CHAPTER 1

Lebesgue Measure and Integration

1.1. SET FUNCTIONS

If A and B are any two sets, we write $A - B$ for the set of all elements x such that $x \in A$, $x \notin B$. The notation $A - B$ does not imply that $B \subset A$. We denote the empty set by $\varnothing$, and say that A and B are disjoint if $A \cap B = \varnothing$.

Definition 1. A family $\mathfrak{R}$ of sets is called a ring if A, $B \in \mathfrak{R}$ implies

$$A \cup B \in \mathfrak{R}, \text{ and } \; A - B \in \mathfrak{R} \tag{1.1.1}$$

Remark 1. Note that

$$A \cap B = A - (A - B) = B - (B - A)$$

so for any $A, B \in \mathfrak{R}$ we also have $A \cap B \in \mathfrak{R}$ if $\mathfrak{R}$ is a ring.

The set $\mathrm{F}(\mathbb{N}) = \{A \subset \mathbb{N} : A \text{ is finite }\}$ is an example of ring.

Definition 2. A ring $\mathfrak{R}$ is called a σ**-**ring if

$$\bigcup_{n=1}^{\infty} A_n \in \mathfrak{R} \tag{1.1.2}$$

whenever $A_n \in \mathfrak{R}$ for all $n = 1, 2, 3, \ldots$.

Since

$$\bigcap_{n=1}^{\infty} A_n = A_1 \cap \bigcap_{n=2}^{\infty} A_n = A_1 - \left(A_1 - \bigcap_{n=2}^{\infty} A_n \right) = A_1 - \bigcup_{n=1}^{\infty} (A_1 - A_n), \tag{1.1.3}$$

for any $A_n \in R$, $n = 1, 2, \ldots$, we also have $\bigcap\limits_{n=1}^{\infty} A_n \in \mathfrak{R}$ if $\mathfrak{R}$ is a σ-ring.

An Example of σ-ring is $\mathcal{P}(X)$ the set of all subsets of and set X.

Note that $\mathrm{F}(\mathbb{N})$ is not a σ-ring, because $E = \bigcup\limits_{n=1}^{\infty} \{2n\}$ the set of all even numbers is not finite.

Definition 3. We say that ϕ is a set function defined on σ-ring $\mathfrak{R}$ if ϕ assigns to every $A \in \mathfrak{R}$ a number $\phi(A)$ of the extended real number system. ϕ is **additive** if $A \cap B = \varnothing$ implies

$$\phi(A \cup B) = \phi(A) + \phi(B), \tag{1.1.4}$$

and ϕ is **countably additive** if $A_i \cap A_j = \varnothing$ for $i \neq j$, (in this case we say that the family A_i is pairwise disjoint) implies

$$\phi\left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \phi(A_n). \tag{1.1.5}$$

REMARK 2. Here we will assume that ϕ is not the constant functions whose only value is $+\infty$ or $-\infty$, and that the range does not contain both $+\infty$ and $-\infty$, because if it did, the right side of (1.1.4) could lose meaning.

REMARK 3. Note that the left side of (1.1.5) is independent of the order in which the A_n's are arranged. Hence, by the rearrangement theorem for series, if the right hand side of (1.1.5)) converges, it converges absolutely. Otherwise, the partial sums tend to $+\infty$ or $-\infty$.

THEOREM 1. *If ϕ is additive, then*

(1) $\phi(\varnothing) = 0$.
(2) $\phi\left(\bigcup_{n=1}^{k} A_n\right) = \sum_{n=1}^{k} \phi(A_n)$, *if* $A_i \cap A_j = \varnothing$ *for* $i \neq j$.
(3) $\phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2)$.
(4) *If* $\phi(A) \geq 0$ *for all* A, *and* $A \subset B$, *then* $\phi(A) \leq \phi(B)$.
added — PS (5) *if* $A \subset B$,*and* $|\phi(A)| < \infty$, *then* $\phi(B - A) = \phi(B) - \phi(A)$.

Note that non-negative additive set functions satisfy item 4, because this fact these functions are called **monotonic**.

THEOREM 2. *Suppose ϕ is countably additive on a ring $\mathfrak{R}$. Suppose $A_n \in \mathfrak{R}$ for $n = 1, 2, 3, \ldots$, $A_1 \subset A_2 \subset A_3 \subset \cdots \subset A \in \mathfrak{R}$ and*

$$A = \bigcup_{n=1}^{\infty} A_n.$$

Then,

$$\phi(A) = \lim_{n\to\infty} \phi(A_n).$$

1.2. CONSTRUCTION OF THE LEBESGUE MEASURE

DEFINITION 4. Let $\mathbb{R}^p$ denote p-dimensional space. By an **interval** in $\mathbb{R}^p$ we mean the set of points $x = (x_1, \ldots, x_p)$ such that

$$a_i \leq x_i \leq b_i \tag{1.2.1}$$

for $i = 1, \ldots, p$, or the set of points which is characterized by (1.2.1) with any or all of the signs $\leq$ replaced by $<$. The possibility that $a_i = b_i$, for any value of i is not ruled out; in particular, the empty set is included among the intervals.

Note that an interval in $\mathbb{R}^p$ is the cartesian product of finite intervals (closed, open, semiopen or degenarate) of $\mathbb{R}$.

