Mathematics

List 1

1. Write down expressions below without the use of the absolute value
   a) $|1-\sqrt{3}|$, b) $|x+y|$, c) $x+|1-x|+2|x-2|$ dla $1 \le x \le 2$,
   d) $|3x-8|$, e) $|x+1|-x$, f) $|x-1|+\frac{x}{|x|}-|x+2|$.

2. Using the geometric meaning of the absolute value, draw the sets of points satisfying conditions below. Write down the solutions of equations or inequalities.
   a) $|x+4|=2$, b) $|3x-2|>1$, c) $|6-2x| \le 3$,
   d) $|x+2|=|3-x|$, e) $|x+3|>|x-1|$, f) $|x|+|x-\sqrt{6}|=1$,
   g) $|x+1|+|x-2|=3$, h) $|x-5|+|x|<5$, i) $|x+1|+|x-3|>4$.

3. Write down the sets below using the absolute value notation $|\cdot|$.
   a) $\{4,18\}$, b) $\{1+\sqrt{3}, 3+\sqrt{3}\}$, c) $-3<x<3$,
   d) $0 \le x \le 2\sqrt{5}$, e) $x \in (-\infty,4) \cup (10,+\infty)$, f) $x \in (-\infty,-\sqrt{2}] \cup [2+\sqrt{2},+\infty)$.

4. Prove that for all $a, b \in R$ we have the following "triangle inequality"
   $|a+b| \le |a|+|b|$.

5. Solve equations and inequalities
   a) $|x|+\sqrt{2}=|x+\sqrt{2}|$, b) $|x+1|+|x-2|=5$, c) $|3x+1|=|3-x|$,
   d) $|x-2|<x$, e) $|3-3x| \ge 6-3x$, f) $|1-2x|-|x+3|>x+4$.

6. Write the quadratic functions below in the product form (if it exists) and in the canonical form; draw the graphs:
   a) $-x^2+x$, b) $2x^2+1$, c) x^2+2x-3,
   d) $x^2+x+\frac{1}{4}$, e) $-2x^2-2x+\frac{3}{2}$, f) $-x^2-3x-\frac{9}{4}$.

7. Determine the values of the parameter m such that the function
   $$f(x)=(m-3)x^2+(m-3)x+m-2$$
   a) is linear. Draw the graph of $f(x)$;
   b) is a quadratic function with one root. Draw the graph of $f(x)$,
   c) the largest value of $f(x)$ is positive.

8. What are the values of m such that the function $f(x)=mx^2+4x+m-3$:
   a) has a root,
   b) has two roots of different signs,
   c) has two positive roots,
   d) has the smallest value and this value is positive.

9. Let $g(m)$ be the number of intersection points of a line $y=mx-3$ and the graph of $y=(m+1)x^2+(2-m)x-2$, depending on m.
   Draw the graph of $g(m)$.

10. Find the coefficients and determine the degree of polynomials:

    a) $(x^4 - 3x^3 + x - 1)(x^2 - x + 4)$, b) $y = (x^3 + 5x^2 - x + 3)(x - 2)^2$,

    c) $W(x) = (x + 2)^3 - (x - 1)^2$, d) $y = (x + 1)^2 - (2x + 3)^3 - 2x$.

11. Calculate the quotient and the remainder in the division of P by Q:
    a) $P(x) = 2x^4 - 3x^3 + 4x^2 - 5x + 6$, $Q(x) = x^2 - 3x + 1$,
    b) $P(x) = x^{16} - 16$, $Q(x) = x^4 + 2$,
    c) $P(x) = x^5 - x^3 + 1$, $Q(x) = (x - 1)^3$.

12. Find the value of a such that the remainder in the division of $W(x) = 2x^3 + (a^2 + 1)x^2 - (a + 2)x - 6$ by $Q(x) = x + 3$ is as small as possible.

13. Find all integer roots of the polynomials:
    a) $x^3 + x^2 - 4x - 4$, b) $3x^3 - 7x^2 + 4x - 4$,

    c) $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$, d) $x^4 + 3x^3 - x^2 + 17x + 99$.

14. Find all rational roots of the polynomials:

    a) $4x^3 + x - 1$, b) $3x^4 - 8x^3 + 6x^2 - 1$,

    c) $x^3 - \frac{7}{6}x^2 - \frac{3}{2}x - \frac{1}{3}$, d) $x^5 + \frac{4}{3}x^3 - x^2 + \frac{1}{3}x - \frac{1}{3}$.

15. Write the polynomials below as products of irreducible components:
    a) $x^6 + 8$, b) $x^4 + x^2 + 1$, c) $x^4 - x^2 + 1$,
    d) $4x^5 - 4x^4 - 13x^3 + 13x^2 + 9x - 9$.

