CHAPTER 1

Lebesgue Measure and Integration

1.1. SET FUNCTIONS

If A and B are any two sets, we write $A - B$ for the set of all elements x such that $x \in A$, $x \notin B$. The notation $A - B$ does not imply that $B \subset A$. We denote the empty set by $\varnothing$, and say that A and B are disjoint if $A \cap B = \varnothing$.

Definition 1. A family $\mathfrak{R}$ of sets is called a ring if $A, B \in \mathfrak{R}$ implies

$$A \cup B \in \mathfrak{R}, \text{ and } \; A - B \in \mathfrak{R} \tag{1.1.1}$$

Remark 1. Note that

$$A \cap B = A - (A - B) = B - (B - A)$$

so for any $A, B \in \mathfrak{R}$ we also have $A \cap B \in \mathfrak{R}$ if $\mathfrak{R}$ is a ring.

The set $\mathrm{F}(\mathbb{N}) = \{A \subset \mathbb{N} : A \text{ is finite } \}$ is an example of ring.

Definition 2. A ring $\mathfrak{R}$ is called a σ**-**ring if

$$\bigcup_{n=1}^{\infty} A_n \in \mathfrak{R} \tag{1.1.2}$$

whenever $A_n \in \mathfrak{R}$ for all $n = 1, 2, 3, \ldots$.

Since

$$\bigcap_{n=1}^{\infty} A_n = A_1 \cap \bigcap_{n=2}^{\infty} A_n = A_1 - \left(A_1 - \bigcap_{n=2}^{\infty} A_n \right) = A_1 - \bigcup_{n=1}^{\infty} \left(A_1 - A_n \right), \tag{1.1.3}$$

for any $A_n \in R$, $n = 1, 2, \ldots$, we also have $\bigcap\limits_{n=1}^{\infty} A_n \in \mathfrak{R}$ if $\mathfrak{R}$ is a σ-ring.

An Example of σ-ring is $\mathcal{P}(X)$ the set of all subsets of and set X.

Note that $\mathrm{F}(\mathbb{N})$ is not a σ-ring, because $E = \bigcup\limits_{n=1}^{\infty} \{2n\}$ the set of all even numbers is not finite.

Definition 3. We say that ϕ is a set function defined on σ-ring $\mathfrak{R}$ if ϕ assigns to every $A \in \mathfrak{R}$ a number $\phi(A)$ of the extended real number system. ϕ is **additive** if $A \cap B = \varnothing$ implies

$$\phi(A \cup B) = \phi(A) + \phi(B), \tag{1.1.4}$$

and ϕ is **countably additive** if $A_i \cap A_j = \varnothing$ for $i \neq j$, (in this case we say that the family A_i is pairwise disjoint) implies

$$\phi\left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \phi(A_n). \tag{1.1.5}$$

REMARK 2. Here we will assume that ϕ is not the constant functions whose only value is $+\infty$ or $-\infty$, and that the range does not contain both $+\infty$ and $-\infty$, because if it did, the right side of (1.1.4) could lose meaning.

REMARK 3. Note that the left side of (1.1.5) is independent of the order in which the A_n's are arranged. Hence, by the rearrangement theorem for series, if the right hand side of (1.1.5)) converges, it converges absolutely. Otherwise, the partial sums tend to $+\infty$ or $-\infty$.

THEOREM 1. *If ϕ is additive, then*
(1) $\phi(\varnothing) = 0$.
(2) $\phi\left(\bigcup_{n=1}^{k} A_n\right) = \sum_{n=1}^{k} \phi(A_n)$, *if* $A_i \cap A_j = \varnothing$ *for* $i \neq j$.
(3) $\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2)$.
(4) *If* $\phi(A) \geq 0$ *for all* A, *and* $A \subset B$, *then* $\phi(A) \leq \phi(B)$.
added — PS (5) *if* $A \subset B$,*and* $|\phi(A)| < \infty$, *then* $\phi(B - A) = \phi(B) - \phi(A)$.

(1) Note that for all $A \in \mathfrak{R}$, $A = A \cup \varnothing$, and $A \cap \varnothing = \varnothing$. Since ϕ is additive, then
$$\phi(A) + 0 = \phi(A) = \phi(A \cup \varnothing) = \phi(A) + \phi(\varnothing)$$
so $0 = \phi(\varnothing)$
(2) The case $n = 2$ is the definition of additive function. Suppose inductively that
$$\phi(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) = \phi(A_1) + \phi(A_2) + \phi(A_3) + \cdots + \phi(A_n),$$
if $A_i \cap A_j = \varnothing$ for $i \neq j$, and let $A_1, A_2, A_3, \ldots, A_{n+1}$ so that $A_i \cap A_j = \varnothing$ for $i \neq j$, and define $B_1 = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$, $B_2 = A_{n+1}$, then $B_1 \cap B_2 = \varnothing$ and
$$\begin{aligned}\phi(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_{n+1}) &= \phi(B_1 \cup B_2) = \phi(B_1) + \phi(B_2)\\ &= \phi(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) + \phi(A_{n+1})\\ &= \phi(A_1) + \phi(A_2) + \phi(A_3) + \cdots + \phi(A_n) + \phi(A_{n+1})\end{aligned}$$
where in the last equality we use the inductive hypothesis.
(3) Note that
$$\begin{aligned}A_1 &= (A_1 - A_2) \cup (A_1 \cap A_2)\\ A_2 &= (A_2 - A_1) \cup (A_1 \cap A_2)\\ A_1 \cup A_2 &= (A_1 - A_2) \cup (A_2 - A_1) \cup (A_1 \cap A_2)\end{aligned}$$
(see figure below), so
$$\begin{aligned}\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) &= (\phi(A_1 - A_2) + \phi(A_2 - A_1) + \phi(A_1 \cap A_2)) + \phi(A_1 \cap A_2)\\ &= (\phi(A_1 - A_2) + \phi(A_1 \cap A_2)) + (\phi(A_2 - A_1) + \phi(A_1 \cap A_2))\\ &= \phi(A_1) + \phi(A_2)\end{aligned}$$

(4) Since $A \subset B$, then $B = A \cup (B - A)$ and $A \cap (B - A) = \varnothing$, so $\phi(B) = \phi(A) + \phi(B - A) \geq \phi(A)$, because $\phi(B - A) \geq 0$

(5) By item 4 we have $\phi(B) = \phi(A) + \phi(B - A)$, since $|\phi(A)| < \infty$, then

$$\phi(B) - \phi(A) = (\phi(A) + \phi(B - A)) - \phi(A) = \phi(B - A)$$

Note that non-negative additive set functions satisfy item 4, because this fact these functions are called **monotonic**.

THEOREM 2. *Suppose ϕ is countably additive on a ring $\mathfrak{R}$. Suppose $A_n \in \mathfrak{R}$ for $n = 1, 2, 3, \ldots$, $A_1 \subset A_2 \subset A_3 \subset \cdots \subset A \in \mathfrak{R}$ and*

$$A = \bigcup_{n=1}^{\infty} A_n.$$

Then,

$$\phi(A) = \lim_{n \to \infty} \phi(A_n).$$

Let $B_1 = A_1$, and $B_n = A_n - A_{n-1}$ for $n = 2, 3, \ldots$. Then $B_i \cap B_j = \varnothing$, for $i \neq j$. Indeed, since $i \neq j$ we can suppose $i < j$, then by hypothesis

$$A_{i-1} \subset A_i \subseteq A_{j-1} \subset A_j.$$

So

$$B_i = A_i - A_{i-1} \subset A_i \subseteq A_{j-1}$$

Since $A_{j-1} \cap B_j = A_{j-1} \cap (A_j - A_{j-1}) = \varnothing$, then $B_i \cap B_j = \varnothing$.

On the other hand, using induction we show that $A_n = \bigcup_{j=1}^{n} B_j$. added — PS

For $n = 1$, the above equality is $A_1 = B_1$, and suppose that this equality is true for $n = k$, since

$$A_{k+1} = (A_{k+1} - A_k) \cup A_k,$$

then

$$A_{k+1} = B_{k+1} \cup A_k = B_{k+1} \cup \bigcup_{j=1}^{k} B_j = \bigcup_{j=1}^{k+1} B_j.$$

Furthermore, $A_1 \subset A_2 \subset A_3 \subset \cdots \subset A_n$, implies $\bigcup_{j=1}^{n} A_j = A_n = \bigcup_{j=1}^{n} B_j$. Then

$$A = \bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} B_j.$$

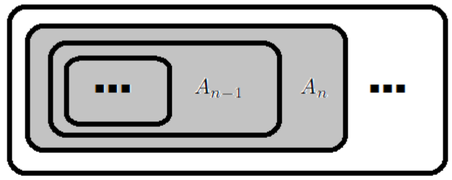
The figure below illustrates this fact

Since ϕ is countably additive, then

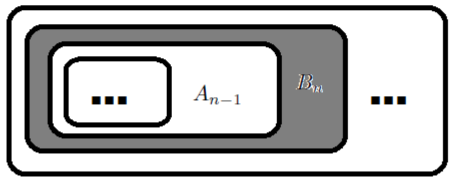
$$\phi(A_n) = \phi\left(\bigcup_{j=1}^{n} B_j\right) = \sum_{j=1}^{n} \phi(B_j),$$

and

$$\phi(A) = \phi\left(\bigcup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} \phi(B_j).$$



$$A=\bigcup_{n=1}^{\infty}A_n=\bigcup_{n=1}^{\infty}B_n,\quad B_n=\begin{cases}A_n-A_{n-1}, & n>1,\\ A_1, & n=1.\end{cases}$$



So

$$\lim_{n\to\infty}\phi(A_n)=\lim_{n\to\infty}\sum_{j=1}^{n}\phi(B_j)=\sum_{j=1}^{\infty}\phi(B_j)=\phi(A)$$

as the theorem states.

THEOREM 3. *Suppose* $\{\phi_n : n=1,2,3,\ldots\}$ *is sequence of countably additive functions on a ring* $\mathfrak{R}$. *Suppose* $\phi_n\le\phi_{n+1}$ *for* $n=2,\ 3,\ \ldots,$ *then*

$$\phi=\sup\{\phi_n : n=1,2,3,\ldots\}$$

is a countably additive function on $\mathfrak{R}$.

Note that for each $B\in\mathfrak{R}$,$\phi_n(B)\le\phi(B)$ for all $n\in\mathbb{N}$. Hence if $A=\bigcup\limits_{j=1}^{\infty}A_j$, with $A_i\cap A_j=\varnothing$, then for each $n\in\mathbb{N}$ we have

$$\phi_n(A)=\sum_{j=1}^{\infty}\phi_n(A_j)\le\sum_{j=1}^{\infty}\phi(A_j).$$

So

$$\phi(A)=\sup_{n\in\mathbb{N}}\phi_n(A)\le\sum_{j=1}^{\infty}\phi(A_j)\tag{1.1.6}$$

On the other hand, Since $B_k=\bigcup\limits_{j=1}^{k}A_j\subset A$, then

$$\phi(B_k)=\sup_{n\in\mathbb{N}}\{\phi_n(B_k)\}\le\sup_{n\in\mathbb{N}}\{\phi_n(A)\}=\phi(A)$$

Fix $k\in N$ and $\varepsilon>0$, for each $j=1,\ 2,\ \ldots,\ k$ there exists n_j so that $\phi(A_j)<\phi_{n_j}(A_j)+\varepsilon/k$. Since $\{\phi_n(A_j) : n\in\mathbb{N}\}$ is increasing, taking $n=\max\{n_j : j=1,2,\ldots,k\}$, and using the countable aditivity of ϕ_n we have

$$\sum_{j=1}^{k}\phi\left(A_j\right)\leq\left(\sum_{j=1}^{k}\phi_n\left(A_j\right)\right)+\varepsilon=\phi_n\left(\bigcup_{j=1}^{k}A_j\right)+\varepsilon\leq\phi\left(\bigcup_{j=1}^{k}A_j\right)+\varepsilon\leq\phi\left(A\right)+\varepsilon.$$

Since ε was arbitrary, then

$$\sum_{j=1}^{k}\phi\left(A_j\right)\leq\phi\left(A\right), \tag{1.1.7}$$

and taking limit when $k\to\infty$, we obtain???

$$\sum_{j=1}^{\infty}\phi\left(A_j\right)\leq\phi\left(A\right) \tag{1.1.8}$$

Equations (1.1.6) and (1.1.8) imply

$$\sum_{j=1}^{\infty}\phi\left(A_j\right)=\phi\left(A\right).$$

1.2. CONSTRUCTION OF THE LEBESGUE MEASURE

Definition 4. Let $\mathbb{R}^p$ denote p-dimensional space. By an **interval** in $\mathbb{R}^p$ we mean the set of points $x=(x_1,\ldots,x_p)$ such that

$$a_i\leq x_i\leq b_i \tag{1.2.1}$$

for $i=1,\ \ldots,\ p$, or the set of points which is characterized by (1.2.1) with any or all of the signs $\leq$ replaced by $<$. The possibility that $a_i=b_i$, for any value of i is not ruled out; in particular, the empty set is included among the intervals.

Note that an interval in $\mathbb{R}^p$ is the cartesian product of finite intervals (closed, open, semiopen or degenarate) of $\mathbb{R}$.

Definition 5. If A is the finite union of intervals, A is said to be an **elementary set**.

If $\mathbf{I}$ is an interval, we define the measure of $\mathbf{I}$ by

$$m\left(\mathbf{I}\right)=\prod_{i=1}^{p}\left(b_i-a_i\right)$$

no matter whether equality is included or excluded in any of the inequalities (1.2.1).

Remark 4. If $\mathbf{I}=I_1\times I_2\times\cdots\times I_p$ and $\mathbf{J}=J_1\times J_2\times\cdots\times J_p$ where $I_1,\ \ldots,$ and $J_1,$ $\ldots$,$\mathbf{I}\cap\mathbf{J}=(I_1\cap J_1)\times(I_2\cap J_2)\times\cdots\times(I_p\cap J_p)$

Remark 5. If I,J are two finite intervals of $\mathbb{R}$, then $I-J$ can be written as the union of two (possible empty) intervals. Indeed, let $a\leq b$ be the extreme points of I and $c\leq d$ be the extreme points of J, then we can have

(a) $a\leq b\leq c\leq d$, in this case $I-J=I$.
(b) $a\leq c\leq b\leq d$, in this case $I-J$ is the interval with extreme points a, c.
(c) $a\leq c\leq d\leq b$, in this case $I-J$ is the union of the intervals with extreme points a, c and d, b.

(d) $c \leq a \leq d \leq b$, in this case $I - J$ is the interval with extreme points d, b.
(e) $c \leq d \leq a \leq b$, in this case $I - J = I$.
added — PS (f) $c \leq a \leq b \leq d$, in this case $I - J = \varnothing$.

Note that, if $\mathbf{I} = I_1 \times I_2 \times \cdots \times I_p$ and $\mathbf{J} = J_1 \times J_2 \times \cdots \times J_p$, then
(1.2.2)
$$\mathbf{I} - \mathbf{J} = (I_1 - J_1) \times I_2 \times \cdots \times I_p \cup (I_1 - J_1) \times (I_2 - J_2) \times \cdots \times I_p \cup \cdots \cup (I_1 - J_1) \times (I_2 - J_2) \times \cdots \times (I_p - J_p)$$
Thus, by Remark 5 $\mathbf{I} - \mathbf{J}$ is the union of intervals disjoint in $\mathbb{R}^p$.

Definition 6. If If the intervals I_j are pairwise disjoint, then for $A = \bigcup_{j=1}^{k} \mathbf{I}_j$, we set

$$m(A) = \sum_{j=1}^{k} m(\mathbf{I}_j) \tag{1.2.3}$$

We denote by $\mathcal{E}$ the family of all elementary subsets of $\mathbb{R}^p$. Note that $\mathcal{E}$ satisfies the following properties.