DEFINITION 5. If A is the finite union of intervals, A is said to be an **elementary set**.
If $\mathbf{I}$ is an interval, we define the measure of $\mathbf{I}$ by

$$m(\mathbf{I}) = \prod_{i=1}^{p} (b_i - a_i)$$

no matter whether equality is included or excluded in any of the inequalities (1.2.1).

REMARK 4. If $\mathbf{I} = I_1 \times I_2 \times \cdots \times I_p$ and $\mathbf{J} = J_1 \times J_2 \times \cdots \times J_p$ where $I_1, \ldots,$ and $J_1, \ldots$,$\mathbf{I} \cap \mathbf{J} = (I_1 \cap J_1) \times (I_2 \cap J_2) \times \cdots \times (I_p \cap J_p)$

REMARK 5. If I, J are two finite intervals of $\mathbb{R}$, then $I - J$ can be written as the union of two (possible empty) intervals. Indeed, let $a \leq b$ be the extreme points of I and $c \leq d$ be the extreme points of J, then we can have

(a) $a \leq b \leq c \leq d$, in this case $I - J = I$.
(b) $a \leq c \leq b \leq d$, in this case $I - J$ is the interval with extreme points a, c.
(c) $a \leq c \leq d \leq b$, in this case $I - J$ is the union of the intervals with extreme points a, c and d, b.
(d) $c \leq a \leq d \leq b$, in this case $I - J$ is the interval with extreme points d, b.
(e) $c \leq d \leq a \leq b$, in this case $I - J = I$.
(f) $c \leq a \leq b \leq d$, in this case $I - J = \varnothing$. added — PS

Note that, if $\mathbf{I} = I_1 \times I_2 \times \cdots \times I_p$ and $\mathbf{J} = J_1 \times J_2 \times \cdots \times J_p$, then

(1.2.2)
$$\mathbf{I} - \mathbf{J} = (I_1 - J_1) \times I_2 \times \cdots \times I_p \cup (I_1 - J_1) \times (I_2 - J_2) \times \cdots \times I_p \cup \cdots \cup (I_1 - J_1) \times (I_2 - J_2) \times \cdots \times (I_p - J_p)$$

Thus, by Remark 5 $\mathbf{I} - \mathbf{J}$ is the union of intervals disjoint in $\mathbb{R}^p$.

DEFINITION 6. If If the intervals I_j are pairwise disjoint, then for $A = \bigcup_{j=1}^{k} \mathbf{I}_j$, we set

$$m(A) = \sum_{j=1}^{k} m(\mathbf{I}_j) \tag{1.2.3}$$

We denote by $\mathcal{E}$ the family of all elementary subsets of $\mathbb{R}^p$. Note that $\mathcal{E}$ satisfies the following properties.

$\mathcal{E}$ 1 $\mathcal{E}$ is a ring, but not a σ-ring. Clearly the if $A = \bigcup_{n=1}^{k} \mathbf{I}_n$ and $B = \bigcup_{m=1}^{l} \mathbf{J}_m$, then $A \cup B = \bigcup_{n=1}^{k} \mathbf{I}_n \cup \bigcup_{m=1}^{l} \mathbf{J}_m$ and

$$A - B = \bigcup_{n=1}^{k} \left(\mathbf{I}_n - \bigcup_{m=1}^{l} \mathbf{J}_m \right) = \bigcup_{n=1}^{k} \left(\bigcap_{n=1}^{l} (\mathbf{I}_n - \mathbf{J}_m) \right)$$

but $\mathbf{I}_n - \mathbf{J}_m$ is the union of at most $2p$ intervals in $\mathbb{R}^p$ and by Remark 4 the intersection of intervals is an interval, then $A - B$ is a finite union of intervals in $\mathbb{R}^p$.

Finally note that $\mathcal{E}$ is not a σ-ring: if $\mathbb{R}^p$ is an element of $\mathcal{E}$, since $\mathbb{R}^p$ can not be written as a finite union of intervals in $\mathbb{R}^p$ then $\mathcal{E}$ is not a σ-ring added — PS

$\mathcal{E}$ 2 If $A \in \mathcal{E}$, then A is the union of a finite number of disjoint intervals. If A is an interval this fact is obvious. Now suppose that all unions of k intervals is the union of a finite number of disjoint intervals, and let $A = \bigcup_{n=1}^{k+1} \mathbf{I}_n$, then

$$\begin{aligned} A &= \bigcup_{n=1}^{k+1} \mathbf{I}_n = \mathbf{I}_{k+1} \cup \bigcup_{n=1}^{k} \mathbf{I}_n = \mathbf{I}_{k+1} \cup \bigcup_{n=1}^{l} \mathbf{J}_n \\ &= \left(\mathbf{I}_{k+1} - \bigcup_{n=1}^{l} \mathbf{J}_n \right) \cup \bigcup_{n=1}^{l} \mathbf{J}_n \\ &= \left(\bigcap_{n=1}^{l} (\mathbf{I}_{k+1} - \mathbf{J}_n) \right) \cup \bigcup_{n=1}^{l} \mathbf{J}_n \end{aligned}$$

where the $\mathbf{J}_n$ are disjoint intervals, since $\mathbf{I}_{k+1} - \mathbf{J}_n$ is the finite union of disjoint intervals (see 5) and the intersection of intervals is an interval, then A is the union of a finite number of disjoint intervals.