16. Solve the equations:
    a) $x^3 - 3x - 2 = 0$, b) $3x^4 - 10x^3 + 10x - 3 = 0$,

    c) $x^6 - 2\sqrt{2}x^3 + 2 = 0$, d) $x^4 - 2x^2 + 3x - 2 = 0$.

17. Solve the inequalities:
    a) $x^3 - x^2 + 4x < 4$, b) $x^3 - 6x^2 + 5x + 12 > 0$,

    c) $(1 - x^2)(4x^2 + 8x - 21) \geq 0$, d) $x^4 + 3x^3 + x^2 \leq 0$.

18. Solve:

    a) $\dfrac{12}{1 - 9x^2} = \dfrac{1 - 3x}{1 + 3x} + \dfrac{1 + 3x}{3x - 1}$, b) $\dfrac{30}{x^2 - 1} - \dfrac{13}{1 + x + x^2} = \dfrac{7 + 18x}{x^3 - 1}$,

    c) $\dfrac{5}{x^2 - 4} + \dfrac{18}{x^2 - 3x + 2} = \dfrac{8}{x^2 - 1}$, d) $\dfrac{x}{x + a} + \dfrac{x}{x - a} = \dfrac{8}{3}$.

19. Solve the inequalities:

    a) $\dfrac{(x - 1)^2}{(x + 1)^3} \leq 0$, b) $\dfrac{x^2 + 2}{x + 1} < 2$, c) $2 + \dfrac{3}{x + 1} > \dfrac{2}{x}$,

    d) $\dfrac{1}{(x + 1)^3} > \dfrac{1}{x + 1}$, e) $\dfrac{x^2 - 5}{x} < x + 1$, f) $|\dfrac{2x - 3}{x - 1}| \geq 2$,

    g) $|\dfrac{x^2 - 5x + 3}{x^2 - 1}| < 1$, h) $\dfrac{x}{|x - 2|} < 3$, i) $\dfrac{\sqrt{x^2 + 6x + 9}}{x} \geq -2$.

20. Discuss the existence of solutions of equations, depending on parameters a i b:

$$\text{a) } a + \frac{b}{x} = \frac{x-2}{x}, \qquad \text{b) } 1 + \frac{b}{x} = \frac{x}{x-a}.$$

21. Prove that no integer can satisfy the inequality

$$\frac{1}{x} + \frac{1}{x+1} < \frac{2}{x+2}.$$

22. Draw the graphs of functions:

    a) $f(x) = |6 - 2x|$, b) $f(x) = 6 - |x|$,

    c) $f(x) = \sqrt{x^2 - 6x + 9} + |x|$, d) $f(x) = x^2 - |x| + 1$,

    e) $f(x) = (2x - 3)/(x + 1)$, f) $f(x) = \operatorname{sgn}(x^2 - 3x)$.

    Note: the function $\operatorname{sgn}(x)$ (*sign* x) takes the value $+1$ for $x > 0$, 0 for $x = 0$ and -1 for $x < 0$.

Mathematics

List 2

1. Evaluate or simplify (write as a power):

$$\frac{3}{\sqrt[3]{3}},\quad \left(\frac{4}{9}\right)^{-\frac{1}{2}},\quad 8^{\frac{5}{3}},\quad 100^{-\frac{3}{2}},\quad 4^{-\frac{1}{4}},\quad \sqrt{2}\cdot\sqrt[3]{4},\quad \frac{1}{\sqrt[3]{5}},\quad 9\sqrt{\sqrt[3]{3}}.$$

2. Solve the equations:

(a) $4^{2x+1}=8^{5x-2}$, (b) $7{\cdot}3^{x+1}-5^{x+2}=3^{x+4}-5^{x+3}$, (c) $2^x{\cdot}4^{2x}{\cdot}8^{3x}=128$,

(d) $(3^x)^{2x}{\cdot}(81^x)^x=9^{x^2+4}$, (e) $5^x-25{\cdot}5^{-x}=24$, (f) $\left(\frac{1}{3}\right)^{x-1}=9^{2x}$.

3. Solve inequalities:

(a) $3^{4x-2}<9^{2-x}$, (b) $|2^x-2^{-x}|\leq\frac{3}{2}$, (c) $x^2\,2^x+x\,2^{x-1}>0$,

(d) $2^{x+2}-2^{x+1}\leq 2^{x-2}-2^{x-1}$, (e) $4^x+8<6{\cdot}2^x$, (f) $3^{2x-1}-3^{x-1}\geq 2$.

4. Find all x such that the expression

$$\frac{1}{2^x+2^{-x}}$$

takes values in the interval $(-1,\frac{2}{5})$.