$\mathcal{E}$ 1 $\mathcal{E}$ is a ring, but not a σ-ring. Clearly the if $A = \bigcup_{n=1}^{k} \mathbf{I}_n$ and $B = \bigcup_{m=1}^{l} \mathbf{J}_m$, then $A \cup B = \bigcup_{n=1}^{k} \mathbf{I}_n \cup \bigcup_{m=1}^{l} \mathbf{J}_m$ and

$$A - B = \bigcup_{n=1}^{k} \left(\mathbf{I}_n - \bigcup_{m=1}^{l} \mathbf{J}_m \right) = \bigcup_{n=1}^{k} \left(\bigcap_{n=1}^{l} (\mathbf{I}_n - \mathbf{J}_m) \right)$$

but $\mathbf{I}_n - \mathbf{J}_m$ is the union of at most $2p$ intervals in $\mathbb{R}^p$ and by Remark 4 the intersection of intervals is an interval, then $A - B$ is a finite union of intervals in $\mathbb{R}^p$.

added — PS Finally note that $\mathcal{E}$ is not a σ-ring: if $\mathbb{R}^p$ is an element of $\mathcal{E}$, since $\mathbb{R}^p$ can not be written as a finite union of intervals in $\mathbb{R}^p$ then $\mathcal{E}$ is not a σ-ring

$\mathcal{E}$ 2 If $A \in \mathcal{E}$, then A is the union of a finite number of disjoint intervals. If A is an interval this fact is obvious. Now suppose that all unions of k intervals is the union of a finite number of disjoint intervals, and let $A = \bigcup_{n=1}^{k+1} \mathbf{I}_n$, then

$$\begin{aligned} A = \bigcup_{n=1}^{k+1} \mathbf{I}_n = \mathbf{I}_{k+1} \cup \bigcup_{n=1}^{k} \mathbf{I}_n &= \mathbf{I}_{k+1} \cup \bigcup_{n=1}^{l} \mathbf{J}_n \\ &= \left(\mathbf{I}_{k+1} - \bigcup_{n=1}^{l} \mathbf{J}_n \right) \cup \bigcup_{n=1}^{l} \mathbf{J}_n \\ &= \left(\bigcap_{n=1}^{l} (\mathbf{I}_{k+1} - \mathbf{J}_n) \right) \cup \bigcup_{n=1}^{l} \mathbf{J}_n \end{aligned}$$

where the $\mathbf{J}_n$ are disjoint intervals, since $\mathbf{I}_{k+1} - \mathbf{J}_n$ is the finite union of disjoint intervals (see 5) and the intersection of intervals is an interval, then A is the union of a finite number of disjoint intervals.

$\mathcal{E}$ 3 If $A \in \mathcal{E}$, $m(A)$ is well defined by (1.2.3); that is, if two different decompositions of A into disjoint intervals are used, each gives rise to the same value of $m(A)$. Indeed, If $A = \bigcup_{r=1}^{k} \mathbf{I}_r = \bigcup_{q=1}^{l} \mathbf{J}_q$, where the intervals $\mathbf{I}_r$ and $\mathbf{J}_q$ are pairwise disjoint, then for each $r = 1, 2, \ldots, k$ and $q = 1, 2, \ldots, l$ we have

$$\mathbf{I}_r = A \cap \mathbf{I}_n = \bigcup_{q=1}^{l} (\mathbf{J}_q \cap \mathbf{I}_r)$$

$$\mathbf{J}_q = A \cap \mathbf{J}_q = \bigcup_{r=1}^{k} (\mathbf{J}_q \cap \mathbf{I}_r)$$

since the family $\{\mathbf{B}_{qr} = \mathbf{J}_q \cap \mathbf{I}_r : r = 1, 2, \ldots, k \text{ and } q = 1, 2, \ldots, l\}$ is pairwise disjoint, then

$$\begin{aligned} m(A) &= \sum_{r=1}^{k} m(\mathbf{I}_r) = \sum_{r=1}^{k} \left(\sum_{q=1}^{l} m(\mathbf{J}_q \cap \mathbf{I}_r) \right) = \sum_{n=1}^{k} \left(\sum_{q=1}^{l} m(\mathbf{B}_{qr}) \right) \\ &= \sum_{q=1}^{l} \left(\sum_{r=1}^{k} m(\mathbf{B}_{qr}) \right) = \sum_{q=1}^{l} \left(\sum_{r=1}^{k} m(\mathbf{J}_q \cap \mathbf{I}_r) \right) = \sum_{q=1}^{l} m(\mathbf{J}_q) \end{aligned}$$

$\mathcal{E}$ 4 m is additive on $\mathcal{E}$. Indeed, If $A = \bigcup_{r=1}^{k} \mathbf{I}_r$ and $B = \bigcup_{q=1}^{l} \mathbf{J}_q$, where the intervals $\mathbf{I}_r$ and $\mathbf{J}_q$ are pairwise disjoint, and $A \cap B = \varnothing$, then $\mathbf{I}_r \cap \mathbf{J}_q = \varnothing$ for each $r = 1, 2, \ldots, k$ and $q = 1, 2, \ldots, l$.

Since $A \cup B = \bigcup_{r=1}^{k} \mathbf{I}_r \cup \bigcup_{q=1}^{l} \mathbf{J}_q$, then

$$m(A \cup B) = m\left(\bigcup_{r=1}^{k} \mathbf{I}_r \cup \bigcup_{q=1}^{l} \mathbf{J}_q \right) = \sum_{r=1}^{k} m(\mathbf{I}_r) + \sum_{q=1}^{l} m(\mathbf{J}_q) = m(A) + m(B)$$

Note that if $p = 1, 2, 3$, then m is length, area, and volume, respectively.

Definition 7. A non-negative additive set function ϕ defined on $\mathcal{E}$ is said to be regular if the following is true: To every A and to every $\varepsilon > 0$ there exist sets $F, G \in \mathcal{E}$ such that F is closed, G is open, $F \subset A \subset G$, and

$$\phi(G) - \varepsilon \leq \phi(A) \leq \phi(F) + \varepsilon \tag{1.2.4}$$

Note that by 1.2 we have $A = \bigcup_{n=1}^{k} \mathbf{I}_n$ where $\mathbf{I}_n$ are intervals pairwise disjoint. So for each $n = 1, 2, \ldots, k$, if F_n is a closed set and G_n is an open set, such that $F_n \subset \mathbf{I}_n \subset G_n$ and

$$\phi(G_n) - \frac{\varepsilon}{k} \leq \phi(\mathbf{I}_n) \leq \phi(F_n) + \frac{\varepsilon}{k}$$

Then $F = \bigcup_{n=1}^{k} F_n$ and $G = \bigcup_{n=1}^{k} G_n$, satisfy requirement in Definition 7 for A. Thus, to show that ϕ is regular on $\mathcal{E}$ It is sufficient to verify the conditions of Definition 7 only in the intervals of $\mathbb{R}^p$.

EXERCISE 1. [Exercise 11.15] Let $\mathcal{R}$ be the ring of all elementary subsets of $(0,1]$. If $0 < a < b \leq 1$, define
$$\phi\left((a,b)\right) = \phi\left((a,b]\right) = \phi\left([a,b)\right) = \phi\left([a,b]\right) = b - a$$
but define
$$\phi\left((0,b)\right) = 1 + b$$
if $0 < b \leq 1$. Show that this gives an additive set function ϕ on $\mathcal{R}$, which is not regular and which cannot be extended to a countably additive set function on a σ-ring.

SOLUTION 1. Here as in Definition 6 we define
$$\phi\left(A\right) = \sum_{j=1}^{k} \phi(\mathbf{I}_j), \tag{1.2.5}$$
if $A = \bigcup_{j=1}^{k} \mathbf{I}_j$, and the intervals $\mathbf{I}_j$ are pairwise disjoint.

First, if A is an elementary set, $\phi(A)$ is well defined by (1.2.5); that is, if two different decompositions of A into disjoint intervals are used, each gives rise to the same value of $\phi(A)$. Indeed, if $A = \bigcup_{n=1}^{k} \mathbf{I}_n = \bigcup_{m=1}^{l} \mathbf{J}_m$, where the intervals $\mathbf{I}_n$ and $\mathbf{J}_m$ are pairwise disjoint, then for each $n = 1, 2, \ldots, k$ and $m = 1, 2, \ldots, l$ we have
$$\mathbf{I}_n = A \cap \mathbf{I}_n = \bigcup_{m=1}^{l} \left(\mathbf{J}_m \cap \mathbf{I}_n\right)$$
$$\mathbf{J}_m = A \cap \mathbf{J}_m = \bigcup_{n=1}^{k} \left(\mathbf{J}_m \cap \mathbf{I}_n\right)$$
since the family $\{\mathbf{B}_{mn} = \mathbf{J}_m \cap \mathbf{I}_n : n = 1, 2, \ldots, k \text{ and } m = 1, 2, \ldots, l\}$ is pairwise disjoint, then
$$\begin{aligned}\phi(A) &= \sum_{n=1}^{k} \phi(\mathbf{I}_n) = \sum_{n=1}^{k}\left(\sum_{m=1}^{l} \phi\left(\mathbf{J}_m \cap \mathbf{I}_n\right)\right) = \sum_{n=1}^{k}\left(\sum_{m=1}^{l} \phi\left(\mathbf{B}_{mn}\right)\right) \\ &= \sum_{m=1}^{l}\left(\sum_{n=1}^{k} \phi\left(\mathbf{B}_{mn}\right)\right) = \sum_{m=1}^{l}\left(\sum_{n=1}^{k} \phi\left(\mathbf{J}_m \cap \mathbf{I}_n\right)\right) = \sum_{m=1}^{l} \phi(\mathbf{J}_m)\end{aligned}$$

Recall that if A is an elementary set, then $A = \bigcup_{j=1}^{k} \mathbf{I}_j$ is the union of a finite number of disjoint intervals (see 1.2). So
$$\phi\left(A\right) = \begin{cases} \sum_{j=1}^{k} l(\mathbf{I}_j) & \text{if 0 is not the end point of any interval in } A \\ 1 + \sum_{j=1}^{k} l(\mathbf{I}_j) & \text{if 0 is the end point of any interval in } A \end{cases}$$
where $l(\mathbf{I}_j)$ is the length of the interval $\mathbf{I}_j$. In particular, $\phi\left(A\right) < 1$ if A is a closed set of $(0,1]$. Indeed, note that $0 \notin A$ for any subset $A \subset (0,1]$.

Now, if 0 is the endpoint of any interval in A, since A is closed then 0 is limit point of A, and $0 \in A$. which is clearly a contradiction.

If two elementary sets A and B are disjoint, at most one of them can have the point 0 as the endpoint of one of its intervals. Then $\phi\left(A \cup B\right)$ is the sum of the lengths of the intervals in $A \cup B$ if neither set contains an interval having 0 as the endpoint, and 1 plus this sum if one of them does contain an interval with 0 as endpoint. In either case $\phi(A \cup B) = \phi(A) + \phi(B)$ when $A \cap B = \varnothing$.Thus, the function ϕ is additive.

The function ϕ is not regular, because by definition $\phi\left(\left(0, \frac{1}{2}\right]\right) = 1 + \frac{1}{2} = \frac{3}{2}$, but $\phi(A) < 1$ if A is closed, so taking $\varepsilon = \frac{1}{3}$, we have $\phi(A) + \frac{1}{3} < 1 + \frac{1}{2} = \phi\left(\left(0, \frac{1}{2}\right]\right)$ for all closed $A \subset \left(0, \frac{1}{2}\right)$. Thus, ϕ does not satisfy Definition 7.

The function also cannot be extended to a countably additive set function on a σ-ring, because

$$\left(0, \frac{1}{2}\right] = \bigcup_{n=1}^{\infty} \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$$

the intervals in this union are pairwise disjoint, but

$$\phi\left(\left(0, \frac{1}{2}\right]\right) = \frac{3}{2} > \frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \sum_{n=1}^{\infty} \phi\left(\left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]\right).$$

EXAMPLE 1.

(a) The set function m is regular. If A is an interval, i.e., $A = I_1 \times I_2 \times \cdots \times I_n$, if a_i, b_i are the extreme points of I_i, then by the continuity of the volume function on $\mathbb{R}^p$, we can choose r such that

$$G = (a_1 - r, b_1 + r) \times (a_2 - r, b_2 + r) \times \cdots \times (a_n - r, b_n + r)$$
$$F = [a_1 + r, b_1 - r] \times [a_2 + r, b_2 - r] \times \cdots \times [a_n + r, b_n - r]$$

and

$$\phi(G) = \prod_{j=1}^{n} (b_j - a_j + 2r) \leq \prod_{j=1}^{n} (b_j - a_j) + \varepsilon = \phi(A) + \varepsilon$$
$$\phi(A) - \varepsilon = \prod_{j=1}^{n} (b_j - a_j) - \varepsilon \leq \prod_{j=1}^{n} (b_j - a_j - 2r) = \phi(F)$$

(b) Take $p = 1$, and let α be a monotonically increasing function defined for all real x. Put

$$\mu([a, b)) = \alpha(b-) - \alpha(a-) = \sup_{t<b} \alpha(t) - \sup_{t<a} \alpha(t)$$
$$\mu([a, b]) = \alpha(b+) - \alpha(a-) = \inf_{b<t} \alpha(t) - \sup_{t<a} \alpha(t)$$
$$\mu((a, b]) = \alpha(b+) - \alpha(a+) = \inf_{b<t} \alpha(t) - \inf_{a<t} \alpha(t)$$
$$\mu((a, b)) = \alpha(b-) - \alpha(a+) = \sup_{t<b} \alpha(t) - \inf_{a<t} \alpha(t)$$

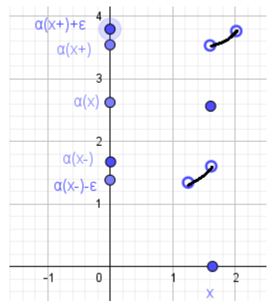
Recall that if α is a monotonically increasing function then the set of points of discontinuity of α is at most countable, and if α is continuous at x, then $\alpha(x-) = \alpha(x) = \alpha(x+)$. Also for each a, $b \in \mathbb{R}$ with $a < b$, by the definition of infimum and supremum given $\varepsilon > 0$ there exists c, x, y, d with $c < a < x < y < b < d$, so that α is continuous at c, x, y, d and

$$\alpha(a-) - \frac{\varepsilon}{2} < \alpha(c) \quad \alpha(x) < \alpha(a+) + \frac{\varepsilon}{2}$$
$$\alpha(b-) - \frac{\varepsilon}{2} < \alpha(y) \quad \alpha(d) < \alpha(b+) + \frac{\varepsilon}{2}$$

or equivalently

$$-\alpha(c)-\frac{\varepsilon}{2}<-\alpha(a-)\qquad-\alpha(a+)<-\alpha(x)+\frac{\varepsilon}{2}$$
$$\alpha(b-)<\alpha(y)+\frac{\varepsilon}{2}\qquad\alpha(d)-\frac{\varepsilon}{2}<\alpha(b+).$$

The behavior of a monotonically increasing function around a discontinuity point x is sketched in the figure below



Now we show that μ is regular on $\mathcal{E}$. Here c, x, y, d are as above

(1) In the case $A=[a,b)$ consider $F=[a,y]$ and $G=(c,b)$, then
$$\phi(G)-\varepsilon<\phi(G)-\frac{\varepsilon}{2}=\alpha(b-)-\alpha(c)-\frac{\varepsilon}{2}\leq\alpha(b-)-\alpha(a-)=\phi(A)$$
$$\phi(A)=\alpha(b-)-\alpha(a-)\leq\alpha(y)+\frac{\varepsilon}{2}-\alpha(a-)=\phi(F)+\frac{\varepsilon}{2}<\phi(F)+\varepsilon$$

(2) In the case $A=[a,b]$ consider $F=[a,b]$ and $G=(c,d)$, then
$$\phi(G)-\varepsilon=\alpha(d)-\frac{\varepsilon}{2}-\alpha(c)-\frac{\varepsilon}{2}\leq\alpha(b+)-\alpha(a-)=\phi(A)$$
$$\phi(A)=\phi(F)<\phi(F)+\varepsilon$$

(3) In the case $A=(a,b]$ consider $F=[x,b]$ and $G=(a,d)$, then
$$\phi(G)-\varepsilon<\phi(G)-\frac{\varepsilon}{2}=\alpha(d)-\frac{\varepsilon}{2}-\alpha(a+)\leq\alpha(b+)-\alpha(a+)=\phi(A)$$
$$\phi(A)=\alpha(b+)-\alpha(a+)\leq\alpha(b+)-\alpha(x)+\frac{\varepsilon}{2}=\phi(F)+\frac{\varepsilon}{2}<\phi(F)+\varepsilon$$

(4) In the case $A=(a,b)$ consider $F=[x,y]$ and $G=(a,b)$, then
$$\phi(G)-\varepsilon<\phi(G)=\phi(A)$$
$$\phi(A)=\alpha(b-)-\alpha(a+)\leq\alpha(y)+\frac{\varepsilon}{2}-\alpha(x)+\frac{\varepsilon}{2}=\phi(F)+\varepsilon$$

Now we show that every regular set function on $\mathcal{E}$ can be extended to a countably additive set function on a σ-ring which contains $\mathcal{E}$.