$\mathcal{E}$ 3 If $A \in \mathcal{E}$, $m(A)$ is well defined by (1.2.3); that is, if two different decompositions of A into disjoint intervals are used, each gives rise to the same value of $m(A)$. Indeed, If $A = \bigcup_{r=1}^{k} \mathbf{I}_r = \bigcup_{q=1}^{l} \mathbf{J}_q$, where the intervals $\mathbf{I}_r$ and $\mathbf{J}_q$ are pairwise disjoint, then for each $r = 1, 2, \ldots, k$ and $q = 1, 2, \ldots, l$ we have

$$\mathbf{I}_r = A \cap \mathbf{I}_n = \bigcup_{q=1}^{l} (\mathbf{J}_q \cap \mathbf{I}_r)$$

$$\mathbf{J}_q = A \cap \mathbf{J}_q = \bigcup_{r=1}^{k} (\mathbf{J}_q \cap \mathbf{I}_r)$$

since the family $\{\mathbf{B}_{qr} = \mathbf{J}_q \cap \mathbf{I}_r : r = 1, 2, \ldots, k \text{ and } q = 1, 2, \ldots, l\}$ is pairwise disjoint, then

$$\begin{aligned} m(A) &= \sum_{r=1}^{k} m(\mathbf{I}_r) = \sum_{r=1}^{k} \left(\sum_{q=1}^{l} m(\mathbf{J}_q \cap \mathbf{I}_r) \right) = \sum_{n=1}^{k} \left(\sum_{q=1}^{l} m(\mathbf{B}_{qr}) \right) \\ &= \sum_{q=1}^{l} \left(\sum_{r=1}^{k} m(\mathbf{B}_{qr}) \right) = \sum_{q=1}^{l} \left(\sum_{r=1}^{k} m(\mathbf{J}_q \cap \mathbf{I}_r) \right) = \sum_{q=1}^{l} m(\mathbf{J}_q) \end{aligned}$$

$\mathcal{E}$ 4 m is additive on $\mathcal{E}$. Indeed, If $A = \bigcup_{r=1}^{k} \mathbf{I}_r$ and $B = \bigcup_{q=1}^{l} \mathbf{J}_q$, where the intervals $\mathbf{I}_r$ and $\mathbf{J}_q$ are pairwise disjoint, and $A \cap B = \varnothing$, then $\mathbf{I}_r \cap \mathbf{J}_q = \varnothing$ for each $r = 1, 2, \ldots, k$ and $q = 1, 2, \ldots, l$.

Since $A \cup B = \bigcup_{r=1}^{k} \mathbf{I}_r \cup \bigcup_{q=1}^{l} \mathbf{J}_q$, then

$$m(A \cup B) = m\left(\bigcup_{r=1}^{k} \mathbf{I}_r \cup \bigcup_{q=1}^{l} \mathbf{J}_q \right) = \sum_{r=1}^{k} m(\mathbf{I}_r) + \sum_{q=1}^{l} m(\mathbf{J}_q) = m(A) + m(B)$$

Note that if $p = 1, 2, 3$, then m is length, area, and volume, respectively.

Definition 7. A non-negative additive set function ϕ defined on $\mathcal{E}$ is said to be regular if the following is true: To every A and to every $\varepsilon > 0$ there exist sets F, $G \in \mathcal{E}$ such that F is closed, G is open, $F \subset A \subset G$, and

$$\phi(G) - \varepsilon \leq \phi(A) \leq \phi(F) + \varepsilon \tag{1.2.4}$$

Note that by 1.2 we have $A = \bigcup_{n=1}^{k} \mathbf{I}_n$ where $\mathbf{I}_n$ are intervals pairwise disjoint. So for each $n = 1, 2, \ldots, k$, if F_n is a closed set and G_n is an open set, such that $F_n \subset \mathbf{I}_n \subset G_n$ and

$$\phi(G_n) - \frac{\varepsilon}{k} \leq \phi(\mathbf{I}_n) \leq \phi(F_n) + \frac{\varepsilon}{k}$$

Then $F = \bigcup_{n=1}^{k} F_n$ and $G = \bigcup_{n=1}^{k} G_n$, satisfy requirement in Definition 7 for A. Thus, to show that ϕ is regular on $\mathcal{E}$ It is sufficient to verify the conditions of Definition 7 only in the intervals of $\mathbb{R}^p$.

Exercise 1. [Exercise 11.15] Let $\mathcal{R}$ be the ring of all elementary subsets of $(0, 1]$. If $0 < a < b \leq 1$, define

$$\phi((a, b)) = \phi((a, b]) = \phi([a, b)) = \phi([a, b]) = b - a$$

but define

$$\phi((0, b)) = 1 + b$$

if $0 < b \leq 1$. Show that this gives an additive set function ϕ on $\mathcal{R}$, which is not regular and which cannot be extended to a countably additive set function on a σ-ring.

Solution 1. Here as in Definition 6 we define

$$\phi(A) = \sum_{j=1}^{k} \phi(\mathbf{I}_j), \tag{1.2.5}$$

if $A = \bigcup_{j=1}^{k} \mathbf{I}_j$, and the intervals $\mathbf{I}_j$ are pairwise disjoint.