5. Calculate or simplify:

$$\log_{\frac{1}{6}}36,\quad \log_2\sqrt{8},\quad \log_5 9^{\log_3 5},\quad \log_{\sqrt{5}}125,\quad \log_{\frac{4}{3}}\frac{27}{64},\quad \log_{\frac{1}{2}}\left(e^{\ln 2}\right)^3,$$

$\log_6 2+\log_6 18$, $\log_3 2-\log_9 2$, $\ln 2+\log_2 e$, $\log_{\frac{1}{2}}3+\log_4 3+\log_8 3$.

(Note: $e\approx 2,718...$ is the Euler constant; $\ln x=\log_e x$)

6. Which number is bigger: $\log_2 a$ or $\log_3 a$?

7. The frequency of occurrence of the leading (first) digit in many real-life statistical data (such as stock prices) shows a peculiar regularity which is called Benford's law. The probability of the occurrence of digit k, $k=1,\ldots,9$, as the leading digit, equals

$$P_k=\log_{10}(\tfrac{k+1}{k}).$$

Benford's law is often used as a criterion for fraud check, as most people try to make data sets look random without being aware that some digits occur more frequently as leading digits than other.
(http://www.mimuw.edu.pl/delta/artykuly/delta2010-12/fenomen.pdf)

Write down frequencies of occurrence of leading digits, as suggested by Benford's law. Calculate the sum of all the P_k's, explain the meaning of the obtained equality. Suppose we have a data set of $N=2000$ numbers. Among those numbers, 1300 numbers begin with $4,5,6,7,8$ or 9. Can we claim that this data set satifies Benford's law?

8. What profit will bring, after 4 years, a 1000 zl deposit at annual interest of 6% in case the interest is capitalized once a year. How will this change in case of monthly capitalization?

9. The nominal annual interest of a deposit is 6% . What is the effective interest in case of a monthly capitalization?

10. You make a long term deposit of 100 zl at annual interest of 6%. How long will it take for the value of your deposit to exceed 1000 zl in case the capitalization is:

    (a) annual, (b) monthly?

11. Suppose the nominal annual interest of a deposit is $r \cdot 100\%$. What is the effective interest after one year in case the capitalization is

    (a) monthly, (b) daily, (c) n times a year (at equal time periods)?

12. Solve equations:

    (a) $\log_3(x+1) = 2$, (b) $\ln^2 x + 3\ln x = 4$, (c) $\log_2 x + \log_8 x = 12$,

    (d) $\log_5 x + \log_5(x+5) = 2 + \log_5 2$, (e) $\log_x 2 - \log_4 x + \dfrac{7}{6} = 0$.

13. Solve inequalities:

    (a) $\log_3 x < -\dfrac{1}{3}$, (b) $\log_{\frac{1}{2}} x \leq 2$, (c) $\log_2 x \geq \log_2 x^2$,

    (d) $\log_{\frac{1}{3}} x + 2\log_{\frac{1}{9}}(x-1) \leq \log_{\frac{1}{3}} 6$, (e) $\log_2 \log_3 \dfrac{x-1}{x+1} > 0$.

14. Determine the values of m such that the equation $x^2 - 2x + \log_{0.5} m = 0$ has two different roots.

15. Solve the systems of equations:

    (a) $\begin{cases} 2\log_3 x - \log_3 y = 2 \\ 10^{y-x} = \frac{1}{100} \end{cases}$, (b) $\begin{cases} xy = 36 \\ x^{\log_3 y} = 16 \end{cases}$, (c) $\begin{cases} x^y = 9 \\ y = \log_3 x + 1 \end{cases}$.

16. Sketch the graphs of functions:

    (a) $y = |3^x - 3|$, (b) $y = 2^{-x}$, (c) $y = 2^{x+|x|}$, (d) $y = 2^{\frac{x^x}{|x|}}$,

    (e) $y = \log_3(x-1)$, (f) $y = \ln|x|$, (g) $y = \log_2(2x)$, (h) $y = \log_x 2$.

17. What is the difference between the graphs of the functions $y = \log_3 x^2$ and $y = 2\log_3 x$?

Hints and answers

2. a) 5/8, b) -1, c) 1/2, d) $-\sqrt{2}$, $\sqrt{2}$, e) 2, f) 1/5. **3.** d) $\emptyset$, e) $(1,2)$, f) $[1,\infty)$. **8.** $1000 \cdot (1.06)^4$, $1000 \cdot (1.005)^{48}$. **9.** $[(1.005)^{12} - 1] \cdot 100\%$. **10.** a) l: $100 \cdot (1.06)^l > 1000$, b) m: $100 \cdot (1.005)^m > 1000$,. **11.** a) $(1 + r/12)^{12} - 1$, b) $(1 + r/365)^{365} - 1$, c) $(1 + r/n)^n - 1$. **12.** a) 8, b) e^{-4}, e, c) 2^9, d) 5, e) 8, $1/\sqrt[3]{4}$. **13.** a) $x \in (0, 1/\sqrt[3]{3})$, b) $x \geq 1/4$, c) $x \in (0,1)$, d) $x \geq 3$, e) $x < -1$. **15.** a) $x = 3, y = 1$ or $x = 6, y = 4$, b) $x = 9, y = 4$ or $x = 4, y = 9$, c) $x = 3, y = 2$ or $x = 1/9, y = -1$.