Definition 8. Let μ be additive, regular, non-negative, and finite on $\mathcal{E}$. Consider countable coverings of any set $E\subset R^p$ by open elementary sets A_n.

$$E\subset\bigcup_{n=1}^{\infty}A_n.$$

Define

$$\mu^*(E) = \inf \sum_{n=1}^{\infty} \mu(A_n), \tag{1.2.6}$$

where the infimum is taken over all countable coverings of E by open elementary sets.

μ^* is called the **outer measure** of E, corresponding to μ.

It is clear that $\mu^*(E) \geq 0$ for all E and that if $E_1 \subset E_2$, then any countable coverings of E_2 by open elementary sets is a countable coverings of E_1 by open elementary sets and by properties of infimum we have

$$\mu^*(E_1) \leq \mu^*(E_2). \tag{1.2.7}$$

Theorem 4.

(a) *For every* $A \in \mathcal{E}$, $\mu^*(E) = \mu(E)$
(b) *if* $E \subset \bigcup_{n=1}^{\infty} E_n$, *then*

$$\mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(E_n) \tag{1.2.8}$$

Remark 6. (a) implies that μ^* is an extension of μ from $\mathcal{E}$ to $\mathcal{P}(\mathbb{R}^p)$. The property (1.2.8) is called subadditivity.

Choose $A \in \mathcal{E}$ and $\varepsilon > 0$. The regularity of μ shows that A is contained in an open elementary set G such that

$$\mu(G) \leq \mu(A) + \varepsilon.$$

Since $\mu^*(A) \leq \mu(G)$ and ε is arbitrary, we have

$$\mu^*(A) \leq \mu(A) \tag{1.2.9}$$

By the properties of the infimum there is a sequence of open elementary sets whose union contains A, such that

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu^*(A) + \frac{\varepsilon}{2}$$

The regularity of μ implies that A contains a closed elementary F such that $\mu(A) \leq \mu(F) + \frac{\varepsilon}{2}$; and note that F is bounded, because it is the finite union of finite intervals, so F is compact, and we have

$$F \subset \bigcup_{n=1}^{k} A_n,$$

for some k. Since μ is additive, using Theorem 1, item 2 we have

$$\mu(A) \leq \mu(F) + \frac{\varepsilon}{2} \leq \mu\left(\bigcup_{n=1}^{k} A_n\right) + \frac{\varepsilon}{2} \leq \sum_{n=1}^{k} \mu(A_n) + \frac{\varepsilon}{2} \leq \sum_{n=1}^{\infty} \mu(A_n) + \frac{\varepsilon}{2} \leq \mu^*(A) + \varepsilon,$$

since ε is arbitrary we obtain

$$\mu(A) \leq \mu^*(A), \tag{1.2.10}$$

Equations (1.2.9) and (1.2.10) prove (a).

Now, suppose $E = \bigcup\limits_{n=1}^{\infty} E_n$. If $\mu^*(E_n) = +\infty$ for some n, then the right side of (1.2.8) is equal to $+\infty$, and (1.2.8) is trivial, so we can assume that $\mu^*(E_n) < +\infty$ for all n.

Given $\varepsilon > 0$, there are coverings $\{A_{nk}\}$ for $k = 1, 2, 3, \cdots$ of E_n by elementary sets, such that

$$\sum_{k=1}^{\infty} \mu(A_{nk}) \leq \mu^*(E_n) + \frac{\varepsilon}{2^n}$$

Since

$$E = \bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{nk}$$

then

$$\mu^*(E) \leq \mu^*\left(\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{nk}\right) \leq \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(A_{nk})\right) \leq \sum_{n=1}^{\infty} \left(\mu^*(E_n) + \frac{\varepsilon}{2^n}\right) = \varepsilon + \sum_{n=1}^{\infty} \mu^*(E_n)$$

and since ε is arbitrary (1.2.8) follows.

Definition 9. For any $A, B \subset \mathbb{R}^p$ We define

$$S(A, B) = (A - B) \cup (B - A) \tag{1.2.11}$$

$$d(A, B) = \mu^*(S(A, B)) \tag{1.2.12}$$

$S(A, B)$ is called **symmetric difference** of A and B. Now we will see some properties of $S(A, B)$ and $d(A, B)$

Lemma 1. *For any A, B, C, A_1, A_2, B_1, B_2 in $\mathbb{R}^p$ we have;*

S1 $S(A, B) = S(B, A)$, $S(A, A) = \varnothing$.

S2 $S(A, B) \subset S(A, C) \cup S(C, B)$.

S3

$$\left.\begin{array}{l} S(A_1 \cup A_2, B_1 \cup B_2) \\ S(A_1 \cap A_2, B_1 \cap B_2) \\ S(A_1 - A_2, B_1 - B_2) \end{array}\right\} \subset S(A_1, B_1) \cup S(A_2, B_2).$$

S1 Since $X \cup Y = Y \cup X$ for all $X, Y \subseteq \mathbb{R}^p$, so

$$S(A, B) = (A - B) \cup (B - A) = (B - A) \cup (A - B) = S(B, A),$$

and $X - X = \varnothing$ for all $X \subseteq \mathbb{R}^p$, implies

$$S(A, A) = (A - A) \cup (A - A) = \varnothing \cup \varnothing = \varnothing.$$

S2 Note that $A \subset A \cup C = (A - C) \cup C$, so

$$A - B \subset ((A - C) - B) \cup (C - B) \subset (A - C) \cup (C - B),$$

interchanging A and B we get

$$B - A \subset (B - C) \cup (C - A)$$

Thus,

$$\begin{aligned} S(A,B) &= (A-B)\cup(B-A) \\ &\subset ((A-C)\cup(C-B))\cup((B-C)\cup(C-A)) \\ &= ((A-C)\cup(C-A))\cup((C-B)\cup(B-C)) \\ &= S(A,C)\cup S(C,B) \end{aligned}$$

S3

(a) If $Y^c = \mathbb{R}^p - Y$ (the complement of Y in $\mathbb{R}^p$), then $X - Y = X \cap Y^c$, so

$$\begin{aligned} (A_1\cup A_2)-(B_1\cup B_2) &= (A_1\cup A_2)\cap(B_1\cup B_2)^c \\ &= (A_1\cup A_2)\cap(B_1^c\cap B_2^c) \\ &= (A_1\cap(B_1^c\cap B_2^c))\cup(A_2\cap(B_1^c\cap B_2^c)) \\ &\subset (A_1\cap B_1^c)\cup(A_2\cap B_2^c) \\ &= (A_1-B_1)\cup(A_2-B_2) \end{aligned}$$

and

$$(B_1\cup B_2)-(A_1\cup A_2)\subset(B_1-A_1)\cup(B_2-A_2)$$

so

$$\begin{aligned} S(A_1\cup A_2,B_1\cup B_2) &= ((A_1\cup A_2)-(B_1\cup B_2))\cup((B_1\cup B_2)-(A_1\cup A_2)) \\ &\subset ((A_1-B_1)\cup(A_2-B_2))\cup((B_1-A_1)\cup(B_2-A_2)) \\ &= ((A_1-B_1)\cup(B_1-A_1))\cup((A_2-B_2)\cup(B_2-A_2)) \\ &= S(A_1,B_1)\cup S(A_2,B_2). \end{aligned}$$

(b) Since

$$\begin{aligned} S(A,B) &= (A\cap B^c)\cup(B\cap A^c) \\ &= ((A^c)^c\cap B^c)\cup((B^c)^c\cap A^c) \\ &= (A^c\cap(B^c)^c)\cup(B^c\cap(A^c)^c) \\ &= S(A^c,B^c), \end{aligned}$$

then

$$\begin{aligned} S(A_1\cap A_2,B_1\cap B_2) &= S((A_1\cap A_2)^c,(B_1\cap B_2)^c) \\ &= S(A_1^c\cup A_2^c,B_1^c\cup B_2^c) \\ &\subset S(A_1^c,B_1^c)\cup S(A_2^c,B_2^c) \\ &= S(A_1,B_1)\cup S(A_2,B_2) \end{aligned}$$

(c)

$$\begin{aligned} S(A_1-A_2,B_1-B_2) &= S(A_1\cap A_2^c,B_1\cap B_2^c) \\ &\subset S(A_1,B_1)\cup S(A_2^c,B_2^c) \\ &= S(A_1,B_1)\cup S(A_2,B_2) \end{aligned}$$

These properties of $S(A,B)$ imply

LEMMA 2. *Forany A, B, C, A_1, A_2, B_1, B_2 in $\mathbb{R}^p$ we have;*

D1 $d(A,B) = d(B,A)$, $d(A,A) = 0$.

D2 $d(A,B) \leq d(A,C) + d(C,B)$.

D3

$$\left.\begin{array}{c} d(A_1 \cup A_2, B_1 \cup B_2) \\ d(A_1 \cap A_2, B_1 \cap B_2) \\ d(A_1 - A_2, B_1 - B_2) \end{array}\right\} \leq d(A_1, B_1) + d(A_2, B_2). \tag{1.2.13}$$

D1

$$d(A,B) = \mu^*(S(A,B)) = \mu^*(S(B,A)) = d(B,A)$$
$$d(A,A) = \mu^*(S(A,A)) = \mu^*(\varnothing) = 0$$

D2 Since $S(A,B) \subset S(A,C) \cup S(C,B)$, by (1.2.7) and (1.2.8)

$$\begin{aligned} d(A,B) = \mu^*(S(A,B)) &\leq \mu^*(S(A,C) \cup S(C,B)) \\ &\leq \mu^*(S(A,C)) + \mu^*(S(C,B)) = d(A,C) + d(C,B) \end{aligned}$$

D3 As in the proof of D2, the inclusions in S3 and (1.2.7) and (1.2.8) imply

$$\left.\begin{array}{c} d(A_1 \cup A_2, B_1 \cup B_2) \\ d(A_1 \cap A_2, B_1 \cap B_2) \\ d(A_1 - A_2, B_1 - B_2) \end{array}\right\} \leq d(A_1, B_1) \cup d(A_2, B_2).$$

The relations D1 and D2 show that $d(A,B)$ satisfies the requirements of definition for a distance except that $d(A,B) = 0$ does not imply $A = B$. For instance, if $p = 1$, $\mu = m$, $A = \{a_n \in \mathbb{R} : n \in \mathbb{N}\}$ is countable, and $B = \varnothing$, then

$$d(A,B) = m^*((A - \varnothing) \cup (\varnothing - A)) = m^*(A)$$

If $\varepsilon > 0$, taken $I_n = \left(a_n - \frac{\varepsilon}{2^{n+1}}, a_n + \frac{\varepsilon}{2^{n+1}}\right)$, then I_n are elementary open sets and $A \subset \bigcup_{n=1}^{\infty} I_n$ and

$$m^*(A) \leq \sum_{n=1}^{\infty} m(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$$

Since ε is arbitrary, then $m^*(A) = 0$.

If $B = \varnothing$, then D2 tells us that

$$\mu^*(A) = d(A,\varnothing) \leq d(A,C) + d(C,\varnothing) = d(A,C) + \mu^*(C)$$

interchanging A and C we get

$$\mu^*(C) \leq d(A,C) + \mu^*(A)$$

So if at least one of $\mu^*(A)$, $\mu^*(C)$ is finite, then

$$|\mu^*(A) - \mu^*(C)| \leq d(A,C) \tag{1.2.14}$$

We write $A_n \to A$, if

$$\lim_{n\to\infty} d(A, A_n) = 0.$$

If there is a sequence $\{A_n\}$ of elementary sets such that $A_n \to A$, we say that A is **finitely μ-measurable** and write $A \in \mathfrak{M}_F(\mu)$.

If A is the union of a countable collection of finitely μ-measurable sets, we say that A is μ-**measurable** and write $A \in \mathfrak{M}(\mu)$.

THEOREM 5. *$\mathfrak{M}(\mu)$ is a σ-ring, and μ^* is countably additive on $\mathfrak{M}(\mu)$.*

Let A, $B \in \mathfrak{M}_F(\mu)$, and $\{A_n\}$, $\{B_n\}$ be sequences of elementary sets such that $A_n \to A$ and $B_n \to B$, then by 1.2.13 and 1.2.14 we have

$$\begin{aligned} d(A_n \cup B_n, A \cup B) &\leq d(A_n, A) + d(B_n, B) \\ d(A_n \cap B_n, A \cap B) &\leq d(A_n, A) + d(B_n, B) \\ d(A_n - B_n, A - B) &\leq d(A_n, A) + d(B_n, B) \\ |\mu^*(A_n) - \mu^*(A)| &\leq d(A_n, A). \end{aligned}$$

Since $\mu^*(A_n) < \infty$, taking limit when $n \to \infty$,we have

(1.2.15a) $$A_n \cup B_n \to A \cup B$$

(1.2.15b) $$A_n \cap B_n \to A \cap B$$

(1.2.15c) $$A_n - B_n \to A - B$$

(1.2.15d) $$\mu^*(A_n) \to \mu^*(A)$$

So $\mu^*(A) < \infty$. By $(1.2.15a)$ and $(1.2.15c)$, $\mathfrak{M}_F(\mu)$ is a ring. Since μ is additive by item 3 in Theorem 1 we have $\mu(A_n) + \mu(B_n) = \mu(A_n \cup B_n) + \mu(A_n \cap B_n)$ taking limit when $n \to \infty$, by $(1.2.15d)$ and Theorem 2 (a) we obtain

$$\mu^*(A) + \mu^*(B) = \mu^*(A \cup B) + \mu^*(A \cap B).$$

Since $A \cap B = \varnothing$, implies $\mu^*(A \cap B) = 0$, then if $A \cap B = \varnothing$ we have

$$\mu^*(A) + \mu^*(B) = \mu^*(A \cup B).$$

And μ^* is additive on $\mathfrak{M}_F(\mu)$.

Now let

$$A = \bigcup_{n=1}^{\infty} A_n$$

with $A_n \in \mathfrak{M}_F(\mu)$. for $n = 1, 2, 3, \ldots$, Then $B_n \in \mathfrak{M}_F(\mu)$ because B_n is a finite union of elements of $\mathfrak{M}_F(\mu)$, $B_1 \subset B_1 \subset \cdots \subset A$, and $A = \bigcup\limits_{n=1}^{\infty} B_n$, if $C_1 = B_1$ and $C_n = B_n - B_{n-1}$ for $n = 2, 3, \cdots$, then $C_n \in \mathfrak{M}_F(\mu)$ because C_n is the difference of elements of $\mathfrak{M}_F(\mu)$, and as we see in the proof of Theorem 2 $C_i \cap C_j = \varnothing$ for $i \neq j$ and $A = \bigcup\limits_{k=1}^{\infty} C_k$.

By (1.2.8)

(1.2.16) $$\mu^*(A) \leq \sum_{k=1}^{\infty} \mu^*(C_k).$$

On the other hand, with the above notation $B_n = \bigcup\limits_{k=1}^{n} C_k \subset A$. the additivity of μ^* implies

(1.2.17) $$\sum_{k=1}^{n} \mu^*(C_k) = \mu^*\left(\bigcup_{k=1}^{n} C_k\right) = \mu^*(B_n) \leq \mu^*(A)$$

Equations (1.2.16) and (1.2.17) imply

$$\sum_{n=1}^{\infty} \mu^*(C_n) = \mu^*(A). \tag{1.2.18}$$

Suppose now that $\mu^*(A) < \infty$. Then (1.2.17) shows

$$\lim_{n\to\infty} \mu^*(B_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu^*(C_k) = \sum_{k=1}^{\infty} \mu^*(C_k) = \mu^*(A).$$

Hence $B_n \to A$; and since $B_n \in \mathfrak{M}_F(\mu)$, there are sequences $\{E_{nk}\}$ for $k = 1, 2, 3, \ldots$ of elementary sets, such that

$$\lim_{k\to\infty} \mu^*(E_{nk}) = \mu^*(B_n),$$

and the double sequence $\{E_{nk} : n, k \in \mathbb{N}\}$ satisfies

$$\lim_{n\to\infty} \lim_{k\to\infty} \mu^*(E_{nk}) = \mu^*(A).$$

So $A \in \mathfrak{M}_F(\mu)$ if $A \in \mathfrak{M}(\mu)$ and $\mu^*(A) < \infty$.