First, if A is an elementary set, $\phi(A)$ is well defined by (1.2.5); that is, if two different decompositions of A into disjoint intervals are used, each gives rise to the same value of $\phi(A)$. Indeed, if $A = \bigcup_{n=1}^{k} \mathbf{I}_n = \bigcup_{m=1}^{l} \mathbf{J}_m$, where the intervals $\mathbf{I}_n$ and $\mathbf{J}_m$ are pairwise disjoint, then for each $n = 1, 2, \ldots, k$ and $m = 1, 2, \ldots, l$ we have

$$\mathbf{I}_n = A \cap \mathbf{I}_n = \bigcup_{m=1}^{l} (\mathbf{J}_m \cap \mathbf{I}_n)$$

$$\mathbf{J}_m = A \cap \mathbf{J}_m = \bigcup_{n=1}^{k} (\mathbf{J}_m \cap \mathbf{I}_n)$$

since the family $\{\mathbf{B}_{mn} = \mathbf{J}_m \cap \mathbf{I}_n : n = 1, 2, \ldots, k \text{ and } m = 1, 2, \ldots, l\}$ is pairwise disjoint, then

$$\begin{aligned} \phi(A) = \sum_{n=1}^{k} \phi(\mathbf{I}_n) &= \sum_{n=1}^{k} \left(\sum_{m=1}^{l} \phi(\mathbf{J}_m \cap \mathbf{I}_n) \right) = \sum_{n=1}^{k} \left(\sum_{m=1}^{l} \phi(\mathbf{B}_{mn}) \right) \\ &= \sum_{m=1}^{l} \left(\sum_{n=1}^{k} \phi(\mathbf{B}_{mn}) \right) = \sum_{m=1}^{l} \left(\sum_{n=1}^{k} \phi(\mathbf{J}_m \cap \mathbf{I}_n) \right) = \sum_{m=1}^{l} \phi(\mathbf{J}_m) \end{aligned}$$

Recall that if A is an elementary set, then $A = \bigcup_{j=1}^{k} \mathbf{I}_j$ is the union of a finite number of disjoint intervals (see 1.2). So

$$\phi(A) = \begin{cases} \sum_{j=1}^{k} l(\mathbf{I}_j) & \text{if 0 is not the end point of any interval in } A \\ 1 + \sum_{j=1}^{k} l(\mathbf{I}_j) & \text{if 0 is the end point of any interval in } A \end{cases}$$

where $l(\mathbf{I}_j)$ is the length of the interval $\mathbf{I}_j$. In particular, $\phi(A) < 1$ if A is a closed set of $(0,1]$. Indeed, note that $0 \notin A$ for any subset $A \subset (0,1]$.

Now, if 0 is the endpoint of any interval in A, since A is closed then 0 is limit point of A, and $0 \in A$. which is clearly a contradiction.

If two elementary sets A and B are disjoint, at most one of them can have the point 0 as the endpoint of one of its intervals. Then $\phi(A \cup B)$ is the sum of the lengths of the intervals in $A \cup B$ if neither set contains an interval having 0 as the endpoint, and 1 plus this sum if one of them does contain an interval with 0 as endpoint. In either case $\phi(A \cup B) = \phi(A) + \phi(B)$ when $A \cap B = \varnothing$.Thus, the function ϕ is additive.

The function ϕ is not regular, because by definition $\phi\left(\left(0, \frac{1}{2}\right]\right) = 1 + \frac{1}{2} = \frac{3}{2}$, but $\phi(A) < 1$ if A is closed, so taking $\varepsilon = \frac{1}{3}$, we have $\phi(A) + \frac{1}{3} < 1 + \frac{1}{2} = \phi\left(\left(0, \frac{1}{2}\right]\right)$ for all closed $A \subset \left(0, \frac{1}{2}\right)$. Thus, ϕ does not satisfy Definition 7.

The function also cannot be extended to a countably additive set function on a σ-ring, because

$$\left(0, \frac{1}{2}\right] = \bigcup_{n=1}^{\infty} \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$$

the intervals in this union are pairwise disjoint, but

$$\phi\left(\left(0, \frac{1}{2}\right]\right) = \frac{3}{2} > \frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \sum_{n=1}^{\infty} \phi\left(\left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]\right).$$

Example 1.

(a) The set function m is regular. If A is an interval, i.e., $A = I_1 \times I_2 \times \cdots \times I_n$, if a_i, b_i are the extreme points of I_i, then by the continuity of the volume function on $\mathbb{R}^p$, we can choose r such that

$$G = (a_1 - r, b_1 + r) \times (a_2 - r, b_2 + r) \times \cdots \times (a_n - r, b_n + r)$$
$$F = [a_1 + r, b_1 - r] \times [a_2 + r, b_2 - r] \times \cdots \times [a_n + r, b_n - r]$$

and

$$\phi(G) = \prod_{j=1}^{n} (b_j - a_j + 2r) \leq \prod_{j=1}^{n} (b_j - a_j) + \varepsilon = \phi(A) + \varepsilon$$
$$\phi(A) - \varepsilon = \prod_{j=1}^{n} (b_j - a_j) - \varepsilon \leq \prod_{j=1}^{n} (b_j - a_j - 2r) = \phi(F)$$