Mathematics

List 3

1. For the matrices below:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 1 & -1 \end{bmatrix}$$

perform these operations $A+B$, $A-C$, $2A-3B$, $A-B+C$, AC, CA, A^T, C^T, $A-C^T$, C^TB^T, $(A^T+C)^T$, $(C-B^T)A$, ABC, ACB, CA^TB, which are well-defined.

2. Find inverse matrices to the given ones (check if $AA^{-1}=I$):

$$\text{(a)} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ -2 & 1 & 2 \end{bmatrix}.$$

3. Solve matricial equations:

$$\text{(a) } X\cdot\begin{bmatrix} -1 & 1 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 3 & 4 \end{bmatrix}, \quad \text{(b) } \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\cdot X\cdot\begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix},$$

$$\text{(c) } \left(\begin{bmatrix} 0 & 3 \\ 5 & -2 \end{bmatrix} + 4X\right)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \text{(d) } 3X + \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\cdot X.$$

4. Use the appropriate inverse matrix and solve the systems:

(a) $2x-y=3,\ \ 3x+y=2$, (b) $x+2y=0,\ 2x-y=5,$

$$\text{(c) } \begin{cases} x + y + z = 5 \\ 2x + 2y + z = 3 \\ 3x + 2y + z = 1 \end{cases}, \quad \text{(d) } \begin{cases} x + y + z = 4 \\ 2x - 3y + 5z = -5 \\ -x + 2y - z = 2 \end{cases}.$$

5. Treating $P=[x,y]^T$ as a point on the plane R^2 give the geometric meaning of the solution of systems of equations in 4 a) and b).

6. Use the considerations in problem 5 and discuss the number of solutions of the system of equations $A_{k\text{x}2}\cdot X_{2\text{x}1} = B_{k\text{x}1}$ for $k=1,2,3,10$.

7. If $P_1=[x_1,y_1]^T$ is a point on the plane R^2 and A is a 2-by-2 matrix then $P_2=A\cdot P_1$ is a point $P_2=[x_2,y_2]^T$ on the plane R^2 which is the image of P_1 in this transformation. Suppose that the matrix A equals

$$\text{(a)} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}, \quad \text{(c)} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}.$$

a) Determine images of several points of the plane. Have you noticed any regularity?
b) Find out the image of the straight line in this transformation (hint: if points P_1 lie on a line, what is the shape of the curve containing images P_2?).
c) Is it possible that the image of a line is the same line?
d) Does this transformation possess fixed points (points such that $P_2{=}P_1$ i.e. points which equal their images)?

8. Solve the systems in 4 using the Gauss (elimination) method.

9. Use the Gauss method to solve the systems of linear equations:

$$\text{(a)}\quad \left\{\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ 2x & + & 3y & + & z & = & 3 \\ 3x & + & y & + & 2z & = & 2 \end{array}\right. ,$$

$$\text{(b)}\quad \left\{\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 14 \\ 4x & + & 3y & - & z & = & 7 \\ x & - & y & + & z & = & 2 \end{array}\right. ,$$

$$\text{(c)}\quad \left\{\begin{array}{rcrcrcrcr} 3x & + & 4y & + & z & + & 2t & = & 3 \\ 6x & + & 8y & + & 2z & + & 5t & = & 7 \\ 9x & + & 12y & + & 3z & + & 10t & = & 13 \end{array}\right. ,$$

$$\text{(d)}\quad \left\{\begin{array}{rcrcrcrcr} 3x & - & 5y & + & 2z & + & 4t & = & 2 \\ 7x & - & 4y & + & z & + & 3t & = & 5 \\ 5x & + & 7y & - & 4z & - & 6t & = & 3 \end{array}\right. ,$$

$$\text{(e)}\quad \left\{\begin{array}{rcrcrcrcr} 3x & - & 2y & + & 5z & + & 4t & = & 2 \\ 6x & - & 4y & + & 4z & + & 3t & = & 3 \\ 9x & - & 6y & + & 3z & + & 2t & = & 4 \end{array}\right. ,$$

