significantly greater as $n \to \infty$:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (f_n(x_1) - f_n(y_1))^2 \kappa(x_1,y_1)\, dy_1\, dx_1 \gtrsim \int_{-n/2}^{0} \int_{-\infty}^{-n} |x_1 - y_1|^{-2-\alpha}\, dy_1\, dx_1$$
$$= \int_{-n/2}^{0} \frac{1}{1+\alpha} \frac{1}{(x_1+n)^{1+\alpha}}\, dx_1 \ge \frac{1}{1+\alpha} \frac{n/2}{n^{1+\alpha}} \approx \frac{1}{n^\alpha}.$$

Lemma 6.2. *Let $\Omega = \mathbb{R} \times (0,1)$. If $f\colon \mathbb{R}^2 \to [0,\infty)$ is radial, then $\int_\Omega (1 \vee |x|) f(x)\, dx \approx \int_{\mathbb{R}^2} f(x)\, dx < \infty$ with a constant independent of f.*

Proof. Note that for $n \in \mathbb{N}$ the area of $\Omega \cap (B_n \setminus B_{n-1})$ is comparable to the $(1/n)$th of the area of the annulus $B_n \setminus B_{n-1}$. Therefore by the rotational symmetry of f we get

$$\int_{\Omega} (1 \vee |x|) f(x)\, dx \approx \sum_{n\in\mathbb{N}} \int_{\Omega\cap(B_n\setminus B_{n-1})} n f(x)\, dx$$
$$\approx \sum_{n\in\mathbb{N}} \int_{B_n\setminus B_{n-1}} f(x)\, dx = \int_{\mathbb{R}^2} f(x)\, dx. \blacksquare$$

The case of $\alpha \in (1,2)$ is included in the following result.

THEOREM 6.3. *Let $\Omega = \mathbb{R} \times (0,1)$. Assume that K satisfies* **A1**–**A3** *and $\sum_{n\ge 1} \int_{B(0,n)^c} K(0,x)\, dx < \infty$. Then the seminorms* (1.2) *and* (1.3) *are comparable.*

Proof. We split the domain Ω into open unit cubes Q_n centered in $(n, 1/2)$, $n \in \mathbb{Z}$, so that $\Omega \subseteq \bigcup_{n\in\mathbb{Z}} \overline{Q_n}$. Let $L_n = \mathrm{Int}[\overline{Q_{n-1} \cup Q_n \cup Q_{n+1}}]$. Then L_n is a uniform domain, hence by Theorem 1.1,

$$\int_{L_n}\int_{L_n} (f(x) - f(y))^2 K(x,y)\, dy\, dx \approx \int_{L_n}\int_{B(x,\theta\delta(x))} (f(x) - f(y))^2 K(x,y)\, dy\, dx$$

with implied constant independent of n. Therefore for every $0 < \theta \le 1$,

$$\int_{\Omega}\int_{B(x,\theta\delta(x))} (f(x) - f(y))^2 K(x,y)\, dy\, dx \tag{6.1}$$
$$\approx \sum_{n\in\mathbb{Z}} \int_{L_n}\int_{L_n} (f(x) - f(y))^2 K(x,y)\, dy\, dx$$
$$\approx \sum_{n\in\mathbb{Z}} \int_{Q_n}\int_{L_n} (f(x) - f(y))^2 K(x,y)\, dy\, dx,$$

so it suffices to show that the last expression is comparable with the integral over $\Omega \times \Omega$. We have

$$\int_{\Omega}\int_{\Omega} (f(x) - f(y))^2 K(x,y)\, dy\, dx = \sum_{i,j\in\mathbb{Z}} \int_{Q_i}\int_{Q_j} (f(x) - f(y))^2 K(x,y)\, dy\, dx$$
$$\approx \sum_{i\in\mathbb{Z}} \sum_{j+1<i} \int_{Q_i}\int_{Q_j} (f(x) - f(y))^2 K(x,y)\, dy\, dx$$
$$+ \sum_{i\in\mathbb{Z}} \int_{Q_i}\int_{L_i} (f(x) - f(y))^2 K(x,y)\, dy\, dx.$$

Clearly it suffices to estimate the first summand. Since the cubes are far apart, we have $|x - y| \approx |i - j|$ for $x \in Q_i$, $y \in Q_j$. Hence

$$\sum_{i\in\mathbb{Z}} \sum_{j+1<i} \int_{Q_i}\int_{Q_j} (f(x) - f(y))^2 K(x,y)\, dy\, dx \tag{6.2}$$
$$\lesssim \sum_{i\in\mathbb{Z}} \sum_{j+1<i} \int_{Q_i}\int_{Q_j} (f(x) - f_{Q_i})^2 K(x,y)\, dy\, dx$$

$$+\sum_{i\in\mathbb{Z}}\sum_{j+1<i}\int_{Q_i}\int_{Q_j}(f(y)-f_{Q_j})^2K(x,y)\,dy\,dx$$
$$+\sum_{i\in\mathbb{Z}}\sum_{j+1<i}\sum_{j\le n<i}\int_{Q_i}\int_{Q_j}(f_{Q_{n+1}}-f_{Q_n})^2|x-y|K(x,y)\,dy\,dx.$$