(b) Take $p = 1$, and let α be a monotonically increasing function defined for all real x. Put

$$\begin{aligned}\mu([a,b)) &= \alpha(b-) - \alpha(a-) = \sup_{t<b}\alpha(t) - \sup_{t<a}\alpha(t)\\ \mu([a,b]) &= \alpha(b+) - \alpha(a-) = \inf_{b<t}\alpha(t) - \sup_{t<a}\alpha(t)\\ \mu((a,b]) &= \alpha(b+) - \alpha(a+) = \inf_{b<t}\alpha(t) - \inf_{a<t}\alpha(t)\\ \mu((a,b)) &= \alpha(b-) - \alpha(a+) = \sup_{t<b}\alpha(t) - \inf_{a<t}\alpha(t)\end{aligned}$$

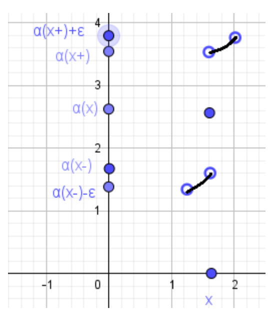
Recall that if α is a monotonically increasing function then the set of points of discontinuity of α is at most countable, and if α is continuous at x, then $\alpha(x-) = \alpha(x) = \alpha(x+)$. Also for each a, $b \in \mathbb{R}$ with $a < b$, by the definition of infimum and supremum given $\varepsilon > 0$ there exists c, x, y, d with $c < a < x < y < b < d$, so that α is continuous at c, x, y, d and

$$\begin{aligned}\alpha(a-) - \frac{\varepsilon}{2} < \alpha(c) \quad & \alpha(x) < \alpha(a+) + \frac{\varepsilon}{2}\\ \alpha(b-) - \frac{\varepsilon}{2} < \alpha(y) \quad & \alpha(d) < \alpha(b+) + \frac{\varepsilon}{2}\end{aligned}$$

or equivalently

$$\begin{aligned}-\alpha(c) - \frac{\varepsilon}{2} < -\alpha(a-) \quad & -\alpha(a+) < -\alpha(x) + \frac{\varepsilon}{2}\\ \alpha(b-) < \alpha(y) + \frac{\varepsilon}{2} \quad & \alpha(d) - \frac{\varepsilon}{2} < \alpha(b+).\end{aligned}$$

The behavior of a monotonically increasing function around a discontinuity point x is sketched in the figure below



Now we show that μ is regular on $\mathcal{E}$. Here c, x, y, d are as above

(1) In the case $A = [a,b)$ consider $F = [a,y]$ and $G = (c,b)$, then

$$\begin{aligned}\phi(G) - \varepsilon < \phi(G) - \frac{\varepsilon}{2} = \alpha(b-) - \alpha(c) - \frac{\varepsilon}{2} \le \alpha(b-) - \alpha(a-) = \phi(A)\\ \phi(A) = \alpha(b-) - \alpha(a-) \le \alpha(y) + \frac{\varepsilon}{2} - \alpha(a-) = \phi(F) + \frac{\varepsilon}{2} < \phi(F) + \varepsilon\end{aligned}$$

(2) In the case $A = [a,b]$ consider $F = [a,b]$ and $G = (c,d)$, then

$$\begin{aligned}\phi(G) - \varepsilon = \alpha(d) - \frac{\varepsilon}{2} - \alpha(c) - \frac{\varepsilon}{2} \le \alpha(b+) - \alpha(a-) = \phi(A)\\ \phi(A) = \phi(F) < \phi(F) + \varepsilon\end{aligned}$$

(3) In the case $A = (a, b]$ consider $F = [x, b]$ and $G = (a, d)$, then

$$\phi(G) - \varepsilon < \phi(G) - \frac{\varepsilon}{2} = \alpha(d) - \frac{\varepsilon}{2} - \alpha(a+) \leq \alpha(b+) - \alpha(a+) = \phi(A)$$

$$\phi(A) = \alpha(b+) - \alpha(a+) \leq \alpha(b+) - \alpha(x) + \frac{\varepsilon}{2} = \phi(F) + \frac{\varepsilon}{2} < \phi(F) + \varepsilon$$

(4) In the case $A = (a, b)$ consider $F = [x, y]$ and $G = (a, b)$, then

$$\phi(G) - \varepsilon < \phi(G) = \phi(A)$$

$$\phi(A) = \alpha(b-) - \alpha(a+) \leq \alpha(y) + \frac{\varepsilon}{2} - \alpha(x) + \frac{\varepsilon}{2} = \phi(F) + \varepsilon$$

Now we show that every regular set function on $\mathcal{E}$ can be extended to a countably additive set function on a σ-ring which contains $\mathcal{E}$.

Definition 8. Let μ be additive, regular, non-negative, and finite on $\mathcal{E}$. Consider countable coverings of any set $E \subset R^p$ by open elementary sets A_n.

$$E \subset \bigcup_{n=1}^{\infty} A_n.$$

Define

$$\mu^*(E) = \inf \sum_{n=1}^{\infty} \mu(A_n), \tag{1.2.6}$$

where the infimum is taken over all countable coverings of E by open elementary sets.