$$\text{(f)}\quad \left\{\begin{array}{rcrcrcrcr} 3x & + & 2y & + & 2z & + & 2t & = & 2 \\ 2x & + & 3y & + & 2z & + & 5t & = & 3 \\ 9x & + & y & + & 4z & - & 5t & = & 1 \\ 2x & + & 2y & + & 3z & + & 4t & = & 5 \\ 7x & + & y & + & 6z & - & t & = & 7 \end{array}\right. ,$$

$$\text{(g)}\quad \left\{\begin{array}{rcrcrcrcrcr} 2x & - & y & + & z & + & 2t & + & 3u & = & 2 \\ 6x & - & 3y & + & 2z & + & 4t & + & 5u & = & 3 \\ 6x & - & 3y & + & 4z & + & 8t & + & 13u & = & 9 \\ 4x & - & 2y & + & z & + & t & + & 2u & = & 1 \end{array}\right. .$$

Mathematics

List 4

1. Find the terms a_3, a_{n+1}, a_{2n} if:

   (a) $a_n = \dfrac{n^2}{n+1}$, (b) $a_n = (-1)^{n+1}\left(\dfrac{2}{3}\right)^n$, (c) $a_n = \dfrac{1}{n} + \cdots + \dfrac{1}{2n}$.

2. Examine monotonicity of the sequence:

   (a) $a_n = -2n + 7$, (b) $b_n = (-1)^n n$, (c) $c_n = 1 - 2/n$.

3. Find the general term of the arithmetic sequence and the sum S_{20} (of the initial 20 terms) if:

   (a) $a_3 = 3,\ a_{12} = 21$, (b) $a_1 + a_2 + a_3 = 18,\ a_1^2 + a_2^2 + a_3^2 = 116$.

4. Evaluate the sum of all three-digit numbers divisible by 3.

5. The lengths of the sides of a right triangle form an arithmetic progression. The shortest side has length p. Compute the surface area of the triangle and of the circumscribed circle as well as of the inscribed circle.

6. Find the general term of the geometric sequence and the sum S_{20} (of the initial 20 terms) if:

   (a) $a_3 = 54,\ a_6 = 2$, (b) $q = \frac{1}{2}$ and $S_7 = \frac{127}{16}$.

7. Change to a fractional form (a) $1.888\ldots$, (b) $0.313131\ldots$.

8. Solve the equation $x^2 - x^3 + x^4 - \cdots = \frac{1}{2}$.

9. We inscribe a square into the circle of radius r. Into this square we inscribe a circle. We repeat this process infinitely many times obtaining a sequence of circles and squares. Evaluate the sum of surface areas of all the squares.

10. Evaluate the limit of the sequence:

    (a) $a_n = \dfrac{2n^2 - n + 1}{3n - n^2 + 2}$, (b) $b_n = \dfrac{n^6 - n^2}{n^7 + 3}$, (c) $c_n = \dfrac{n^4 - n + 2}{2n^3 + 3}$,

    (d) $d_n = \sqrt[3]{n^2 + 1} - \sqrt[3]{n^2 - 1}$, (e) $e_n = n(\sqrt{n^2 + 2} - n)$,

    (f) $f_n = \dfrac{3^n + 2^n}{3^n - 2^n}$, (g) $g_n = \left(1 + \dfrac{2}{n}\right)^n$, (h) $h_n = \left(1 - \dfrac{1}{n}\right)^n$,

    (i) $i_n = \dfrac{1 + 2 + \cdots + n}{1 + 2 + \cdots + 2n}$, (j) $j_n = \dfrac{1}{n^2 + 1} + \dfrac{2}{n^2 + 1} + \cdots + \dfrac{n}{n^2 + 1}$,

    (k) $k_n = \sqrt[n]{3^n + 2^n}$, (l) $l_n = \dfrac{1}{n^2 + 1} + \dfrac{1}{n^2 + 2} + \cdots + \dfrac{1}{n^2 + n}$.

11. The annual interest rate of a bank deposit is $r \cdot 100\%$. Find the effective annual interest rate in case of continuous capitalization (which is defined as the limit, as $n \to \infty$, of the capitalization n times per year in equal time distances). What is the effective interest rate after time t years ($t \geq 0$) in continuous capitalization?

12. Evaluate the limit as $x \to +\infty$ and as $x \to -\infty$ of the function $f(x)$:

$$\text{(a) } x^7 - x^4 + x, \quad \text{(b) } \sqrt{x^2+2} - x, \quad \text{(c) } \sqrt[3]{x+3} - \sqrt[3]{x-1}, \quad \text{(d) } \frac{|x|}{x+1},$$

$$\text{(e) } \frac{x^3-2}{(x^2+1)(x+3)}, \qquad \text{(f) } \frac{x^2}{x+1} - x + 2, \qquad \text{(g) } \frac{x^3}{3x^2-4} - \frac{x^2}{3x+2}.$$

13. Evaluate the limits (if they exist):

$$\text{(a) } \lim_{x\to -3} \frac{x^2-9}{x+3}, \quad \text{(b) } \lim_{x\to 1} \frac{x^3-1}{x^2-1}, \quad \text{(c) } \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}, \quad \text{(d) } \lim_{x\to 0} \frac{\sqrt{x+1}}{x}.$$

14. Let $R(x)$ denote the radius of the ball circumscribed around the cone whose base has a (fixed) radius r and (a variable) height x. Evaluate the limits $\lim_{x\to 0+} R(x)$, $\lim_{x\to\infty} R(x)$. Is it possible to find these limits without determining the function $R(x)$?