To see that μ^* is countably additive on $\mathfrak{M}(\mu)$, note that if $A = \bigcup_{n=1}^{\infty} C_n$,where $\{C_n\}$ is a sequence of disjoint sets of $\mathfrak{M}(\mu)$. Then

If $\mu^*(C_k) = \infty$ for some $k \in \mathbb{N}$, since $C_k \subset A$, then

$$\infty = \mu^*(C_k) \le \mu^*(A) \le \sum_{n=1}^{\infty} \mu^*(C_n) = \infty \tag{1.2.19}$$

therefore the inequalities in (1.2.19) are all equalities.

If $\mu^*(C_n) < \infty$ for all $n \in \mathbb{N}$, above we see that $A_n \in \mathfrak{M}_F(\mu)$ and by (1.2.18) we have

$$\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(C_n).$$

Finally, if $A = \bigcup_{n=1}^{\infty} A_n$, with $A_n \in \mathfrak{M}(\mu)$, then for each n there exists$\{B_{nk}\}$ for $k = 1, 2, 3, \ldots$ with $B_{nk} \in \mathfrak{M}_F(\mu)$ and $A_n = \bigcup_{k=1}^{\infty} B_{nk}$, so

$$A = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} B_{nk},$$

Note that A is a countable union of elements in $\mathfrak{M}_F(\mu)$, then $A \in \mathfrak{M}(\mu)$.

Now suppose that A, $B \in \mathfrak{M}(\mu)$ and

$$A = \bigcup_{n=1}^{\infty} A_n, \qquad B = \bigcup_{m=1}^{\infty} B_m,$$

where A_n, $B_n \in \mathfrak{M}_F(\mu)$ for all $n \in \mathbb{N}$. Note that

$$A_n \cap B = A_n \cap \bigcup_{m=1}^{\infty} B_m = \bigcup_{m=1}^{\infty} (A_n \cap B_m).$$

So $A_n \cap B_m \in \mathfrak{M}_F(\mu)$, and $A_n \cap B \in \mathfrak{M}(\mu)$ for all $n \in \mathbb{N}$. Since $A_n \cap B \subset A_n$, then

$$\mu^*(A_n \cap B) \leq \mu^*(A_n) < \infty.$$

Thus, $A_n \cap B \in \mathfrak{M}_F(\mu)$, so $A_n - B = A_n - (A_n \cap B) \in \mathfrak{M}_F(\mu)$ for all $n \in \mathbb{N}$. Hence

$$A - B = \left(\bigcup_{n=1}^{\infty} A_n\right) - B = \bigcup_{n=1}^{\infty}(A_n - B) \in \mathfrak{M}(\mu)$$

We now replace $\mu^*(A)$ by $\mu(a)$, if $A \in \mathfrak{M}(\mu)$. So μ, initially defined on $\mathcal{E}$, is extended to a countably additive set function on the σ-ring $\mathfrak{M}(\mu)$. This extended set function is called a measure. The case $\mu = m$ is called the Lebesgue measure on $\mathbb{R}^p$.

Remark 7.

(a) If A is open, then $A \in \mathfrak{M}(\mu)$. Because every open set in $\mathbb{R}^p$ is the union of a countable collection of intervals. To see this, using the density of $\mathbb{Q}$ in $\mathbb{R}$, we can see that $\beta = \{I = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_p, b_p) : a_i, b_i \in \mathbb{Q}\}$ is a countable base whose elements are open intervals. Since $\mathbb{R}^p$ is an open set, taking complements we obtain that closed set is in $\mathfrak{M}(\mu)$

(b) If $A \in \mathfrak{M}(\mu)$ and $\varepsilon > 0$, there exist sets F and G that $F \subset A \subset G$, F is closed, G is open, and

$$\mu(G - A) < \varepsilon, \qquad \mu(A - F) < \varepsilon. \tag{1.2.20}$$

Indeed, if $\mu(A) < \infty$, i.e., $A \in \mathfrak{M}_F(\mu)$ by (1.2.6), there exists a sequence $\{A_n\}$ of open elementary sets, so that

$$A \subset \bigcup_{n=1}^{\infty} A_n \qquad \text{and} \quad \sum_{n=1}^{\infty} \mu(A_n) < \mu(A) + \varepsilon,$$

taking $G = \bigcup\limits_{n=1}^{\infty} A_n$, then

$$\mu(G - A) = \mu(G) - \mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n) - (\mu(A)) < \varepsilon.$$

If $\mu(A) = \infty$ there exists a sequence $\{A_n\}$ with $A_n \in \mathfrak{M}_F(\mu)$ for all $n \in \mathbb{N}$, so that $A = \bigcup\limits_{n=1}^{\infty} A_n$. Now, for each $n \in \mathbb{N}$ by show above there exists an open set G_n with $A_n \subset G_n$ so that

$$\mu(G_n - A_n) < \frac{\varepsilon}{2^n},$$

taking $G = \bigcup\limits_{n=1}^{\infty} G_n$, then, G is open, $A \subset G$, $G - A = \bigcup\limits_{n=1}^{\infty}(G_n - A_n)$, and

$$\mu(G - A) \leq \mu\left(\bigcup_{n=1}^{\infty}(G_n - A_n)\right) \leq \sum_{n=1}^{\infty} \mu(G_n - A_n) < \varepsilon.$$

For the second inequality, since $\mathfrak{M}(\mu)$ is a σ-ring, then $A^c \in \mathfrak{M}(\mu)$, so there exists G

$$\mu(G - A^c) < \varepsilon.$$

Taking $F = G^c$, then F is closed $A \subset F$, using Remark 1 we have

$$\mu(A - F) = \mu(A - G^c) = \mu(A \cap G) = \mu(G - A^c) < \varepsilon$$

(c) We say that E is a Borel set if E can be obtained by a countable number of operations, starting from open sets, each operation consisting in taking unions, intersections, or complements. The collection $\mathcal{B}$ of all Borel sets in $\mathbb{R}^p$ is a σ-ring; in fact, it is the smallest σ-ring which contains all open sets. By Remark (a) if $\mathcal{B} \subset \mathfrak{M}(\mu)$.

(d) By (b) If $A \in \mathfrak{M}(\mu)$, for each $n \in \mathbb{N}$, there exist Borel sets F_n so that that $F_n \subset A$, F_n is closed for all $n \in \mathbb{N}$, and

$$\mu(A - F_n) < \frac{1}{n},$$

If $F = \bigcup\limits_{n=1}^{\infty} F_n$, then F is a Borel set, $F \subset A$, $A - F \subset A - F_n$ for all $n \in \mathbb{N}$, in consequence we have

$$\mu(A - F) \leq \mu(A - F_n) < \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

Thus,

$$\mu(A - F) = 0. \tag{1.2.21}$$

Since $A = F \cup (A - F)$, we see that every $A \in \mathfrak{M}(\mu)$ is the union of a Borel set and a set of measure zero.

The Borel sets are always μ-measurable for all μ. But the sets of measure zero, i.e., the sets E for which $\mu^*(E) = 0$ may be different for different measures μ's.

(e) For every μ, the sets of measure zero form a σ-ring, Indeed, recall that E has measure zero, if for a given $\varepsilon > 0$, there exists a sequence $\{A_n\}$ of open elementary sets, so that

$$E \subset \bigcup_{n=1}^{\infty} A_n \qquad \text{and} \quad \sum_{n=1}^{\infty} \mu(A_n) < \varepsilon, \text{ i}$$

Since $E_1 - E_2 \subset E_1$ for all E_1, E_2, then

$$0 \leq \mu^*(E_1 - E_2) \leq \mu^*(E_1) = 0$$

On the other hand, if $E = \bigcup\limits_{n=1}^{\infty} E_n$, with $\mu^*(E_n) = 0$, for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let $\{A_{nk} : k \in \mathbb{N}\}$ a sequence of open elementary sets, so that

$$E_n \subset \bigcup_{k=1}^{\infty} A_{nk} \qquad \text{and} \quad \sum_{k=1}^{\infty} \mu(A_{nk}) < \frac{\varepsilon}{2^n},$$

Thus, $E = \bigcup\limits_{n=1}^{\infty} E_n \subset \bigcup\limits_{n=1}^{\infty} \bigcup\limits_{k=1}^{\infty} A_{nk}$, and

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(A_{nk}) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon,$$

(f) In case of the Lebesgue measure, every countable set has measure zero. In effect, for each $\varepsilon > 0$, the interval centered on x and of volume $\frac{\varepsilon}{2}$ is a covering of $A = \{x\}$ by elementary sets, with measure less than ε. As any countable set B is the countable union of its elements, the remark 6 (e) shows that B has zero measurement. But there are uncountable sets of measure zero. The Cantor set P is an example: Recall that P is defined as

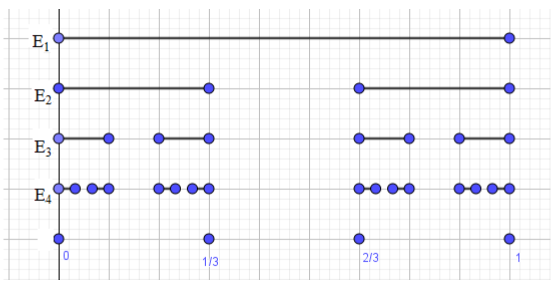
$$P = \bigcap_{n=0}^{\infty} E_n,$$

where $E_0 = [0,1]$. Removing the the middle thirds ibterval of this intervals we obtain $E_1 = E_0 - (\frac{1}{3}, \frac{2}{3})$, so E_1 is $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Removing the middle thirds interval of these intervals we obtain E_2 and continuing in this way, we obtain a sequence of compact sets E_n, such that

$$E_0 \supset E_1 \supset E_2 \cdots$$

and E_n is the union of 2^n intervals, each of length 3^{-n}. So $m(E_n) = \left(\frac{2}{3}\right)^n$.

Below we show the E_1, E_2, E_3 and E_4.



Now P can be identified with the set of the sequences $(a_0 a_1 \dots a_n \dots)$ where $a_n = 0$ or $a_n = 2$, and using the same argument as the one that shows that $[0,1]$ is uncountable, we get that P is uncountable.

Since $P \subset E_n$ for all $n \in \mathbb{N}$, then $m(P) \leq m(E_n) = \left(\frac{2}{3}\right)^n$, and taking limit when $n \to \infty$ we obtain $m(P) = 0$.

Moreover, if we denote by $\mathfrak{c}$ the cardinality of $\mathbb{R}$, we obtain that cardinality of P is equal to $\mathfrak{c}$. Because every subset of the set of measure 0 is the set of measure 0, we have at least $2^{\mathfrak{c}}$ measurable sets on $\mathbb{R}$. Because the cardinality of the family of all subset of $\mathbb{R}$ is also $2^{\mathfrak{c}}$, the natural question is: ‘are there unmeasurable sets?’. added — PS

(g) A **Vitali set** is a subset V of the interval $[0,1]$ of real numbers such that, for each real number r, there is exactly one number $v \in V$ such that $v - r$ is a rational number. Vitali sets are a set of representative of the group $\mathbb{R}/\mathbb{Q}$ in $[0,1]$. added — PS

Every Vitali set V is uncountable, and

$$v - u \text{ is irrational for any } u, v \in V,\ u \neq v. \tag{1.2.22}$$

A Vitali set is non-measurable. Indeed, assume that V is measurable and let q_1, q_2, $\dots$ be an enumeration of the rational numbers in $[-1,1]$. And let V_n be the translated sets defined by

$$V_n = V + q_n = \{v + q_n : v \in V\},$$

Note that $V_n \cap V_m = \varnothing$, because if $y \in V_n \cap V_m$, then $v + q_n = y = u + q_m$, implies $v - u$ is rational in contradiction with (1.2.22).

Also note that $[0,1] \subseteq \bigcup_{n=1}^{\infty} V_n \subseteq [-1,2]$.

To see the first inclusion, consider any real number $r \in [0,1]$ and let v be the representative in V for the equivalence class $[r]$; then $r - v = q_n$ for some rational number $q_n \in [-1,1]$ which implies that $r \in V_n$

Since the Lebesgue measure is countably additive, then

$$1 \leq \sum_{n=1}^{\infty} m(V_n) \leq 3.$$

Because the Lebesgue measure is translation invariant, we have $m(V_n) = m(V)$ for all $n \in \mathbb{N}$, and therefore

$$1 \leq \sum_{n=1}^{\infty} m(V) \leq 3.$$

But this is impossible. Summing infinitely many copies of the constant $m(V)$ yields either zero or infinity, according to whether the constant is zero or positive. In neither case is the sum in [1, 3]. So V cannot bee measurable. An adequate change of the above argument shows that, for all measurable set A with $m(A) > 0$ there exists a non-measurable set B with $B \subset A$.

1.3. MEASURE SPACES

Definition 10. Suppose X is a set, not necessarily a subset of an Euclidean space, or indeed of any metric space, X is said to be a **measure space** if there exists a σ-ring $\mathfrak{M}$ of subsets of X, which are called **measurable sets**, and a non-negative countably additive set function μ which is called a **measure**, defined on $\mathfrak{M}$.

If, in addition, $X \in \mathfrak{M}$ then X is said to be a measurable space.

For example, we can take $X = \mathbb{R}^{\ltimes}$, and $\mathfrak{M}$. the collection of all Lebesgue measurable subsets of $\mathbb{R}^{\ltimes}$, and $\mu = m$ the Lebesgue measure.

Or, let $X = \mathbb{N}$ the set of all positive integers, $\mathfrak{M}$ the collection of all subsets of X, and $\mu(E)$ is the number of elements of E. μ is know as the **counting measure**

Another example is provided by probability theory, where events may be considered as sets, and the probability of the occurrence of events is an additive (or countably additive) set function.

1.4. MEASURABLE FUNCTIONS

Definition 11. Let f be a function defined on the measurable space X, with values in the extended real number system. The function f is said to measurable if the set

$$\{x : f(x) > a\} \tag{1.4.1}$$

is measurable for every real a.

Example 2. If $X = \mathbb{R}^p$ and $\mathfrak{M} = \mathfrak{M}(\mu)$ as in Theorem 5, every continuous f is measurable, because then (1.4.1) is an open set, and the open sets belong to $\mathfrak{M}$.

Theorem 6. *Let f be a function defined on the measurable space X. The followings conditions are equivalent:*

(1.4.2) $\{x : f(x) > a\}$ is measurable for every real a

(1.4.3) $\{x : f(x) \geq a\}$ is measurable for every real a

(1.4.4) $\{x : f(x) < a\}$ is measurable for every real a

(1.4.5) $\{x : f(x) \leq a\}$ is measurable for every real a

Let $a \in \mathbb{R}$.

(1) If $f(x) \geq a$, then $f(x) > a - \frac{1}{n}$ for all $n \in \mathbb{N}$. So

$$\{x : f(x) \geq a\} \subset \bigcap_{n\in\mathbb{N}} \left\{x : f(x) > a - \frac{1}{n}\right\}.$$

Reciprocally, if $f(x) > a - \frac{1}{n}$ for all $n \in \mathbb{N}$, then $f(x)$ is an upper bound of the set $\left\{a - \frac{1}{n} : n \in \mathbb{N}\right\}$, since $a = \sup\left\{a - \frac{1}{n} : n \in \mathbb{N}\right\}$ we have $f(x) \geq a$ and

$$\bigcap_{n\in\mathbb{N}} \left\{x : f(x) > a - \frac{1}{n}\right\} \subset \{x : f(x) \geq a\}.$$

So $\{x : f(x) \geq a\} = \bigcap_{n\in\mathbb{N}} \{x : f(x) > a - \frac{1}{n}\}$. Since the intersections of measurable sets is a measurable set, then 1.4.1 implies 1.4.2

(2) Note that

$$\{x : f(x) \geq a\}^c = X - \{x : f(x) \geq a\} = \{x : f(x) < a\}$$

Since the complement of measurable sets is a measurable set, then (1.4.2) implies (1.4.3)

(3) If $f(x) \leq a$, then $f(x) < a + \frac{1}{n}$ for all $n \in \mathbb{N}$. So

$$\{x : f(x) \leq a\} \subset \bigcap_{n\in\mathbb{N}} \{x : f(x) < a + \frac{1}{n}\}.$$

Reciprocally, If $f(x) < a + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $f(x)$ is lower bound of the set $\left\{a + \frac{1}{n} : n \in \mathbb{N}\right\}$, since $a = \inf\left\{a + \frac{1}{n} : n \in \mathbb{N}\right\}$ we have $f(x) \geq a$ and

$$\bigcap_{n\in\mathbb{N}} \left\{x : f(x) < a + \frac{1}{n}\right\} \subset \{x : f(x) \leq a\}.$$

So $\{x : f(x) \leq a\} = \bigcap_{n\in\mathbb{N}} \{x : f(x) < a + \frac{1}{n}\}$. Since the intersections of measurable sets is a measurable set, then (1.4.3) implies (1.4.4).