μ^* is called the **outer measure** of E, corresponding to μ.

It is clear that $\mu^*(E) \geq 0$ for all E and that if $E_1 \subset E_2$, then any countable coverings of E_2 by open elementary sets is a countable coverings of E_1 by open elementary sets and by properties of infimum we have

$$\mu^*(E_1) \leq \mu^*(E_2). \tag{1.2.7}$$

Theorem 3.

(a) *For every* $A \in \mathcal{E}$, $\mu^*(E) = \mu(E)$

(b) *if* $E \subset \bigcup_{n=1}^{\infty} E_n$, *then*

$$\mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(E_n) \tag{1.2.8}$$

Remark 6. (a) implies that μ^* is an extension of μ from $\mathcal{E}$ to $\mathcal{P}(\mathbb{R}^p)$. The property (1.2.8) is called subadditivity.

Definition 9. For any $A, B \subset \mathbb{R}^p$ We define

$$S(A, B) = (A - B) \cup (B - A) \tag{1.2.9}$$

$$d(A, B) = \mu^*(S(A, B)) \tag{1.2.10}$$

$S(A,B)$ is called **symmetric difference** of A and B. Now we will see some properties of $S(A,B)$ and $d(A,B)$

Lemma 1. *For any A, B, C, A_1, A_2, B_1, B_2 in $\mathbb{R}^p$ we have;*

S1 $S(A,B) = S(B,A)\,, \quad S(A,A) = \varnothing.$
S2 $S(A,B) \subset S(A,C) \cup S(C,B)\,.$
S3

$$\left.\begin{array}{r} S(A_1 \cup A_2, B_1 \cup B_2) \\ S(A_1 \cap A_2, B_1 \cap B_2) \\ S(A_1 - A_2, B_1 - B_2) \end{array}\right\} \subset S(A_1,B_1) \cup S(A_2,B_2)\,.$$

These properties of $S(A,B)$ imply

Lemma 2. *Forany A, B, C, A_1, A_2, B_1, B_2 in $\mathbb{R}^p$ we have;*

D1 $d(A,B) = d(B,A), \quad d(A,A) = \varnothing.$
D2 $d(A,B) \le d(A,C) + d(C,B).$
D3

$$\left.\begin{array}{r} d(A_1 \cup A_2, B_1 \cup B_2) \\ d(A_1 \cap A_2, B_1 \cap B_2) \\ d(A_1 - A_2, B_1 - B_2) \end{array}\right\} \le d(A_1,B_1) + d(A_2,B_2)\,. \tag{1.2.11}$$

The relations D1 and D2 show that $d(A,B)$ satisfies the requirements of definition for a distance except that $d(A,B) = 0$ does not imply $A = B$. For instance, if $p = 1$, $\mu = m$, $A = \{a_n \in \mathbb{R} : n \in \mathbb{N}\}$ is countable, and $B = \varnothing$, then

$$d(A,B) = m^*\left((A - \varnothing) \cup (\varnothing - A)\right) = m^*(A)$$

If $\varepsilon > 0$, taken $I_n = \left(a_n - \frac{\varepsilon}{2^{n+1}}, a_n + \frac{\varepsilon}{2^{n+1}}\right)$, then I_n are elementary open sets and $A \subset \bigcup_{n=1}^{\infty} I_n$ and

$$m^*(A) \le \sum_{n=1}^{\infty} m(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$$

Since ε is arbitrary, then $m^*(A) = 0$.

If $B = \varnothing$, then D2 tells us that

$$\mu^*(A) = d(A,\varnothing) \le d(A,C) + d(C,\varnothing) = d(A,C) + \mu^*(C)$$

interchanging A and C we get

$$\mu^*(C) \le d(A,C) + \mu^*(A)$$

So if at least one of $\mu^*(A)\,,\ \mu^*(C)$ is finite, then

$$|\mu^*(A) - \mu^*(C)| \le d(A,C) \tag{1.2.12}$$

We write $A_n \to A$, if

$$\lim_{n\to\infty} d(A,A_n) = 0.$$

If there is a sequence $\{A_n\}$ of elementary sets such that $A_n \to A$, we say that A is **finitely μ-measurable** and write $A \in \mathfrak{M}_F(\mu)\,.$

If A is the union of a countable collection of finitely μ-measurable sets, we say that A is **μ-measurable** and write $A \in \mathfrak{M}(\mu)\,.$

Theorem 4. *$\mathfrak{M}(\mu)$ is a σ-ring, and μ^* is countably additive on $\mathfrak{M}(\mu)\,.$*

We now replace $\mu^*(A)$ by $\mu(a)$, if $A \in \mathfrak{M}(\mu)$. So μ, initially defined on $\mathcal{E}$, is extended to a countably additive set function on the σ-ring $\mathfrak{M}(\mu)$. This extended set function is called a measure. The case $\mu = m$ is called the Lebesgue measure on $\mathbb{R}^p$.

Remark 7.