15. Find all asymptotes of the function:

$$\text{(a) } y = \frac{2x^3+2}{x^3+x^2}, \quad \text{(b) } y = \frac{|x^2-1|}{x^2-2}, \quad \text{(c) } y = \frac{x^3+8}{x^2-4}, \quad \text{(d) } y = \sqrt{1-\frac{1}{x}}.$$

16. Find the values of the parameters a, $b \in R$ for which the function $f(x)$ is continuous:

$$\text{(a) } f(x) = \begin{cases} bx+3 & : x < 1, \\ 2x^2+x+a & : x \geq 1, \end{cases} \quad \text{(b) } f(x) = \begin{cases} x & : |x| \leq 1, \\ x^2+ax+b & : |x| > 1. \end{cases}$$

17. Prove that each of the equations below has a solution:

$$\text{(a) } x^3 + x = 3, \qquad \text{(b) } x + \sqrt{x} = 3 \text{(exactly one solution)},$$

$$\text{(c) } \sqrt{x+3} = x^2+x+2, \qquad \text{(d) } x^3+3x^2 = 3 \ \text{ (exactly three solutions)}.$$

18. Prove that the equation $x^4 + x = 5$ has exactly one positive solution. Find this solution using a calculator, up to an error 0.05.

Hints and answers

2. a) $\searrow$, b) not monotone., c)$\nearrow$. **3.** a) $a_1 = -1$, $r = 2$, b) $r = 2$, $a_1 = 4$ lub $r = -2$, $a_1 = 8$. **4.** $S_{300} = ((102+999)/2)300 = 165150$. **5.** $2p^2/3$, $5p/6$, $p/3$. **6.** a) $a_1 = 486$, $q = 1/3$, b) $a_1 = 4$. **7.** a) 17/9, b) 31/99. **8.** x=1/2. **9.** $4r^2$ **10.** a) -2, b) 0, c) $+\infty$, d) 0, e) 1, f) 1, g) e^2, h) $1/e$, i) 1/4, j) 1/2, k) 3, l) 0. **11.** $e^r - 1$; $e^{rt} - 1$. **12.** a) $+\infty$, $-\infty$, b) 0, $+\infty$, c) 0, 0, d) 1, -1, e) 1, 1 f) 1, 1, g) 2/9, 2/9. **13.** a) -6, b) 3/2, c) 1/2, d) does not exist. **14.** ∞, ∞. **15.** a) $y = 2$ at $\pm\infty$, $x = 0$, b) $y = 1$ at $\pm\infty$, $x = -\sqrt{2}$, $x = \sqrt{2}$, c) $y = x$ at $\pm\infty$, $x = 2$, d) $y = 1$ at $\pm\infty$, $x = 0$ left-sided. **16.** a) $b = a$, b) $a = 1$, $b = -1$.

Mathematics

List 5

1. Find the increment Δy of the function $y = x^2/2$, where $x = 2$, assuming that the increment Δx of the independent variable x is equal to (a) 0.5, (b) -0.2. Sketch an appropriate figure.

2. Find the increment Δy and the difference quotient $\Delta y/\Delta x$ corresponding to the increment Δx of the argument x for the function: (a) $y = ax + b$, (b) $y = 1/(2x+1)$. Find the derivative of the function $y = y(x)$ as the limit of the difference quotient.

3. Find the derivative function:

   (a) $y = ax^3 + \dfrac{b}{x} + c$, (b) $y = 9x^7 + 3x^{-5} - 3x^{-11}$, (c) $y = \dfrac{3}{3x-2}$,

   (d) $y = \sqrt[5]{x^2}$, (e) $y = \dfrac{x+1}{x-1}$, (f) $y = \sqrt{x^2-4}$, (g) $y = \dfrac{\sqrt[3]{x}}{1-\sqrt[3]{x}}$,

   (h) $(x-2)\ln x$, (i) $y = x^3 e^x$, (j) $\sqrt[3]{x}(\ln x - e^x)$, (k) $\dfrac{x^2 - \ln x}{e^x + x}$,

   (l) $v = (4z^2 - 5z + 13)^5$, (m) $s = 2\left(7t^2 - \dfrac{4}{t} + 6\right)^6$, (n) $y = 5e^{2x}$,

   (o) $y = 5^x + 2^x$, (p) $y = 3^x x^3$, (r) $y = \ln\ln x$, (s) $y = \ln\dfrac{5}{x-2}$,

   (t) $s = \ln\sqrt{\dfrac{1+t}{1-t}}$, (u) $y = \arctan(3x)$, (w) $y = \arctan(x - \sqrt{x^2+1})$.