(4) Note that

$$\{x : f(x) \leq a\}^c = X - \{x : f(x) \leq a\} = \{x : f(x) > a\}$$

Since the complement of measurable sets is a measurable set, then (1.4.4) implies (1.4.1)

Corollary 1. *Let f be a measurable function on a measurable space X, then $-f$ is a measurable function on X.*

Note that

$$\{x : -f(x) > a\} = \{x : f(x) < -a\}$$

Hence $\{x : -f(x) > a\}$ is measurable for every real a if f is a measurable function.

Hence any of these conditions may be used instead of (1.2.21) to define measurability

THEOREM 7. *Let f be a measurable function on a measurable space X, then $|f|$ is a measurable function on X.*

If $a < 0$, then $\{x : |f(x)| > a\} = X$, hence measurable.

Recall that $|f(x)| > a$, if and only if $f(x) > a$ or $-f(x) < -a$. So

$$\{x : |f(x)| > a\} = \{x : f(x) > a\} \cup \{x : -f(x) < -a\},$$

and by Theorem 6 and corollary 1 $\{x : |f(x)| > a\}$ is measurable for every real a if f is a measurable function.

THEOREM 8. *Let $\{f_n\}$ be a sequence of measurable functions on a measurable space X. Then*

$$\begin{aligned} g(x) &= \sup\{f_n(x) : n \in \mathbb{N}\} \\ h(x) &= \inf\{f_n(x) : n \in \mathbb{N}\} \\ k(x) &= \lim_{n\to\infty} \sup\{f_n(x)\} \\ l(x) &= \lim_{n\to\infty} \inf\{f_n(x)\} \end{aligned}$$

are measurable functions.

Note $g(x) \leq a$, if and only if $f_n(x) \leq a$ for all $n \in \mathbb{N}$. Thus, since $\{x : g(x) \leq a\} = \bigcap_{n\in\mathbb{N}} \{x : f_n(x) \leq a\}$, $\{x : g(x) \leq a\}$ is measurable for every real a if f_n are measurable functions for all n.

If f_n are measurable functions for all n, then $-f_n$ are measurable functions, so $p(x) = \sup\{-f_n(x)\}$ is a measurable function. Since $h(x) = -p(x)$, then h is also a measurable function.

Note that

$$\begin{aligned} k(x) &= \inf g_m(x) \\ l(x) &= \sup h_m(x), \end{aligned}$$

where $h_m(x) = \inf\{f_n(x) : n > m\}$ and $g_m(x) = \sup\{f_n(x) : n > m\}$. Then by the show above, g_m, h_m are measurable functions, in consequence k, l are also measurable functions.

COROLLARY 2. *Let X be a measurable space, then*

(1) *If f, g are measurable functions on X, then*

$$\max\{f, g\}, \quad \min\{f, g\}, \quad f^+, \quad f^-$$

are also measurable functions.

(2) *The limit of a convergent sequence of measurable functions on X is measurable on X.*

(a) Recall that

$$\begin{aligned} \max\{f, g\} &= \sup\{f, g\}, \\ \min\{f, g\} &= \inf\{f, g\}, \\ f^+ &= \max\{f, \mathbf{0}\}, \\ f^- &= \max\{-f, \mathbf{0}\}, \end{aligned}$$

where $\mathbf{0}(x) = 0$.

(b) If $f(x) = \lim_{n\to\infty} f_n(x)$, then

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sup \{f_n(x)\}.$$

THEOREM 9. *Let f and g be measurable real valued functions defined on X, let F be a real valued and continuous function on $\mathbb{R}^2$, and put*

$$h(x) = F(f(x), g(x))$$

then h is measurable.

In particular, $f + g$ and $f \cdot g$ are measurable.

Let

$$G_a = \{(u, v) : F(u, v) > a\} = F^{-1}(a, +\infty)$$

Since F is continuous, then G_a is an open subset of $\mathbb{R}^2$, recall that every open set in $\mathbb{R}^2$ is the union of a countable collection of intervals (because both open squares and open disks form a topological basis in $\mathbb{R}^2$), so we can write $G_a = \bigcup_{n=1}^{\infty} \mathbf{I}_n$, where $\{\mathbf{I}_n\}$ is a sequence of open intervals: added — PS

$$\mathbf{I}_n = (a_n, b_n) \times (c_n, d_n).$$

Since f and g are measurable, then the sets

$$\{x : a_n < f(x) < b_n\} = f^{-1}(a_n, b_n),$$
$$\{x : c_n < g(x) < d_n\} = g^{-1}(c_n, d_n),$$

are measurable. So

$$\{x : (f(x), g(x)) \in \mathbf{I}_n\} = \{x : a_n < f(x) < b_n\} \cap \{x : c_n < g(x) < d_n\}$$

is measurable, Hence the same is true for

$$\bigcup_{n=1}^{\infty} \{x : (f(x), g(x)) \in \mathbf{I}_n\} = \{x : (f(x), g(x)) \in G_a\} = \{x : h(x) = F(f(x), g(x)) > a\}.$$

Above we see that all ordinary operations of analysis when we apply to measurable functions, lead to measurable functions. but the next example shows that there exits continuous function g and a measurable function h such that $h \circ g$ is non-measurable.

EXERCISE 2. [Exercise 11.3] If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points at which $\{f_n(x)\}$ converges is measurable.

SOLUTION 2. Let $A = \{x : \{f_n(x)\} \text{ converges}\}$, if $x \in A$ then $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Thus, given $n \in \mathbb{N}$, there exists$n_0 \in \mathbb{N}$ such that $|f_p(x) - f_q(x)| < \frac{1}{n}$ if $p,\ q > n_0$.

Note that for each $n \in \mathbb{N}$. we have

$$x \in \bigcap_{p,q>n_0} \left\{z : |f_p(z) - f_q(z)| < \frac{1}{n}\right\} \subset \bigcup_{n_0\in\mathbb{N}} \bigcap_{p,q>n_0} \left\{z : |f_p(z) - f_q(z)| < \frac{1}{n}\right\}$$

and x belong to the set

$$C = \bigcap_{n\in\mathbb{N}} \bigcup_{n_0\in\mathbb{N}} \bigcap_{p,q>n_0} \left\{z : |f_p(z) - f_q(z)| < \frac{1}{n}\right\}$$

Clearly by construction if $x \in C$, then $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$, so $\{f_n(x)\}$ converges and $x \in A$. in consequence $A = C$.

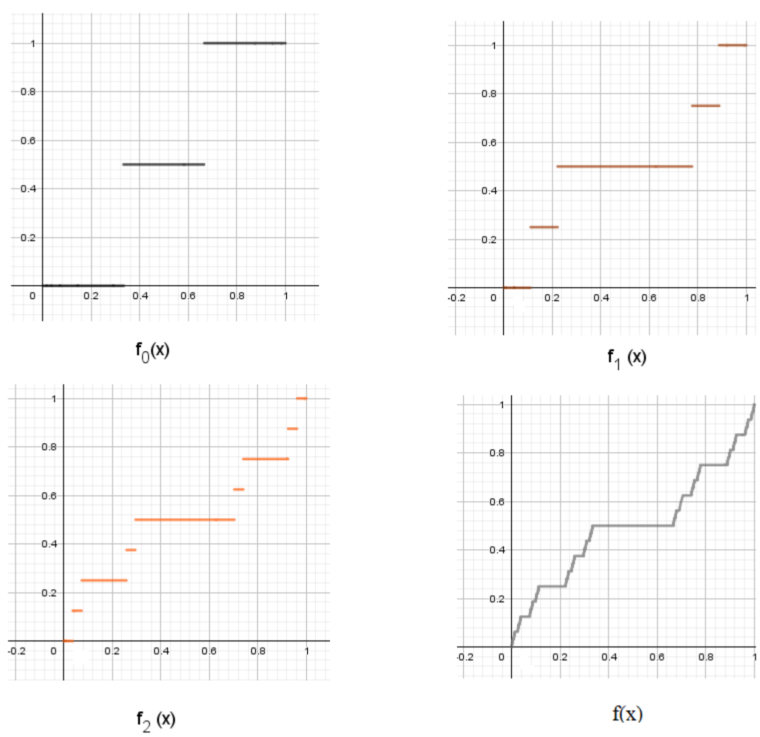
Finally, since f_n is measurable for all n, then $|f_p - f_q|$ is measurable for all p, q, in consequence the sets $\left\{z : |f_p(z) - f_q(z)| < \frac{1}{n}\right\}$ are measurable for all p, q and n. Since the measurable sets form a σ-ring then $A = C$ is a measurable set.

Example 3. Consider the following sequence $\{f_n\}$ of functions defined by

$$f_0(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{3} \\ \frac{1}{2}, & \frac{1}{3} \leq x < \frac{2}{3} \\ 1, & \frac{2}{3} \leq x \leq 1 \end{cases}$$

$$f_n(x) = \begin{cases} \frac{1}{2} f_{n-1}(3x), & 0 \leq x < \frac{1}{3} \\ \frac{1}{2}, & \frac{1}{3} \leq x < \frac{2}{3} \\ \frac{1}{2}\left(1 + f_{n-1}(3x-2)\right), & \frac{2}{3} \leq x \leq 1 \end{cases}$$

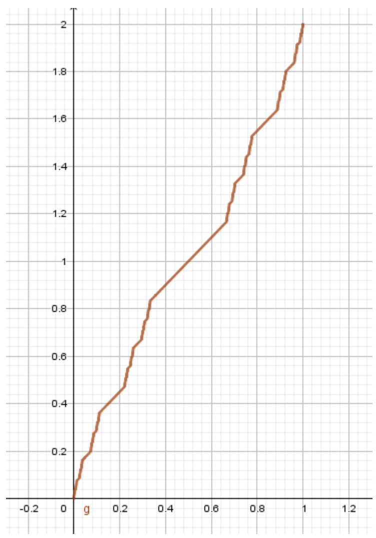
This sequence of functions converges to the continuous function f know as the **devil's staircase function of Cantor,** below we can see the graph of the first three functions f_n and a sketch of the graph of f.



Consider the function $g(x) = f(x) + x$, where f is the devil's staircase function of Cantor. The function g is strictly increasing homeomorphism from $[0, 1]$ onto $[0, 2]$ and note that the measure of $[0,2]-g(C)$, where C is the Cantor set, is equal to two minus the sum of the measures of the images of the middle thirds of the intervals in E_n, that we use to defined C, i.e.,

$$m([0,2]-g(C)) = 2-\sum_{n=0}^{\infty}\left(\frac{2^n}{3^{n+1}}\right)=1$$

in the figure below we can see a sketch of the graph of g,



If A is a non-measurable $A\subset g(C)$. Note that $B=g^{-1}(A)$ is measurable, because $B\subset C$, so $m(B)=0$. Since g is a homeomorphism, then g^{-1} is continuous. If $h=\chi_B$, then h is measurable on $[0,1]$ but

$$h(g^{-1}(x)) = \begin{cases} 1 & g^{-1}(x)\in B \\ 0 & g^{-1}(x)\notin B \end{cases} = \begin{cases} 1 & x\in A \\ 0 & x\notin A \end{cases} = \chi_A,$$

is the non-measurable characteristic function of the non-measurable set A.

Note that measure μ has not been mentioned in above discussion of measurable functions. In fact, the class of measurable functions on X depends only on the σ-ring $\mathfrak{M}$.

For instance, we speak of Borel-measurable functions on $\mathbb{R}$, that is, of function f for which

$$\{x : f(x) < a\}$$

is always a Borel set, without reference to any particular measure.

1.5. SIMPLE FUNCTIONS

Definition 12. Let s be a real-valued function defined on X. If the range of s is finite, we say that s a **simple function**.

Let $E\subset X$, and put $\chi_E\colon X\to\mathbb{R}$

$$\chi_E = \begin{cases} 1 & x\in E \\ 0 & x\notin E \end{cases} \tag{1.5.1}$$

χ_E is called the **characteristic function** of E. Note that $\chi_E=\chi_E^2=|\chi_E|$.

Suppose the range of s consists of the distinct numbers $c_1, \ldots, c_r$. Let

$$E_i = \{x : s(x) = c_i\},$$

then, $X = \bigcup_{i=1}^{r} E_i$, $E_i \cap E_j = \varnothing$ if $i \neq j$ and

$$s = \sum_{i=1}^{r} c_i \chi_{E_i} \tag{1.5.2}$$

REMARK 8. So every simple function is a finite linear combination of characteristic functions. It is noted that if s is measurable, then

$$E_i = \{x : s(x) = c_i\} = \{x : s(x) \leq c_i\} \cap \{x : s(x) \geq c_i\}$$

are measurable sets. And reciprocally, if E_i are measurable sets for all $i = 1, 2, \ldots, r$, then

$$\{x : s(x) < a\} = \bigcup_{c_i < a} E_i$$

So s is measurable, iff E_i are measurable sets for all $i = 1, 2, \ldots, r$.

Now we see that every function can be approximated by simple functions

THEOREM 10. *Let f be a real function on X. There exists a sequence $\{s_n\}$ of simple functions such that*

$$\lim_{n\to\infty} s_n(x) = f(x) \text{ for every } x \in X \tag{1.5.3}$$

If f is measurable, s_n may be chosen to be measurable functions. If $f \geq 0$, then $\{s_n\}$ may be chosen to be a monotonically increasing sequence.

If $f \geq 0$, for each $n \in \mathbb{N}$ define

$$F_n = \{x : f(x) \geq n\} = f^{-1}([n, +\infty)), \tag{1.5.4}$$

and for $k = 1, 2, \ldots, n2^n$ define

$$E_{n,k} = \left\{x : \frac{k-1}{2^n} \leq f(x) \leq \frac{k}{2^n}\right\} = f^{-1}\left(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right), \tag{1.5.5}$$

and let

$$s_n(x) = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{E_{n,k}} + n\chi_{F_n}$$

Note that $F_{n+1} \subset F_n$, so if $x \in F_{n+1}$, then

$$s_{n+1}(x) = n + 1 > n = s_n(x).$$

Since

$$E_{n+1,2k} = f^{-1}\left(\left[\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}\right)\right) = f^{-1}\left(\left[\frac{2k-1}{2^{n+1}}, \frac{k}{2^n}\right)\right) \subset f^{-1}\left(\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right) = E_{n,k}$$

$$E_{n+1,2k+1} = f^{-1}\left(\left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right)\right) = f^{-1}\left(\left[\frac{k}{2^n}, \frac{2k+1}{2^{n+1}}\right)\right) \subset f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right) = E_{n,k+1}$$

so if $x \in E_{n+1,2k}$, then

$$s_{n+1}(x) = \frac{2k-1}{2^{n+1}} > \frac{2k-2}{2^{n+1}} = \frac{k-1}{2^n} = s_n(x),$$

and if $x \in E_{n+1,2k+1}$, then

$$s_{n+1}(x) = \frac{2k}{2^{n+1}} = \frac{k}{2^n} = s_n(x).$$

So $\{s_n\}$ is a monotonically increasing sequence.

On the other hand, let $x \in X$, since $f(x) \in \mathbb{R}$, then there exists$m \in \mathbb{N}$ so that $f(x) < m$, and let $n, k \in \mathbb{N}$ so that $n > m$ and $\frac{k-1}{2^n} < f(x) < \frac{k}{2^n}$, for some $k \in \{1, 2, \ldots, n2^n\}$. Then, $x \in E_{n,k}$ and by definition of $s_n(x)$ we have

$$0 \leq f(x) - s_n(x) < \frac{1}{2^n} \tag{1.5.6}$$

taking limit when n$\rightarrow \infty$, we obtain

$$\lim_{n\rightarrow\infty} s_n(x) = f(x)$$

In the general case, consider $f = f^+ - f^-$ and apply the preceding construction to f^+ and f^-.

Note that if f is measurable, then F_n, $E_{n,k}$ are measurable sets, so s_n are measurable functions.

Note that if $f \geq 0$, then functions s_n in the proof satisfy $s_n \geq 0$, and if f is bounded, i.e., $|f(x)| < M$ for all $x \in X$ and some $M \in \mathbb{R}$, then equation (1.5.6) tells us that the sequence $\{s_n\}$ converges uniformly to f.

1.6. INTEGRATION

Now we define integration on a measurable space X, in which $\mathfrak{M}$ is the σ-ring of measurable sets, and μ is the measure.