(a) If A is open, then $A \in \mathfrak{M}(\mu)$. Because every open set in $\mathbb{R}^p$ is the union of a countable collection of intervals. To see this, using the density of $\mathbb{Q}$ in $\mathbb{R}$, we can see that $\beta = \{I = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_p, b_p) : a_i, b_i \in \mathbb{Q}\}$ is a countable base whose elements are open intervals. Since $\mathbb{R}^p$ is an open set, taking complements we obtain that closed set is in $\mathfrak{M}(\mu)$

(b) If $A \in \mathfrak{M}(\mu)$ and $\varepsilon > 0$, there exist sets F and G that $F \subset A \subset G$, F is closed, G is open, and

$$\mu(G - A) < \varepsilon, \qquad \mu(A - F) < \varepsilon. \tag{1.2.13}$$

Indeed, if $\mu(A) < \infty$, i.e., $A \in \mathfrak{M}_F(\mu)$ by (1.2.6), there exists a sequence $\{A_n\}$ of open elementary sets, so that

$$A \subset \bigcup_{n=1}^{\infty} A_n \qquad \text{and} \quad \sum_{n=1}^{\infty} \mu(A_n) < \mu(A) + \varepsilon,$$

taking $G = \bigcup\limits_{n=1}^{\infty} A_n$, then

$$\mu(G - A) = \mu(G) - \mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n) - (\mu(A)) < \varepsilon.$$

If $\mu(A) = \infty$ there exists a sequence $\{A_n\}$ with $A_n \in \mathfrak{M}_F(\mu)$ for all $n \in \mathbb{N}$, so that $A = \bigcup\limits_{n=1}^{\infty} A_n$. Now, for each $n \in \mathbb{N}$ by show above there exists an open set G_n with $A_n \subset G_n$ so that

$$\mu(G_n - A_n) < \frac{\varepsilon}{2^n},$$

taking $G = \bigcup\limits_{n=1}^{\infty} G_n$, then, G is open, $A \subset G$, $G - A = \bigcup\limits_{n=1}^{\infty} (G_n - A_n)$, and

$$\mu(G - A) \leq \mu\left(\bigcup_{n=1}^{\infty} (G_n - A_n)\right) \leq \sum_{n=1}^{\infty} \mu(G_n - A_n) < \varepsilon.$$

For the second inequality, since $\mathfrak{M}(\mu)$ is a σ-ring, then $A^c \in \mathfrak{M}(\mu)$, so there exists G

$$\mu(G - A^c) < \varepsilon.$$

Taking $F = G^c$, then F is closed $A \subset F$, using Remark 1 we have

$$\mu(A - F) = \mu(A - G^c) = \mu(A \cap G) = \mu(G - A^c) < \varepsilon$$

(c) We say that E is a Borel set if E can be obtained by a countable number of operations, starting from open sets, each operation consisting in taking unions, intersections, or complements. The collection $\mathcal{B}$ of all Borel sets in $\mathbb{R}^p$ is a σ-ring; in fact, it is the smallest σ-ring which contains all open sets. By Remark (a) if $\mathcal{B} \subset \mathfrak{M}(\mu)$.

(d) By (b) If $A \in \mathfrak{M}(\mu)$, for each $n \in \mathbb{N}$, there exist Borel sets F_n so that that $F_n \subset A$, F_n is closed for all $n \in \mathbb{N}$, and

$$\mu(A - F_n) < \frac{1}{n},$$

If $F = \bigcup_{n=1}^{\infty} F_n$, then F is a Borel set, $F \subset A$, $A - F \subset A - F_n$ for all $n \in \mathbb{N}$, in consequence we have

$$\mu(A - F) \leq \mu(A - F_n) < \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

Thus,

$$\mu(A - F) = 0. \tag{1.2.14}$$

Since $A = F \cup (A - F)$, we see that every $A \in \mathfrak{M}(\mu)$ is the union of a Borel set and a set of measure zero.

The Borel sets are always μ-measurable for all μ. But the sets of measure zero, i.e., the sets E for which $\mu^*(E) = 0$ may be different for different measures μ's.

(e) For every μ, the sets of measure zero form a σ-ring, Indeed, recall that E has measure zero, if for a given $\varepsilon > 0$, there exists a sequence $\{A_n\}$ of open elementary sets, so that

$$E \subset \bigcup_{n=1}^{\infty} A_n \qquad \text{and} \quad \sum_{n=1}^{\infty} \mu(A_n) < \varepsilon, \text{ i}$$

Since $E_1 - E_2 \subset E_1$ for all E_1, E_2, then

$$0 \leq \mu^*(E_1 - E_2) \leq \mu^*(E_1) = 0$$

On the other hand, if $E = \bigcup_{n=1}^{\infty} E_n$, with $\mu^*(E_n) = 0$, for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let $\{A_{nk} : k \in \mathbb{N}\}$ a sequence of open elementary sets, so that

$$E_n \subset \bigcup_{k=1}^{\infty} A_{nk} \qquad \text{and} \quad \sum_{k=1}^{\infty} \mu(A_{nk}) < \frac{\varepsilon}{2^n},$$

Thus, $E = \bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_{nk}$, and

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(A_{nk}) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon,$$

(f) In case of the Lebesgue measure, every countable set has measure zero. In effect, for each $\varepsilon > 0$, the interval centered on x and of volume $\frac{\varepsilon}{2}$ is a covering of $A = \{x\}$ by elementary sets, with measure less than ε. As any countable set B is the countable union of its elements, the remark 6 (e) shows that B has zero measurement. But there are uncountable sets of measure zero. The Cantor set P is an example: Recall that P is defined as