4. (a) At which point does the tangent to the curve $y = (x-8)/(x+1)$ cross the x-axis at the angle $\pi/4$?

   (b) Find the point on the curve $y = e^x$ at which the tangent line is parallel to the line $x - y + 7 = 0$.

   (c) Under what conditions on the coefficients of the quadratic equation $y = x^2 + px + q$ is this parabola tangent to the x-axis?

   (d) At which point of the logarithmic curve $y = \ln x$ is the tangent parallel to the line $y = 2x$?

5. Using the differential, find the approximative value of:

   (a) $\sqrt[3]{63}$, (b) $e^{-0.07}$, (c) $\dfrac{1}{\sqrt{3.98}}$, (d) $\ln 0.9993$.

6. Prove the inequality:

   (a) $x > \ln(1+x)$, $x > 0$, (b) $e^x \geq x+1$, (c) $2x\arctan x \geq \ln(1+x^2)$.

7. Find the intervals of monotonicity of the function:

   (a) $y = x(3 - 2\sqrt{x})$, (b) $y = x/(1+x^2)$, (c) $y = 2x^3 - 12x + 5$.

8. Find the intervals of convexity, concavity and inflexion points:

   (a) $y = x^3 - 3x^2$, (b) $y = x/(1+x^2)$, (c) $y = \arctan x$, (d) $y = x + 1/x$.

9. Find all local extrema of the function:

   (a) $y = x^3 + 12x^2 + 36x - 50$, (b) $y = x\sqrt{1-x}$, (c) $y = x^2 + \dfrac{1}{x^2}$.

10. Find the largest and smallest value of the given function in the given range:

    (a) $y = x^4 - 2x^2 + 5$ in $[-2, 2]$, (b) $y = 1 - 24x + 15x^2 - 2x^3$ in $[1, 3]$.

11. Say as much as you can about the functions (domain, extrema, intervals of monotonicity, intervals of concavity, etc.):

    (a) $y = x^3 + 3x^2 - 9x - 2$, (b) $y = \dfrac{x^2 - 3}{x - 2}$, (c) $y = \dfrac{\ln x}{\sqrt{x}}$.

12. Decompose the number 20 a sum of two numbers with the smallest possible sum of their squares.

13. A cylinder is inscribed in a sphere of radius R. Find the radius r of cylinder's base circle for which the side surface area S of the cylinder attains its largest value.

Hints and answers

3. a) $3ax^2 - b/x^2$, b) $63x^6 - 15x^{-6} + 33x^{-12}$, c) $-9/(3x-2)^2$, d) $2/(5\sqrt[5]{x^3})$, e) $-2/(x-1)^2$, f) $x/\sqrt{x^2-4}$, g) $1/(3(\sqrt[3]{x})^2(1-\sqrt[3]{x})^2)$, h) $(x-1)/x + \ln x$, i) $x^2(x+3)e^x$, j) $(\ln x - e^x)/(3\sqrt[3]{x^2}) + \sqrt[3]{x}(1/x - e^x)$, k) $((2x-1/x)(e^x+x) - (x^2 - \ln x)(e^x+1))/(e^x+x)^2$, l) $5(4z^2-5z+13)(8z-5)$, m) $12(7t^2-4/t+6)^5(14t + 4/t^2)$, n) $10e^{2t}$, o) $5^x \ln 5 + 2^x \ln 2$, p) $(3^x \ln 3)\cdot x^3 + 3^x \cdot 3x^2$, r) $(1/\ln x)\cdot(1/x)$ s) $-1/(x-2)$, t) $1/(1-t^2)$, u) $3/(1+9x^2)$, w) $1/2(x^2+1)$. **4.** a) $(-4, 4)$ lub $(2, -2)$, b) $(0, 1)$, c) $y = 0$, $y' = 0$: $p^2 + 4q = 2$, d) $(2, \ln 2)$. **5.** a) $4 - 1/48$, b) $1 - 0.07$, c) $1/2 + 1/800$, d) -0.0007. **6.** a) Let $f(x) = x - \ln(1+x)$ for $x \in [0, \infty)$; $f'(x) = x/(1+x) > 0$ for $x > 0$, that is $f(x)$ increases on $[0, \infty)$; $f(0) = 0$ so $f(x) > 0$ for $x > 0$. b) Let $f(x) = e^x - (x+1)$ for $x \in R$; $f'(x) = e^x - 1$, hence $f(x)$ decreases on $(-\infty, 0]$ and increases on $[0, \infty)$; $f_{min} = f(0) = 0$, and thus $f(x) \geq 0$ for $x \in R$. c) like b). **7.** a) $\nearrow$: $x \in [0, 1]$, $\searrow$: $x > 1$, b) $\nearrow$: $x \in [-1, 1]$, $\searrow$: for $x < -1$ and for $x > 1$, c) $\nearrow$: on $(-\infty, -\sqrt{2}$ and on $(\sqrt{2}, \infty)$, $\searrow$: $x \in [-\sqrt{2}, \sqrt{2}]$. **8.** a) $\smile$: $(1, \infty)$, $\frown$: $(-\infty, 1)$, infl: $x = 1$, b) $\smile$: $(-\infty, 0)$, $\frown$: $(0, \infty)$, infl: $x = 0$, c) $\smile$: $(0, \infty)$, $\frown$: $(-\infty, 0)$, infl: $x = 0$. **9.** a) $y_{max} = y(-6)$, $y_{min} = y(-2)$, b) $y_{max} = y(2/3)$, c) $y_{min} = y(-1)$, $y_{min} = y(1)$. **10.** a) max: $y(-2) = y(2) = 13$, min: $y(-1) = y(1) = 4$, b) max: $y(3) = 10$, min: $y(1) = -10$. **12.** $10 + 10$; (a maximum of an appropriate function). **13.** $S = 4\pi r\sqrt{R^2 - r^2}$ achieves its max at $r = R/\sqrt{2}$.