In this inequality we have used $(a_1+\cdots+a_m)^2\le m(a_1^2+\cdots+a_m^2)$ and $|Q_i|=|Q_j|=1$. For the first term we use Jensen's inequality and the fact that the sum over j is uniformly bounded with respect to i and $x\in Q_i$:

$$\sum_{i\in\mathbb{Z}}\int_{Q_i}(f(x)-f_{Q_i})^2\sum_{j+1<i}\int_{Q_j}K(x,y)\,dy\,dx\lesssim\sum_{i\in\mathbb{Z}}\int_{Q_i}\int_{Q_i}(f(y)-f(x))^2\,dy\,dx.$$

The latter expression does not exceed (6.1). The second term can be estimated in a similar way after changing the order of summation.

By Lemma 6.2 the additional assumption on K is equivalent to

$$\sum_{n\ge1}\int_{B(0,n)^c\cap\Omega}|x|K(0,x)\,dx<\infty.$$

We change the order of summation and use that fact to estimate the last term on the right hand side of (6.2):

$$\begin{aligned}\sum_{i\in\mathbb{Z}}\sum_{j+1<i}\sum_{j\le n<i}(f_{Q_{n+1}}-f_{Q_n})^2\int_{Q_i}\int_{Q_j}|x-y|K(x,y)\,dy\,dx\\
=\sum_{n\in\mathbb{Z}}(f_{Q_{n+1}}-f_{Q_n})^2\sum_{i>n}\sum_{\substack{j+1<i\\ j\le n}}\int_{Q_i}\int_{Q_j}|x-y|K(x,y)\,dy\,dx\\
\lesssim\sum_{n\in\mathbb{Z}}(f_{Q_{n+1}}-f_{Q_n})^2\le\sum_{n\in\mathbb{Z}}\int_{Q_n}\int_{Q_{n+1}}(f(x)-f(y))^2\,dy\,dx\\
\lesssim\sum_{n\in\mathbb{Z}}\int_{Q_n}\int_{Q_{n+1}}(f(x)-f(y))^2K(x,y)\,dy\,dx.\ \blacksquare\end{aligned}$$

Proof of Theorem 1.2. The idea is similar to the above. We split Ω into a family $(Q_i)_{i\in\mathbb{Z}^k}$ of unit cubes and we let

$$L_i=\operatorname{Int}\Big[\bigcup\{\overline{Q_j}\colon B(x_{Q_i},\sqrt{d})\cap Q_j\ne\emptyset\}\Big].$$

By Theorem 1.1, for $0<\theta\le1$ we have

$$\begin{aligned}\int_\Omega\int_{B(x,\theta\delta(x))}(f(x)-f(y))^2|x-y|^{-d-\alpha}\,dy\,dx\\
\approx\sum_{i\in\mathbb{Z}^k}\int_{Q_i}\int_{L_i}(f(x)-f(y))^2|x-y|^{-d-\alpha}\,dy\,dx.\end{aligned}$$

For $i=(i_1,\ldots,i_k)$, $j=(j_1,\ldots,j_k)$ and $m\in\mathbb{N}_0$, we write $j>i+m$ if $j_1>i_1+m,\ldots,j_k>i_k+m$. By $j>m$ we mean $j>0+m$, and $j\ge i+m$ is defined by replacing *all* the inequalities by weak ones. By the radial symmetry of $|x-y|^{-d-\alpha}$ it suffices to show that under our assumptions on l and α we have

$$\sum_{i\in\mathbb{Z}^k}\int_{Q_i}\int_{L_i}(f(x)-f(y))^2|x-y|^{-d-\alpha}\,dy\,dx \gtrsim \sum_{i\in\mathbb{Z}^k}\int_{Q_i}\sum_{j>i+1}\int_{Q_j}(f(x)-f(y))^2|x-y|^{-d-\alpha}\,dy\,dx.$$

In order to perform a decomposition similar to (6.2) we fix a method of communication from Q_i to Q_j, $j>i$: first we move on the coordinate i_1 until we reach j_1, and then we do the same with the next coordinates. The set of indices of the cubes connecting Q_i and Q_j in the way presented above, with Q_i included and Q_j excluded, will be denoted $i\to j$. Note that $|i\to j| \approx |i-j|$. Let $\mathcal{N}(Q)$ be the successor of Q on the way from Q_i to Q_j. As before, we have $|i-j|\approx|x-y|$ for $x\in Q_i$, $y\in Q_j$, therefore