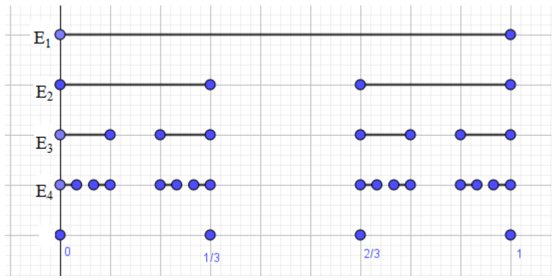
$$P = \bigcap_{n=0}^{\infty} E_n,$$

where $E_0 = [0, 1]$. Removing the the middle thirds ibterval of this intervals we obtain $E_1 = E_0 - (\frac{1}{3}, \frac{2}{3})$, so E_1 is $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Removing the middle thirds interval of these intervals we obtain E_2 and continuing in this way, we obtain a sequence of compact sets E_n, such that

$$E_0 \supset E_1 \supset E_2 \cdots$$

and E_n is the union of 2^n intervals, each of length 3^{-n}. So $m(E_n) = \left(\frac{2}{3}\right)^n$.

Below we show the E_1, E_2, E_3 and E_4.



Now P can be identified with the set of the sequences $(a_0 a_1 \dots a_n \dots)$ where $a_n = 0$ or $a_n = 2$, and using the same argument as the one that shows that $[0,1]$ is uncountable, we get that P is uncountable.

Since $P \subset E_n$ for all $n \in \mathbb{N}$, then $m(P) \leq m(E_n) = \left(\frac{2}{3}\right)^n$, and taking limit when $n \to \infty$ we obtain $m(P) = 0$.

Moreover, if we denote by $\mathfrak{c}$ the cardinality of $\mathbb{R}$, we obtain that cardinality of P is equal to $\mathfrak{c}$. Because every subset of the set of measure 0 is the set of measure 0, we have at least $2^{\mathfrak{c}}$ measurable sets on $\mathbb{R}$. Because the cardinality of the family of all subset of $\mathbb{R}$ is also $2^{\mathfrak{c}}$, the natural question is: 'are there unmeasurable sets?'. added — PS

(g) A **Vitali set** is a subset V of the interval $[0,1]$ of real numbers such that, for each real number r, there is exactly one number $v \in V$ such that $v - r$ is a rational number. Vitali sets are a set of representative of the group $\mathbb{R}/\mathbb{Q}$ in $[0,1]$. added — PS

Every Vitali set V is uncountable, and

$$v - u \text{ is irrational for any } u, v \in V,\ u \neq v. \tag{1.2.15}$$

A Vitali set is non-measurable. Indeed, assume that V is measurable and let q_1, q_2, $\dots$ be an enumeration of the rational numbers in $[-1,1]$. And let V_n be the translated sets defined by

$$V_n = V + q_n = \{v + q_n : v \in V\},$$

Note that $V_n \cap V_m = \varnothing$, because if $y \in V_n \cap V_m$, then $v + q_n = y = u + q_m$, implies $v - u$ is rational in contradiction with (1.2.15).

Also note that $[0,1] \subseteq \bigcup_{n=1}^{\infty} V_n \subseteq [-1,2]$.

To see the first inclusion, consider any real number $r \in [0,1]$ and let v be the representative in V for the equivalence class $[r]$; then $r - v = q_n$ for some rational number $q_n \in [-1,1]$ which implies that $r \in V_n$

Since the Lebesgue measure is countably additive, then

$$1 \leq \sum_{n=1}^{\infty} m(V_n) \leq 3.$$

Because the Lebesgue measure is translation invariant, we have $m(V_n) = m(V)$ for all $n \in \mathbb{N}$, and therefore

$$1 \leq \sum_{n=1}^{\infty} m(V) \leq 3.$$

But this is impossible. Summing infinitely many copies of the constant $m(V)$ yields either zero or infinity, according to whether the constant is zero or positive. In neither case is the sum in $[1, 3]$. So V cannot bee measurable. An adequate change of the above argument shows that, for all measurable set A with $m(A) > 0$ there exists a non-measurable set B with $B \subset A$.

1.3. MEASURE SPACES

Definition 10. Suppose X is a set, not necessarily a subset of an Euclidean space, or indeed of any metric space, X is said to be a **measure space** if there exists a σ-ring $\mathfrak{M}$ of subsets of X, which are called **measurable sets**, and a non-negative countably additive set function μ which is called a **measure**, defined on $\mathfrak{M}$.

If, in addition, $X \in \mathfrak{M}$ then X is said to be a measurable space.

For example, we can take $X = \mathbb{R}^{\ltimes}$, and $\mathfrak{M}$. the collection of all Lebesgue measurable subsets of $\mathbb{R}^{\ltimes}$, and $\mu = m$ the Lebesgue measure.

Or, let $X = \mathbb{N}$ the set of all positive integers, $\mathfrak{M}$ the collection of all subsets of X, and $\mu(E)$ is the number of elements of E. μ is know as the **counting measure**

Another example is provided by probability theory, where events may be considered as sets, and the probability of the occurrence of events is an additive (or countably additive) set function.