Definition 13. Suppose $s \geq 0$

$$s(x) = \sum_{k=1}^{n} c_k \chi_{E_k}(x) \quad x \in X$$

is measurable simple function, and suppose $E \in \mathfrak{M}$, we define the **integral** of s on E by

$$I_E(s) = \sum_{k=1}^{n} c_k \mu(E \cap E_k) \tag{1.6.1}$$

In the above definition we need $s \geq 0$, i.e., $c_k \geq 0$ for $k = 1, 2, \ldots, n$, to avoid indefinitions in the sum of the form $+\infty - \infty$, For example, if $\mu = m$ is the Lebesgue measure on $\mathbb{R}$, and we do not have this restrictions, consider $s(x) = 1 \cdot \chi_{(-\infty,0)} + (-1) \cdot \chi_{[0,+\infty)}$, then $I_{\mathbb{R}}(s)$ is not defined.

Now we see some properties of this integral

Theorem 11. *Let $s \geq 0$ be a measurable simple function on X, and suppose $E \in \mathfrak{M}$, with*

$$E = \bigcup_{j=1}^{\infty} F_j$$

and $F_j \in \mathfrak{M}$ are pairwise disjoint. Then

$$I_E(s) = \sum_{j=1}^{\infty} I_{F_j}(s) \tag{1.6.2}$$

Suppose that $s(x)=\sum_{k=1}^{n}c_k\chi_{E_k}(x) \;\; x\in X$, then

$$\begin{aligned}I_E(s)&=\sum_{k=1}^{n}c_k\mu(E\cap E_k)=\sum_{k=1}^{n}c_k\mu\left(\left(\bigcup_{j=1}^{\infty}F_j\right)\cap E_k\right)\\&=\sum_{k=1}^{n}c_k\mu\left(\bigcup_{j=1}^{\infty}(F_j\cap E_k)\right)=\sum_{k=1}^{n}c_k\sum_{j=1}^{\infty}\mu(F_j\cap E_k)\\&=\sum_{k=1}^{n}\left(\sum_{j=1}^{\infty}c_k\mu(F_j\cap E_k)\right)=\sum_{j=1}^{\infty}\left(\sum_{k=1}^{n}c_k\mu(F_j\cap E_k)\right)\\&=\sum_{j=1}^{\infty}I_{F_j}(s).\end{aligned}$$

REMARK 9. The above theorem tells us that if $\phi_s(E)=I_E(s)$ for each $E\in\mathfrak{M}$, then ϕ_s is countably additive.

THEOREM 12. *Let $s_1,\ s_2\geq 0$ be measurable simple functions on X,*

(1) *if $s_1\leq s_2$, then for each $E\in\mathfrak{M}$ we have*

$$I_E(s_1)\leq I_E(s_2) \tag{1.6.3}$$

(2) *$s_1+s_2\geq 0$ is a measurable simple function on X, and for each $E\in\mathfrak{M}$ we have*

$$I_E(s_1+s_2)=I_E(s_1)+I_E(s_2) \tag{1.6.4}$$

Suppose that

$$s_1(x)=\sum_{k=1}^{n}c_k\chi_{E_k},\quad s_2(x)=\sum_{j=1}^{k}d_j\chi_{F_j}$$

where $c_k\geq 0$, $d_j\geq 0$.

Since $X=\bigcup\limits_{i=1}^{n}E_i=\bigcup\limits_{j=1}^{k}F_j$, then for $1\leq i\leq n$, $1\leq j\leq k$, let $A_{ij}=F_j\cap E_i\cap E$

$$E_i\cap E=\bigcup_{j=1}^{k}A_{ij},\quad F_j\cap E=\bigcup_{i=1}^{n}A_{ij}$$

$$E=\bigcup_{i=1}^{n}(E_i\cap E)=\bigcup_{i=1}^{n}\bigcup_{j=1}^{k}A_{ij}$$

(1) Note that $s_1(x)=c_i\leq d_j=s_2(x)$ for $x\in A_{ij}$. So

$$I_{A_{ij}}(s_1)=c_i\mu(A_{ij})\leq d_j\mu(A_{ij})=I_{A_{ij}}(s_2)$$

By theorem (11)

$$I_E(s_1)=\sum_{i=1}^{n}\sum_{j=1}^{k}I_{A_{ij}}(s_1)\leq\sum_{i=1}^{n}\sum_{j=1}^{k}I_{A_{ij}}(s_2)=I_E(s_2)$$

(2) Note that $s_1(x)+s_2(x)=c_i+d_j$ for $x\in A_{ij}$. So

$I_{A_{ij}}(s_1+s_2)=(c_i+d_j)\mu(A_{ij})=c_i\mu(A_{ij})+d_j\mu(A_{ij})=I_{A_{ij}}(s_1)+I_{A_{ij}}(s_2)$;

By theorem 11

$$I_E(s_1+s_2)=\sum_{i=1}^{n}\sum_{j=1}^{k}I_{A_{ij}}(s_1+s_2)=\sum_{i=1}^{n}\sum_{j=1}^{k}\left(I_{A_{ij}}(s_1)+I_{A_{ij}}(s_2)\right)=I_E(s_1)+I_E(s_2).$$

Note that if $0\le s_1\le s_2$ are measurable simple functions and A_{ij} are as the above proof, then

$$s_2-s_1=\sum_{j=1}^{m}(d_j-c_i)\chi_{A_{ij}}\ge 0$$

and

$$I_E(s_2-s_1)=I_E(s_2)-I_E(s_1) \tag{1.6.5}$$

Definition 14. If f is measurable and non-negative, we define

$$\int_E f d\mu=\sup I_E(s), \tag{1.6.6}$$

where the supremum is taken over all measurable simple functions s such that $0\le s\le f$.

The left member of (1.6.6) is called the **Lebesgue integral** of f, with respect to the measure μ, over the set E. Note that the integral may have the value $+\infty$.

Note that if $0\le s$ is a measurable simple function, then $s\le s$, and by Theorem 12 (1) $I_E(s_1)\le I_E(s)$ for all measurable simple functions s_1 such that $0\le s_1\le s$ so

$$\int_E s d\mu=I_E(s). \tag{1.6.7}$$

Definition 15. Let f be measurable, and consider the two integrals

$$\int_E f^+ d\mu,\quad \int_E f^- d\mu, \tag{1.6.8}$$

where $f=f^+-f^-$ and f^+ and f^- are defined as Corollary 2. If at least one of the integrals (1.6.8) is finite, we define

$$\int_E f d\mu=\int_E f^+ d\mu-\int_E f^- d\mu. \tag{1.6.9}$$

If both integrals in (1.6.8) are finite, then the integral in (1.6.9) is finite, and we say that f is **integrable** (or **summable**) on E in the Lebesgue sense, with respect to μ and we write $f\in\mathcal{L}(\mu)$ on E. If $\mu=m$ the Lebesgue measure, we say that f is **Lebesgue integrable** on E and that $f\in\mathcal{L}$ on E.

Note that if the integral in (1.6.9) is $+\infty$ or $-\infty$, then the integral of f over E is defined, although f is not integrable on E in the above sense; f is integrable on E only if its integral over E is finite.

All properties in Remark 11.23 of Rudin's books are true, but not are evidently as he say, here we only show those we need, and lead to later the rest

Remark 10. The integral satisfies the following properties:

(a) If $f \in \mathcal{L}(\mu)$ on E, and $\lambda \in \mathbb{R}$, then

$$\int_E \lambda f \, d\mu = \lambda \int_E f \, d\mu \tag{1.6.10}$$

We first see the case $0 \le s = \sum_{k=1}^{n} c_k \chi_{E_k}$

1. If $\lambda \ge 0$, then $0 \le \lambda s$ is a measurable simple function and $\int_E \lambda s \, d\mu = I_E(\lambda s) = \sum_{k=1}^{n} \lambda c_k \mu(E \cap E_k) = \lambda \sum_{k=1}^{n} c_k \mu(E \cap E_k) = \lambda I_E(s) = \lambda \int_E s \, d\mu$
2. If $\lambda < 0$, then $\lambda s \le 0$, so $(\lambda s)^- = (-\lambda) s$

$$\int_E \lambda s \, d\mu = -\int_E (\lambda s)^- \, d\mu = -\int_E (-\lambda) s \, d\mu = -(-\lambda) \int_E s d\mu = \lambda \int_E s \, d\mu$$

In the case $0 \le f$;

1. If $\lambda \ge 0$, then s_1 is a simple function with $0 \le s_1 \le \lambda f$, if and only if $s_1 = \lambda s$ where s is a simple function with $0 \le s \le f$. So

$$\int_E \lambda f \, d\mu = \sup \{I_E(\lambda s)\} = \sup \left\{ \int_E \lambda s \, d\mu \right\} = \sup \left\{ \lambda \int_E s \, d\mu \right\} = \lambda \sup \left\{ \int_E s \, d\mu \right\} = \lambda \int_E f \, d\mu$$

and consequently $f \, d\mu$ to the end

2. If $\lambda < 0$, then $\lambda f \le 0$, so $(\lambda f)^- = (-\lambda) f$

$$\int_E \lambda f d\mu = -\int_E (\lambda f)^- \, d\mu = -\int_E (-\lambda) f d\mu = -(-\lambda) \int_E f d\mu = \lambda \int_E f d\mu$$

In the general case;

1. If $\lambda \ge 0$, then $(\lambda f)^+ = \lambda f^+, (\lambda f)^- = \lambda f^-$ and

$$\int_E \lambda f d\mu = \int_E (\lambda f)^+ - \int_E (\lambda f)^- = \int_E \lambda f^+ - \int_E \lambda f^- = \lambda \int_E f^+ - \lambda \int_E f^- = \lambda \left(\int_E f^+ - \int_E f^- \right) = \lambda \int_E f d\mu$$

2. If $\lambda < 0$, then $(\lambda f)^+ = (-\lambda) f^-, (\lambda f)^- = (-\lambda) f^+$

$$\begin{aligned} \int_E \lambda f d\mu &= \int_E (\lambda f)^+ - \int_E (\lambda f)^- = \int_E (-\lambda) f^- - \int_E (-\lambda) f^+ = (-\lambda) \int_E f^- - (-\lambda) \int_E f^+ \\ &= \lambda \left(\int_E f^+ - \int_E f^- \right) = \lambda \int_E f d\mu \end{aligned}$$

(b) If f is measurable and bounded on E, and if $\mu(E) < \infty$, then $f \in \mathcal{L}(\mu)$ on E.

Since f is bounded, then there exists M, so that $|f(x)| \le M$, $f^+(x) \le M$ and $f^-(x) \le M$.

If $s_1 = \sum_{k=1}^{n} c_k \chi_{E_k}$ and $s_2 = \sum_{k=1}^{m} d_k \chi_{F_k}$ are simple functions so that $0 \le s_1 \le f^+$ and $0 \le s_2 \le f^-$, then $0 \le c_k, d_k \le M$ and

$$\int_E s_1 d\mu = \sum_{k=1}^{n} c_k \mu(E \cap E_k) \le \sum_{k=1}^{n} M \cdot \mu(E \cap E_k) \le M \cdot \mu(E)$$

$$\int_E s_2 d\mu = \sum_{k=1}^{m} d_k \mu(E \cap F_k) \le \sum_{k=1}^{m} M \cdot \mu(E \cap F_k) \le M \cdot \mu(E)$$

So

$$\int_E f^+ d\mu \le M \cdot \mu(E) < \infty$$
$$\int_E f^+ d\mu \le M \cdot \mu(E) < \infty$$

and $f \in \mathcal{L}(\mu)$ on E.

(c) If $f, g \in \mathcal{L}(\mu)$, and $0 \le f(x) \le g(x)$, then

$$\int_E f d\mu \le \int_E g d\mu. \tag{1.6.11}$$

Note that $0 \le f(x) \le g(x)$ implies that

$$\{s : s \text{ is a simple function, } 0 \le s \le f\ \} \subset \{s : s \text{ is a simple function, } 0 \le s \le g\}.$$

So

$$\int_E f d\mu = \sup_{0 \le s \le f} I_E(s) \le \sup_{0 \le s \le g} I_E(s) = \int_E f d\mu.$$

(d) If $\mu(E) = 0$, and f is measurable, then

$$\int_E f d\mu = 0. \tag{1.6.12}$$

Note that $E \cap A \subset E$ implies $\mu(E \cap A) = 0$,So if $0 \le f$, and $s = \sum_{k=1}^n c_k \chi_{E_k}$ is a simple function, $0 \le s \le f$ on E, then

$$I_E(s) = \sum_{k=1}^n c_k \mu(E \cap E_k) = \sum_{k=1}^n c_k \cdot 0 = 0.$$

Thus,

$$\int_E f d\mu = \sup_{0 \le s \le f} I_E(s) = 0.$$

In the general case, we have $\int_E f^+ d\mu = \int_E f^- d\mu = 0$, so

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu = 0.$$

(e) If $f \in \mathcal{L}(\mu)$ on E, and $A \in \mathfrak{M}$, $A \subset E$, then $f \in \mathcal{L}(\mu)$ on A.

if $0 \le f$, and $s = \sum_{k=1}^n c_k \chi_{E_k}$ is a simple function, $0 \le s \le f$, since $A \subset E$, then $A \cap E_k \subset E \cap E_k$ and $\mu(A \cap E_k) \le \mu(E \cap E_k)$. Thus,

$$I_A(s) = \sum_{k=1}^n c_k \mu(A \cap E_k) \le \sum_{k=1}^n c_k \mu(A \cap E_k) = I_E(s).$$

and taking supremun we have,

$$\int_A f d\mu = \sup_{0 \le s \le f} I_A(s) \le \sup_{0 \le s \le f} I_A(s) = \int_E f d\mu.$$

In the general case, we have $\int_A f^+ d\mu \le \int_E f^+ d\mu < \infty$, $\int_A f^- d\mu \le \int_E f^- d\mu < \infty$, so $f \in \mathcal{L}(\mu)$ on A.

THEOREM 13 (Lebesgue's monotone convergence theorem). *Suppose* $E \in \mathfrak{M}$. *Let* $\{f_n\}$ *be a sequence of real, measurable functions such that*

$$0 \leq f_1(x) \leq f_2(x) \leq \dots \tag{1.6.13}$$

for all $x \in E$. *Let* f *be defined by*

$$f(x) = \lim_{n\to\infty} f_n(x), \tag{1.6.14}$$

then

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu \tag{1.6.15}$$

First note that the sequence $\{f_n(x)\}$ is an increasing sequence of real functions so

$$f(x) = \lim_{n\to\infty} f_n(x) = \sup_{n\in\mathbb{N}} \{f_n(x)\}$$

and f is a measurable non-negative function with $f_n(x) \leq f(x)$ for all $x \in E$ and $n \in \mathbb{N}$. Also we have $\left\{\int_E f_n(x)d\mu\right\}$ is an increasing sequence of real extended numbers, so

end of the proof not here

$$\lim_{n\to\infty} \int_E f_n d\mu = \sup_{n\in\mathbb{N}} \left\{\int_E f_n d\mu\right\} = \alpha$$

for some α, and since $f_n(x) \leq f(x)$ for all $x \in E$ and $n \in \mathbb{N}$, we have

$$\alpha \leq \int_E f d\mu. \tag{1.6.16}$$

Choose c such that $0 < c < 1$, and let s be a simple measurable function such that $0 \leq s \leq f$ and let

$$E_n = \{x : f_n(x) \geq cs(x)\},$$

for $n \in \mathbb{N}$. Since

By (1.6.13), $E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$. If $x \in E$, since $0 < c < 1$ then

$$cs(x) < f(x) = \sup_{n\in\mathbb{N}} \{f_n(x)\},$$

so there existsn $n \in \mathbb{N}$, so that $cs(x) \leq f_n(x)$. Thus,

$$E = \bigcup_{n=1}^{\infty} E_n. \tag{1.6.17}$$

Remark 10 (c) implies that for every n,

$$\int_E f_n d\mu \geq \int_{E_n} f_n d\mu \geq \int_{E_n} cs d\mu = c\int_{E_n} s d\mu = cI_{E_n}(s) = c\phi_s(E_n) \tag{1.6.18}$$

Since the ϕ_s is a countably additive set function (see Remark 9) by (1.6.17) we can apply Theorem 11 and Remark 10 (a) to ϕ_s in (1.6.18), and taking limit when $n \to \infty$ in(1.6.18), we obtain

$$\alpha = \lim_{n\to\infty} \int_E f_n d\mu \geq \lim_{n\to\infty} c\phi_s(E_n) = c\phi_s(E) = c\int_E s d\mu \tag{1.6.19}$$

Taking limit when $c \to 1$, we see that

$$\alpha \geq \int_E s d\mu \tag{1.6.20}$$

so

$$\alpha \geq \sup_{0 \leq s \leq f} \int_E s d\mu = \int_E f d\mu \tag{1.6.21}$$

Inequalities (1.6.16) and (1.6.21) and definition of α imply

$$\int_E f d\mu = \alpha = \lim_{n \to \infty} \int_E f_n d\mu.$$

Theorem 14. (a) *Suppose f is measurable and non-negative on X. For $A \in \mathfrak{M}$, define*

$$\phi_f(A) = \int_A f d\mu \tag{1.6.22}$$

then ϕ_f is countably additive.