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}^k}\int_{Q_i}\sum_{j>i+1}\int_{Q_j}(f(x)-f(y))^2|x-y|^{-d-\alpha}\,dy\,dx\\
&\quad\lesssim \sum_{i\in\mathbb{Z}^k}\int_{Q_i}\sum_{j>i+1}\int_{Q_j}(f(x)-f_{Q_i})^2|x-y|^{-d-\alpha}\,dy\,dx\\
&\qquad+\sum_{i\in\mathbb{Z}^k}\int_{Q_i}\sum_{j>i+1}\int_{Q_j}(f(y)-f_{Q_j})^2|x-y|^{-d-\alpha}\,dy\,dx\\
&\qquad+\sum_{i\in\mathbb{Z}^k}\int_{Q_i}\sum_{j>i+1}\int_{Q_j}\sum_{n\in i\to j}(f_{Q_n}-f_{\mathcal{N}(Q_n)})^2|x-y|^{-d-\alpha+1}\,dy\,dx.
\end{aligned}$$

The first two terms can be handled as in the previous theorem. In the last one we change the order of summation and find that up to a constant it does not exceed

$$\sum_{n\in\mathbb{Z}^k}\Big(\int_{L_n}|f_{Q_n}-f(\xi)|\,d\xi\Big)^2\sum_{j\ge n}\sum_{\substack{i\le n\\ i+1<j}}\int_{Q_i}\int_{Q_j}|x-y|^{-d-\alpha+1}\,dy\,dx.$$

To finish the proof we note that the double sum over i,j does not depend on n, hence we take $n=(1,\dots,1)$ (for short, $n=1$) and we estimate as follows:

$$\begin{aligned}
\sum_{j\ge1}\sum_{\substack{i\le1\\ i+1<j}}\int_{Q_i}\int_{Q_j}|x-y|^{-d-\alpha+1}\,dy\,dx &\approx \sum_{j\ge1}\int_{Q_j}\int_{B(y,|j|)^c\cap\Omega}|x-y|^{-d-\alpha+1}\,dx\,dy\\
&=\sum_{j\ge1}\int_{Q_j}\sum_{m=0}^{\infty}\int_{(B(0,2^{m+1}|j|)\setminus B(0,2^m|j|))\cap\Omega}|x|^{-d-\alpha+1}\,dx\,dy
\end{aligned}$$

$$\approx \sum_{j\geq 1} \int_{Q_j} \sum_{m=0}^{\infty} (2^m|j|)^k (2^m|j|)^{-d-\alpha+1}\, dy$$
$$\approx \sum_{j\geq 1} |j|^{k-d-\alpha+1} = \sum_{j\geq 1} |j|^{-l-\alpha+1} \approx \sum_{j\in\mathbb{Z}^k\setminus\{0\}} |j|^{-l-\alpha+1},$$

which is finite provided that $k-l-\alpha<-1$. ∎

7. Application: a new class of Markov processes. In this section we show how our comparability results can be applied to prove the existence of Markov stochastic processes corresponding to the truncated seminorms (1.3). Hereafter we work with Sobolev spaces, i.e. $p=q=2$.

We will discuss several cases which depend on various results concerning Sobolev spaces and censored/reflected Markov processes, each with its own assumptions. Therefore we refrain from formulating any theorems here, as they would be unnecessarily complicated. Interested readers may gather the assumptions from the references provided for each case.

We will gradually introduce some notions concerning Dirichlet forms in *italics*; for details we refer to Fukushima, Oshima, and Takeda [14, Chapter 1.1]. Let $\mathcal{E}$ be a symmetric bilinear form with domain $D[\mathcal{E}]\subseteq L^2(\Omega)$ for some $\Omega\subseteq\mathbb{R}^d$. Let $\mathcal{E}_1(u,u)=\mathcal{E}(u,u)+\|u\|^2_{L^2(\Omega)}$. We say that $(\mathcal{E},D[\mathcal{E}])$ (this pair will also be called form below) is *closed* if, with respect to $\mathcal{E}_1$, every Cauchy sequence has a limit in $D[\mathcal{E}]$. We say that the form is *closable* if it has a closed extension. In what follows we write

$$\mathcal{E}^{\mathrm{cen}}(u,u)=\int_\Omega\int_\Omega (u(x)-u(y))^2K(x,y)\,dy\,dx,$$

and for $\theta\in(0,1]$,

$$\mathcal{E}^{\mathrm{tr}}(u,u)=\int_\Omega\int_{B(x,\theta\delta(x))} (u(x)-u(y))^2K(x,y)\,dy\,dx.$$

The symbol $\mathcal{E}^{\mathrm{cen}}$ refers to the censored stable processes introduced by Bogdan, Burdzy, and Chen [4]. There, the kernel was the one known from the fractional Sobolev spaces: $K(x,y)=c|x-y|^{-d-\alpha}$. Censored processes for more general K corresponding to a class of subordinated Brownian motions were studied by Wagner [28].