(b) *The same conclusion holds if $f \in \mathcal{L}(\mu)$ on X.*

Note that if $f \in \mathcal{L}(\mu)$ on X, then $\phi_{f^+}(A) = \int_A f^+ d\mu$ and $\phi_{f^-}(A) = \int_A f^- d\mu$ are both finite, then write

$$\phi_f(A) = \int_A f d\mu = \int_A f^+ d\mu - \int_A f^- d\mu = \phi_{f^+}(A) - \phi_{f^-}(A) \tag{1.6.23}$$

and apply (a) to ϕ_{f^+} and ϕ_{f^-} we obtain (b).

To prove (a), by Theorem (10) there exists a sequence $\{s_n\}$ of simple functions such that

$$0 \leq s_1(x) \leq s_2(x) \leq ... \leq f \tag{1.6.24}$$

for all $x \in X$. So that

$$f = \lim_{n \to \infty} s_n = \sup\{s_n : n = 1, 2, 3, \ldots\} \tag{1.6.25}$$

Remark 10 (c) shows that $\phi_{s_n} \leq \phi_{s_{n+1}}$ for $n = 1, 2, 3, \ldots$. So $\{\phi_{s_n} : n = 1, 2, 3, \ldots\}$ is a sequence of countably additive functions on $\mathfrak{M}$. By Lebesgue's monotone convergence theorem

$$\phi_f(A) = \int_A f d\mu = \lim_{n \to \infty} \int_A s_n = \sup_{n \in \mathbb{N}} \left\{ \int_A s_n d\mu \right\} = \sup_{n \in \mathbb{N}} \{\phi_{s_n}(A)\}, \tag{1.6.26}$$

and by Theorem 3, ϕ_f is a countably additive function on $\mathfrak{M}$.

Corollary 3. *If $A, B \in \mathfrak{M}$, $B \subset A$ and $\mu(B - A) = 0$, then*

$$\int_A f d\mu = \int_B f d\mu \tag{1.6.27}$$

Since $B \subset A$, then $A = B \cup (A - B)$, $B \cap (A - B) = \varnothing$. The condition $\mu(B - A) = 0$ implies $\int_{A-B} f d\mu = 0$ (see Remark 10 (d)), so

$$\int_A f d\mu = \phi_f(A) = \phi_f(B) + \phi_f(A - B) = \int_B f d\mu + \int_{A-B} f d\mu = \int_B f d\mu. \tag{1.6.28}$$

Remark 11. The preceding corollary shows that sets of measure zero are negligible in integration

Let us write $f \sim g$ on E if the set $\{x : f(x) \neq g(x)\} \cap E$ has measure zero.

Clearly $f \sim f$ .and $f \sim g$ implies $g \sim f$; and note that

$$y \in \{x : f(x) = g(x)\} \cap \{x : g(x) = h(x)\} \tag{1.6.29}$$

then

$$y \in \{x : f(x) = h(x)\}$$

so

$$\begin{aligned}\{x : f(x) \neq h(x)\} \cap E &= \{x : f(x) = h(x)\}^c \cap E \\ &\subset (\{x : f(x) = g(x)\}^c \cup \{x : g(x) = h(x)\}^c) \cap E \\ &= (\{x : f(x) \neq g(x)\} \cap E) \cup (\{x : g(x) \neq h(x)\} \cap E)\end{aligned}$$

Thus, $\mu(\{x : f(x) \neq h(x)\} \cap E) = 0$ if $\mu(\{x : f(x) \neq g(x)\} \cap E) = 0$ and $\mu(\{x : g(x) \neq h(x)\} \cap E) = 0$. Hence $f \sim g$ and $g \sim h$ implies $f \sim h$. In conclusion, $\sim$ is an equivalence relation on E

If $f \sim g$ on E, then $f \sim g$ on A, for all $A \subset E$. let, $B = \{x : f(x) = h(x)\} \cap A$ and $C = \{x : f(x) \neq h(x)\}$, then $A = B \cup C$, $B \cap C = \varnothing$. So if f and g are integrable on A then

$$\int_A f d\mu = \int_B f d\mu = \int_B g d\mu = \int_A g d\mu. \tag{1.6.30}$$

If a property p holds for $x \in E - A$ and if $\mu(A) = 0$, we say that p holds for almost all $x \in E$, or that p holds almost everywhere on E.

Note that if $\mu(A) > 0$, where $A = \{x : f(x) = +\infty\}$, then $s_n(x) = \frac{n}{\mu(A)} \chi_A \leq f(x)$, so

$$n = \int_A s_n d\mu \leq \int_A f d\mu. \tag{1.6.31}$$

Thus, $\int_A f d\mu = +\infty$ and $f \notin \mathcal{L}(\mu)$ on E, in consequence if $f \in \mathcal{L}(\mu)$ on E, then $f(x)$ must be finite almost everywhere on E. In most cases we therefore do not lose generality if we assume the given functions to be finite-valued from the outset.

Exercise 3. Let f be a measurable function on E. Then, if $f \geq 0$ and $\int_E f d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E. Hint: Let E_n be the subset of E on which $f(x) > \frac{1}{n}$. Write $A = \bigcup_{n=1}^{\infty} E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n.

Solution 3. Note that for every n, $E_n \subset A$, so if $\mu(A) = 0$, then $\mu(E_n) = 0$ also. Conversely, if $\mu(E_n) = 0$ for every n, by Corollary **??** we have

Not labelled

$$\mu(A) = \mu(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0$$

On the other hand, if $\mu(E_n) > 0$ for any n, since $f \geq \frac{1}{n}\chi_{E_n}$ on E,then

$$0 = \int_E f d\mu \geq \int_E \frac{1}{n}\chi_{E_n} d\mu = \frac{1}{n}\mu(E \cap E_n) = \frac{1}{n}\mu(E_n) > 0$$

which is a clearly contradiction.

Exercise 4. [Exercise 11.2] Let f be a measurable function on E. Then, if $\int_A f d\mu = 0$ for every measurable subset A of a measurable set E, then $f(x) = 0$ almost everywhere on E.

SOLUTION 4. Consider $A_+ = \{x \in E : f(x) \geq 0\}$ and $A_- = \{x \in E : f(x) < 0\}$, since f is measurable, then A_+ and A_- are measurable sets and

$$A_+ \cap A_- = \varnothing, \quad E = A_+ \cup A_-$$

Note that

$$f(x) = \begin{cases} f^+(x) & x \in A_+ \\ -f^-(x) & x \in A_- \end{cases},$$
$$f^+(x) = \begin{cases} f(x) & x \in A_+ \\ 0 & x \in A_- \end{cases}$$
$$f^-(x) = \begin{cases} 0 & x \in A_+ \\ -f(x) & x \in A_- \end{cases}$$

If A is a measurable subset of E, then

$$A = (A_+ \cap A) \cup (A_- \cap A)$$

Since $A_+ \cap A$ and $A_- \cap A$ are measurable sets of E, we have

$$0 = \int_{A_+ \cap A} f d\mu = \int_{A_+ \cap A} f^+ d\mu$$
$$0 = \int_{A_- \cap A} f d\mu = \int_{A_- \cap A} -f^- d\mu = -\int_{A_- \cap A} f^- d\mu$$

Since

$$\int_A f^+ d\mu = \int_{A_+ \cap A} f^+ d\mu + \int_{A_- \cap A} f^+ d\mu = \int_{A_+ \cap A} f^+ d\mu + \int_{A_- \cap A} 0 d\mu = 0 + 0 = 0$$
$$\int_A f^- d\mu = \int_{A_+ \cap A} f^- d\mu + \int_{A_- \cap A} f^- d\mu = \int_{A_+ \cap A} 0 d\mu + \int_{A_- \cap A} f^- d\mu = 0 + 0 = 0,$$

Since f^+ and f^- are non-negative measurable functions, then exercise 11.1 implies that $f^+(x) = 0$ and $f^-(x) = 0$ almost everywhere on E, so

$$f(x) = f^+(x) - f^-(x) = 0$$

almost everywhere on E.

THEOREM 15. *If $f \in \mathcal{L}(\mu)$ on E, then $|f| \in \mathcal{L}(\mu)$ on E and*

$$\left| \int_E f d\mu \right| \leq \int_E |f| \, d\mu \tag{1.6.32}$$

Write $E = A \cup B$, where $A = \{x \in E : f(x) \geq 0\}$ and $B = \{x \in E : f(x) < 0\}$, then A and B are measurable and $A \cap B = \varnothing$, and note that

(1) If $x \in A$, then $|f|(x) = f(x) = f^+(x)$, $f^-(x) = 0$.
(2) If $x \in B$, then $|f|(x) = -f(x) = f^-(x)$, $f^+(x) = 0$.

By Theorem 14 we have

$$\int_E f^+ d\mu = \phi_{f^+}(E) = \phi_{f^+}(A) + \phi_{f^+}(B) = \int_A f^+ d\mu + \int_B f^+ d\mu = \int_A f^+ d\mu \tag{1.6.33}$$

$$\int_E f^- d\mu = \phi_{f^-}(E) = \phi_{f^-}(A) + \phi_{f^-}(B) = \int_A f^- d\mu + \int_B f^- d\mu = \int_B f^{\cdot} d\mu \tag{1.6.34}$$

and,

$$\begin{aligned}\int_E |f|\, d\mu &= \phi_{|f|}(E) = \phi_{|f|}(A) + \phi_{|f|}(B) \\ &= \int_A |f|\, d\mu + \int_B |f|\, d\mu = \int_A f^+ d\mu + \int_B f^- d\mu \\ &= \int_E f^+ d\mu + \int_E f^- d\mu < \infty.\end{aligned}$$

Since $0 \le f^+$, and $0 \le f^-$ we have $0 \le \int_E f^+ \, d\mu$, and $0 \le \int_E f^- \, d\mu$. So

$$\begin{aligned}\left|\int_E f d\mu\right| &= \left|\int_E f^+ d\mu - \int_E f^- d\mu\right| \le \left|\int_E f^+ d\mu\right| + \left|\int_E f^- d\mu\right| \\ &= \int_E f^+ d\mu + \int_E f^- d\mu = \int_E |f|\, d\mu.\end{aligned}$$

Since the integrability of f implies that $|f|$ s integrable, the Lebesgue integral is called an absolutely convergent integral.

Theorem 16. *Let f be a measurable function on E and $g \in \mathcal{L}(\mu)$ on E. If $|f| \le g$, then $f \in \mathcal{L}(\mu)$ on E.*

Note that $0 \le f^+, f^- \le |f| \le g$ so

$$\begin{aligned}0 \le \int_E f^+ d\mu \le \int_E g d\mu < \infty \\ 0 \le \int_E f^- d\mu \le \int_E g d\mu < \infty\end{aligned}$$

and $f \in \mathcal{L}(\mu)$ on E.

Exercise 5. [Exercise 11.4] If $f \in \mathcal{L}(\mu)$ on E and g is bounded and measurable on E, then $fg \in \mathcal{L}(\mu)$ on E.

Solution 5. Since g is bounded, let M a positive constant such that $|g(x)| \le M$ for all $x \in E$. Since $f \in \mathcal{L}(\mu)$ on E, by Remark 10 (a) and Theorem **??** we have $M|f| \in \mathcal{L}(\mu)$ on E. Since f and g are measurable, fg is also measurable. Now

$$|fg|(x) = |g(x)||f(x)| \le M|f(x)| = M|f|(x)$$

so by Theorem **??** we have $fg \in \mathcal{L}(\mu)$ on E.

Theorem 17. *If f, $g \in \mathcal{L}(\mu)$ on E, then*

$$\int_E (f+g)\, d\mu = \int_E f d\mu + \int_E g d\mu \tag{1.6.35}$$

In the case $0 \leq f$, g let $\{s_n\}$, $\{p_n\}$ be sequences of simple functions such that

$$0 \leq s_1(x) \leq s_2(x) \leq \ldots \leq f,$$
$$0 \leq p_1(x) \leq p_2(x) \leq \ldots \leq g,$$

for all $x \in E$. and

$$f(x) = \lim_{n\to\infty} s_n(x), \quad g(x) = \lim_{n\to\infty} p_n(x) \quad x \in E. \tag{1.6.36}$$

Then

$$0 \leq s_1(x) + p_1(x) \leq s_2(x) + p_2(x) \leq \ldots \leq f + g,$$

for all $x \in E$ and

$$(f+g)(x) = \lim_{n\to\infty} (s_n + p_n)(x). \quad x \in E. \tag{1.6.37}$$

By Lebesgue's monotone convergence theorem and item 2 in Theorem 12 we have

$$\begin{aligned}\int_E (f+g)\, d\mu &= \lim_{n\to\infty} \int_E (s_n + p_n)\, d\mu = \lim_{n\to\infty} \left(\int_E s_n d\mu + \int_E p_n d\mu \right) \\ &= \lim_{n\to\infty} \int_E s_n d\mu + \lim_{n\to\infty} \int_E p_n d\mu = \int_E f d\mu + \int_E g d\mu\end{aligned}$$

In the general case, since $f, g \in \mathcal{L}(\mu)$ we can suppose that $f(x)$ and $g(x)$ are finite for all $x \in E$. Let $h = f + g$, then

$$h^+ - h^- = h = f + g = f^+ - f^- + g^+ - g^-$$

or equivalently

$$h^+ + f^- + g^- = f^+ + g^+ + h^- \tag{1.6.38}$$

Since both sides in equation (1.6.38) are the sum of three positive functions of $\mathcal{L}(\mu)$ we conclude

$$\begin{aligned}\int_E h^+ d\mu + \int_E f^- d\mu + \int_E g^- d\mu &= \int_E \left(h^+ + f^- + g^-\right) d\mu \\ &= \int_E \left(f^+ + g^+ + h^-\right) d\mu \\ &= \int_E f^+ d\mu + \int_E g^+ d\mu + \int_E h^- d\mu\end{aligned}$$

Thus,

$$\begin{aligned}\int_E (f+g)\, d\mu &= \int_E h d\mu = \int_E h^+ d\mu - \int_E h^- d\mu = \int_E f^+ d\mu - \int_E f^- d\mu + \int_E g^+ d\mu - \int_E g^- d\mu \\ &= \int_E f d\mu + \int_E g d\mu.\end{aligned}$$

THEOREM 18 (Absolute Continuity). *Let E be a measurable set and $f \in \mathcal{L}(\mu)$ on E. Then, given any $\varepsilon > 0$, there is a $\delta > 0$ such that for any $A \subset E$ measurable with $\mu(A) < \delta$, we have*

$$\phi_{|f|}(A) = \int_A |f|\, d\mu < \varepsilon.$$

If f is bounded, i.e., $|f(x)| \le M$ for some real positive M and all $x \in E$, then if $\mu(A) < \delta = \frac{\varepsilon}{M}$, since $\mu(A) < \mu(E)$ we have

$$\phi_{|f|}(A) = \int_A |f|\, d\mu < \int_A M d\mu = \mu(A) M < \varepsilon.$$

Otherwise, define

$$f_n(x) = \begin{cases} |f(x)| & f(x) \le n \\ n & f(x) > n \end{cases}$$

Then $\{f_n\}$ is an increasing sequence of bounded, measurable functions with $f = \lim_{n\to\infty} f_n$ on E. By Lebesgue's monotone convergence theorem, we can find $N \in \mathbb{N}$ so that

$$\int_E f_N d\mu > \int_E |f|\, d\mu - \frac{\varepsilon}{2}$$

and hence

$$\frac{\varepsilon}{2} > \int_E |f|\, d\mu - \int_E f_N d\mu = \int_E (|f| - f_N)\, d\mu.$$

If $0 < \delta < \frac{\varepsilon}{2N}$ and $\mu(A) < \delta$, since $0 < f_N(x) < N$ for all $x \in E$, then

$$\begin{aligned} \int_A |f|\, d\mu &= \int_A (|f| - f_n) + f_n d\mu = \int_A (|f| - f_n) + \int_A f_n d\mu \\ &\le \int_E (|f| - f_n) + \int_A N d\mu \\ &< \frac{\varepsilon}{2} + N\mu(A) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and the proof is complete.

THEOREM 19. *Suppose $E \in \mathfrak{M}$. Let $\{f_n\}$ be a sequence of non-negative measurable functions on E. Let f be defined by*

$$f(x) = \sum_{k=1}^{\infty} f_k(x), \tag{1.6.39}$$

then

$$\int_E f d\mu = \sum_{k=1}^{\infty} \int_E f_k d\mu \tag{1.6.40}$$

Note that the sequence $\{g_n = \sum_{k=1}^{n} f_k\}$ of the partial sums of (1.6.39) form a monotonically increasing sequence such that

$$f = \lim_{n\to\infty} g_n,$$

and by Theorem 17

$$\int_E g_n d\mu = \sum_{k=1}^{n} \int_E f_k d\mu \tag{1.6.41}$$

By Lebesgue's monotone convergence theorem we have

$$\int_E f d\mu = \lim_{n\to\infty} \int_E g_n d\mu = \lim_{n\to\infty} \sum_{k=1}^{n} \int_E f_k d\mu = \sum_{k=1}^{\infty} \int_E f_k d\mu.$$

THEOREM 20 (Fatou's Lemma). *Suppose $E \in \mathfrak{M}$. Let $\{f_n\}$ be a sequence of non-negative measurable functions on E. Let f defined by*

$$f(x) = \liminf_{n\to\infty} f_n(x), \tag{1.6.42}$$

then

$$\int_E f d\mu \le \liminf_{n\to\infty} \int_E f_n d\mu. \tag{1.6.43}$$

Consider the sequence $\{g_n = \inf_{k\ge n} f_k\}$, by Theorem 8 the functions g_n are measurable functions on E. On the other hand: or comma

$$0 \le g_1(x) \le g_2(x) \le \dots \tag{1.6.44}$$

$$g_n \le f_n \tag{1.6.45}$$

$$f = \sup_{n\in\mathbb{N}} g_n = \lim_{n\to\infty} g_n, \tag{1.6.46}$$

Inequality (1.6.45) and Remark 10 (c) imply

$$\int_E g_n d\mu \le \int_E f_n d\mu$$

So

$$\liminf_{n\to\infty} \int_E g_n d\mu \le \liminf_{n\to\infty} \int_E f_n d\mu \tag{1.6.47}$$

Equations (1.6.44) and (1.6.46) imply that $\{g_n\}$ satisfies the hypothesis of Lebesgue's monotone convergence theorem, so using (1.6.47) we have

$$\int_E f d\mu = \lim_{n\to\infty} \int_E g_n d\mu = \liminf_{n\to\infty} \int_E g_n d\mu \le \liminf_{n\to\infty} \int_E f_n d\mu. \tag{1.6.48}$$

EXERCISE 6. [Exercise 11.5] Put

$$\begin{aligned} g(x) &= \begin{cases} 0 & 0 \le x < \frac{1}{2} \\ 1 & \frac{1}{2} \le x \le 1 \end{cases}, \\ f_{2k}(x) &= g(x), \\ f_{2k+1}(x) &= g(1-x). \end{aligned}$$

Show that

$$\liminf_{n\to\infty} f_n(x) = 0,$$

but

$$\int_0^1 f_n(x) dm = \frac{1}{2}$$

[Compare with (1.6.43).]

SOLUTION 6. Since for each x with $0 \le x < \frac{1}{2}$, we have $f_{2k}(x) = g(x) = 0$ for all k, then for these x, by the definition of the inferior limit we have $\liminf_{n\to\infty} f_n(x) = 0$.

Analogously, for each x with $\frac{1}{2} \le x \le 1$, we have $f_{2k+1}(x) = g(1-x) = 0$ for all k, thus $\liminf f_n(x) = 0$ for $\frac{1}{2} \le x \le 1$.

So $\liminf_{n\to\infty} f_n(x) = 0$ for all $0 \le x \le 1$.

Note that $g(x)=\chi_{\left[\frac{1}{2},1\right]}(x)$, so

$$f_{2k}\left(x\right)=g(x)=\chi_{\left[\frac{1}{2},1\right]}(x)$$
$$f_{2k+1}\left(x\right)=g(1-x)=\chi_{\left[0,\frac{1}{2}\right)}(x)$$

Thus,

$$\int_0^1 f_n(x)dm=\begin{cases}\int_0^1 f_{2k}(x)dm=\int_0^1\chi_{\left[\frac{1}{2},1\right]}(x)dm=m\left(\left[\frac{1}{2},1\right]\right)=\frac{1}{2} & n=2k\\ \int_0^1 f_{2k+1}(x)dm=\int_0^1\chi_{\left[0,\frac{1}{2}\right)}(x)dm=m\left(\left[0,\frac{1}{2}\right)\right)=\frac{1}{2} & n=2k+1\end{cases},$$

This exercise show that we can obtain strict inequality in equation (1.6.43) in Fatou's Lemma.

THEOREM 21 (Lebesgue dominated convergence theorem). *Suppose $E\in\mathfrak{M}$. Let $\{f_n\}$ be a sequence of measurable functions on E and f a measurable function on E such that*

$$f=\lim_{n\to\infty}f_n \quad \text{in } E \tag{1.6.49}$$

If there exists a function $g\in\mathcal{L}(\mu)$ on E, such that

$$\left|f_n\left(x\right)\right|\le g\left(x\right) \tag{1.6.50}$$

too early end of the theorem

for all $n\in\mathbb{N}$ and $x\in E$, then $f\in\mathcal{L}(\mu)$ on E and

$$\int_E f d\mu=\lim_{n\to\infty}\int_E f_n d\mu \tag{1.6.51}$$

Note that (1.6.50) implies

$$|f(x)|=\lim_{n\to\infty}|f_n(x)|\le f(x) \quad \text{in } E$$

and by Theorem 16 we have f, $f_n\in\mathcal{L}(\mu)$ on E, (1.6.50) also implies

$$-f_n\left(x\right)\le g\left(x\right) \tag{1.6.52}$$

$$f_n\left(x\right)\le g\left(x\right) \tag{1.6.53}$$

So the sequences $\{g+f_n\}$, $\{g-f_n\}$ are measurable non-negative functions such that

$$g+f=\lim_{n\to\infty}\left(g+f_n\right)=\liminf_{n\to\infty}\left(g+f_n\right)$$
$$g-f=\lim_{n\to\infty}\left(g-f_n\right)=\liminf_{n\to\infty}\left(g-f_n\right)$$

By Fatou's theorem and Theorem 17 we have

$$\int_E gd\mu+\int_E fd\mu=\int_E\left(g+f\right)d\mu\le\liminf_{n\to\infty}\int_E\left(g+f_n\right)d\mu=\int_E gd\mu+\liminf_{n\to\infty}\int_E f_n d\mu$$
$$\int_E gd\mu-\int_E fd\mu=\int_E\left(g-f\right)d\mu\le\liminf_{n\to\infty}\int_E\left(g-f_n\right)d\mu=\int_E gd\mu-\liminf_{n\to\infty}\int_E f_n d\mu$$

So

$$\int_E fd\mu\le\liminf_{n\to\infty}\int_E f_n d\mu \tag{1.6.54}$$

$$-\int_E fd\mu\le-\liminf_{n\to\infty}\int_E f_n d\mu \tag{1.6.55}$$

Note that (1.6.55) can be rewritten as

$$\limsup_{n\to\infty}\int_E f_n d\mu \le \int_E f d\mu \tag{1.6.56}$$

Thus,

$$\limsup_{n\to\infty}\int_E f_n d\mu \le \int_E f d\mu \le \liminf_{n\to\infty}\int_E f_n d\mu \le \limsup_{n\to\infty}\int_E f_n d\mu \tag{1.6.57}$$

So all inequalities in equation (1.6.57) are equalities which implies that the limit in (1.6.51) exist and the equality in (1.6.51) is holds.

Corollary 4. *If* $\mu(E) < \infty$, *and* $\{f_n\}$ *is uniformly bounded on E, and* $f(x) = \lim_{n\to\infty} f_n(x)$ *on* E, *then*

$$\int_E f d\mu = \lim_{n\to\infty}\int_E f_n d\mu. \tag{1.6.58}$$

Since $\{f_n\}$ is uniformly bounded on E, then there exists M such that $|f_n(x)| \le M = M\chi_E(x)$ for all $n \in \mathbb{N}$ and $x \in E$. note that $\int_E M\chi_E(x)\, d\mu = M\mu(E) < \infty$, So $M\chi_E(x) \in \mathcal{L}(\mu)$ on E, and we can apply the Lebesgue's dominated convergence theorem with $g = M\chi_E$.

A uniformly bounded convergent sequence is said to be **boundedly convergent**.

Exercise 7. [Exercise 11.6] Let

$$f_n(x) = \begin{cases} \frac{1}{n} & |x| \le n \\ 0 & |x| > n \end{cases}.$$

Then $f_n(x) \to 0$ uniformly on $\mathbb{R}^1$, but

$$\int_{-\infty}^{\infty} f_n\, dm = 2 \quad (n = 1, 2, 3, \ldots).$$

(We write $\int_{-\infty}^{\infty}$ in place of $\int_{\mathbb{R}^1}$). Thus uniform convergence does not imply dominated convergence in the sense of Lebesgue dominated convergence theorem. However, on sets of finite measure, uniformly convergent sequences of bounded functions do satisfy the Lebesgue dominated convergence theorem.

Solution 7. Recall that $\{f_n\}$ converges to g uniformly on $\mathbb{R}^1$ if for given $\varepsilon > 0$ there exist a natural number N, such that $|f_n(x) - g(x)| < \varepsilon$ for all $x \in \mathbb{R}^1$ and $n > N$.

In this case given $\varepsilon > 0$. Let N be natural number such that $\frac{1}{N} < \varepsilon$. Note that for all $x \in \mathbb{R}^1$and all integers $n > N$, we have

$$|f_n(x) - 0| \le \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

So $\{f_n\}$ converges to 0 uniformly on $\mathbb{R}^1$.

On the other hand, we have that $f_n = \frac{1}{n}\chi_{[-n,n]}$. So for all $n \in \mathbb{N}$

$$\int_{-\infty}^{\infty} f_n(x)\, dm = \int_{-\infty}^{\infty} \frac{1}{n}\chi_{[-n,n]}(x)\, dm = \frac{1}{n} m([-n,n]) = \frac{1}{n}2n = 2.$$

and the proof is complete.

Example 4. In this example we have a sequence $\{f_k\}$ of continuous function that converge to a continuous function f, but

$$\lim_{k\to\infty}\int_a^b f_k dx \neq \int_a^b f dx$$

In this case consider

$$f_k(x) = \begin{cases} k^2 x & 0 \le x \le \dfrac{1}{k} \\ k^2\left(\dfrac{2}{k} - x\right) & \dfrac{1}{k} < x \le \dfrac{2}{k} \\ 0 & \frac{2}{k} < x \le 1 \end{cases}$$

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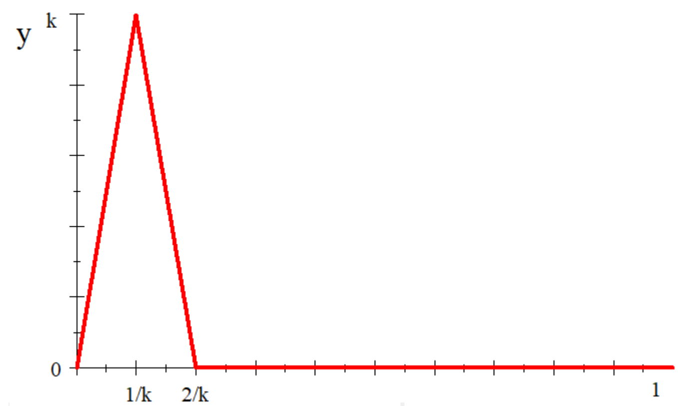
This sequence punctually converge to the function $f(x) = 0$ for all $x \in [0,1]$, below we can see the graph of f_k, this graph shows that the integral of f_k is 1 equal to the area of the triangle in red on the graph, so

$$\lim_{k\to\infty}\int_a^b f_k dx = 1 \neq 0 = \int_a^b f dx.$$



The last example shows that the condition $|f_k(x)| \le g(x)$, in the dominated converge theorem is necessary to obtain the conclusion.

Exercise 8. [Exercise 11.12] Suppose

(a) $|f(x,y)| \le 1$, for $0 \le x \le 1, \quad 0 \le y \le 1$
(b) for fixed x, $f(x,y)$ is a continuous function of y,
(c) for fixed y, $f(x,y)$ is a continuous function of x.

Put

$$g(x) = \int_0^1 f(x,y)dy \qquad (0 \le x \le 1)$$

Is g continuous?

Solution 8. The answer is yes, g is continuous. We only need to show that if $\{x_n\}$ is a sequence so that $x_n \to x$, then $f(x_n) \to f(x)$.

Indeed, first note that the continuity condition (b) implies that the right side on the definition of $g(x)$ is well defined (really we only need that for fixed x, $f_x(y) = f(x,y)$ is integrable on $[0,1]$).

On the other hand, by (c), $f(x_n, y) \to f(x,y)$ for each $y \in [0,1]$, in particular for almost every y.

Since the set $[0,1]$ has finite measure, then the function $h = \chi_{[0,1]} \in L(m)$

Let $f_n(y) = f(x_n, y)$ and $f_x(y) = f(x,y)$, then $f_n(y) \to f_x(y)$ and by (a)

$$|f_n(y)| = |f(x_n, y)| \leq 1 = h(y)$$

for all n and $y \in [0,1]$, using the dominated convergence theorem we have

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} \int_0^1 f(x_n, y)dy = \int_0^1 \lim_{n\to\infty} f(x_n, y)dy = \int_0^1 f(x,y)dy = g(x).$$

This concludes the proof.

1.7. COMPARISON WITH THE RIEMANN INTEGRAL

Our next theorem will show that every function which is Riemann-integrable on an interval is also Lebesgue-integrable, and that Riemann-integrable functions are subject to rather stringent continuity conditions. Quite apart from the fact that the Lebesgue theory therefore enables us to integrate a much larger class of functions, its greatest advantage lies perhaps in the ease with which many limit operations can be handled; from this point of view, Lebesgue's convergence theorems may well be regarded as the core of the Lebesgue theory. with hyphen?

One of the difficulties which is encountered in the Riemann theory is that limits of Riemann-integrable functions, for example, If $\{q_n \in [a,b] : n \in \mathbb{N}\}$ is an enumeration of the rationals number in the interval $[a,b]$ and $E_k = \{q_1, q_2, \ldots, q_k\}$ consider $f_k = \chi_{E_k}$, $f = \chi_{\mathbb{Q}\cap[a,b]}$, then the discontinuities of f_k are the points of E_k, since E_k is finite, then f_k are Riemann-integrable functions, also $\lim_{k\to\infty} f_k = f$ and note that f is not Riemann-integrable, because if $P = \{x_1, x_2, \ldots, x_n\}$ is a partition of $[a,b]$, $L(f,P) = 0$, $U(f,P) = 1$, So

$$\underline{\int_a^b} f dx = \sup\{L(f,P)\}\, 0 < 1 = \inf\{U(f,P)\} = \overline{\int_a^b} f dx.$$

This difficulty is eliminated with the Lebesgue theory, because limits of measurable functions are always measurable.

Let the measure space X be the interval $[a,b]$ of the real line, with $\mu = m$ (the Lebesgue measure), and $\mathfrak{M}$ the family of Lebesgue-measurable subsets of $[a,b]$. Instead of

$$\int_X f dm$$

it is customary to use the familiar notation

$$\int_a^b f dx$$

for the Lebesgue integral of f over $[a,b]$. The notation $f \in \mathcal{R}$ on $[a,b]$ means that f is Riemann integrable on $[a,b]$, To distinguish Riemann integrals from Lebesgue integrals, we shall now denote the former by

$$\mathcal{R}\int_a^b f dx.$$

THEOREM 22. (a) *If* $f \in \mathcal{R}$ *on* $[a,b]$, *then* $f \in \mathcal{L}$ *on* $[a,b]$ *and*

$$\int_a^b f dx = \mathcal{R}\int_a^b f dx. \tag{1.7.1}$$

(b) *Suppose* f *is bounded on* $[a,b]$. *Then* $f \in \mathcal{R}$ *on* $[a,b]$ *if and only if* f *is continuous almost everywhere on* $[a,b]$.