We will consider the above forms in two contexts. In the first one, we start with the space $C_c^\infty(\Omega)$ of smooth functions, compactly supported in Ω. Using the arguments which follow equation (2.4) in [4, p. 93] it can be shown that $(\mathcal{E}^{\mathrm{cen}},C_c^\infty(\Omega))$ is closable and *Markovian* for an arbitrary Lévy kernel K (in particular, any which satisfies **A1**) and set Ω. If we let

$$\mathcal{F}:=\text{completion of } C_c^\infty(\Omega) \text{ with respect to } \mathcal{E}_1^{\mathrm{cen}},$$

then by [14, Theorem 3.1.1], $(\mathcal{E}^{\mathrm{cen}}, \mathcal{F})$ is closed and Markovian, that is, a *Dirichlet* form. Furthermore, by construction, it is obvious that $C_c^\infty(D)$ is a *core* for $(\mathcal{E}^{\mathrm{cen}}, \mathcal{F})$, hence the form is *regular* and by [14, Theorem 7.2.1] to every regular Dirichlet form corresponds a Hunt process. Thus, when $\mathcal{E}^{\mathrm{cen}}$ is comparable to $\mathcal{E}^{\mathrm{tr}}$ we obtain the existence of a Hunt process with the Dirichlet form $(\mathcal{E}^{\mathrm{tr}}, \mathcal{F})$. We note that the arguments from [4, p. 93] may be used directly with $\mathcal{E}^{\mathrm{tr}}$ because it has a similar structure. Then, independently of comparability results, we obtain a regular Dirichlet form $(\mathcal{E}^{\mathrm{tr}}, \mathcal{F}^{\mathrm{tr}})$, where

$$\mathcal{F}^{\mathrm{tr}} := \text{completion of } C_c^\infty(\Omega) \text{ with respect to } \mathcal{E}_1^{\mathrm{tr}}.$$

The second approach is by considering the domain corresponding to the active reflected form $\mathcal{F}^{\mathrm{ref}} := F_{2,2}(\Omega)$. Here the argument becomes more tedious, since in general $C_c^\infty(\Omega)$ (or even $C_c(\Omega)$) need not be dense in $\mathcal{F}^{\mathrm{ref}}$. However, for some K and Ω the density holds true (see e.g., [4, Corollary 2.6] and [28, Corollary 2.9]). In that case we get $\mathcal{F} = F_{2,2}(\Omega)$ and when the comparability holds, we in fact have $\mathcal{F}^{\mathrm{tr}} = F_{2,2}(\Omega)$. Thus, the form $(\mathcal{E}^{\mathrm{tr}}, F_{2,2}(\Omega))$ is a regular Dirichlet form and there exists an associated Hunt process. If the density does not hold, the technical remedy is to change the reference set to $\overline{\Omega}$ (cf. [4, Remark 2.1]). If K and Ω are sufficiently regular, then there exist extension (and trace) operators between $F_{2,2}(\Omega)$ and $F_{2,2}(\mathbb{R}^d)$ (see e.g. Jonsson and Wallin [17, Chapter V] or Rutkowski [23, Section 6]). Thanks to them we may show that $C_c^\infty(\overline{\Omega})$ is dense with respect to $\mathcal{E}_1^{\mathrm{cen}}$ in $F_{2,2}(\Omega)$ by using the results for the functions on the whole space $\mathbb{R}^d$, available for very general Lévy kernels (see, e.g., Bogdan, Grzywny, Pietruska-Pałuba, and Rutkowski [5, Lemma A.5] or Fiscella, Servadei, and Valdinoci [13]). Then we obtain the existence of a process on $\overline{\Omega}$ corresponding to the regular Dirichlet form $(\mathcal{E}^{\mathrm{cen}}, F_{2,2}(\Omega))$ and the comparability yields the existence of the process corresponding to $(\mathcal{E}^{\mathrm{tr}}, F_{2,2}(\Omega))$.

The last case seems more interesting in terms of applying the comparability results as we build regular Dirichlet forms from the truncated form $\mathcal{E}^{\mathrm{tr}}$ on the well-established Sobolev/Triebel–Lizorkin space $F_{2,2}(\Omega)$, which is then its natural domain.

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