Mathematics

List 6

1. Calculate the indefinite integrals:

$$\text{(a)} \int (3x^3+2\sqrt{x}-1)dx, \quad \text{(b)} \int x(x-1)(x-2)dx, \quad \text{(c)} \int \frac{x+3}{x^2}dx,$$

$$\text{(d)} \int \frac{2\sqrt[3]{x}-3}{x}dx, \quad \text{(e)} \int \frac{x^2+2}{x^2+1}dx, \quad \text{(f)} \int \frac{x^3+8}{x^2}dx, \quad \text{(g)} \int \frac{x^2}{x^3+8}dx,$$

$$\text{(h)} \int (9x^2-x+1)^2dx, \quad \text{(i)} \int \frac{x^2-\sqrt{x}}{\sqrt[3]{x}}dx, \quad \text{(j)} \int \frac{e^x-2^x}{5^x}dx.$$

2. Integrate by parts:

$$\text{(a)} \int xe^{-3x}dx, \quad \text{(b)} \int \ln x\, dx, \quad \text{(c)} \int x^2e^xdx, \quad \text{(d)} \int x\ln x\, dx,$$

$$\text{(e)} \int \frac{\ln x}{x^2}dx, \quad \text{(f)} \int \sqrt{x}\ln x\, dx, \quad \text{(g)} \int \operatorname{arctg} x\, dx, \quad \text{(h)} \int (\ln x)^2dx.$$

3. Apply the appropriate substitutions and compute antiderivatives:

$$\text{(a)} \int x\sqrt{x^2+1}\, dx, \quad \text{(b)} \int (5-3x)^{10}dx, \quad \text{(c)} \int \sqrt{a+bx}\, dx,$$

$$\text{(d)} \int xe^{x^2}\, dx, \quad \text{(e)} \int \frac{x}{x^4+1}\, dx, \quad \text{(f)} \int \frac{\ln^2 x}{x}dx, \quad \text{(g)} \int \frac{\ln x}{x}\, dx.$$

4. Compute the definite integrals:

$$\text{(a)} \int_0^2 \frac{3x-1}{3x+1}dx, \quad \text{(b)} \int_{-1}^1 (x^3-x+1)dx, \quad \text{(c)} \int_{-1}^2 |x|dx.$$

5. Find the formulae for velocity $v(t)$ and distance travelled $s(t)$ in linear motion with constant acceleration $a(t) = a$, given the initial conditions $v(0) = v_0$ and $s(0) = s_0$.

6. Calculate the integrals by substitution:

$$\text{(a)} \int_0^4 \frac{dx}{1+\sqrt{x}}, \quad x = t^2 \quad \text{(b)} \int_0^1 \frac{e^x}{e^{2x}+1}dx, \quad \text{(c)} \int_{-3}^{-2} \frac{dx}{x^2+2x+1}.$$

7. Integrate by parts:

$$\text{(a)} \int_0^2 xe^{-x}dx, \quad \text{(b)} \int_0^1 x^2\operatorname{arctg} x dx, \quad \text{(c)} \int_1^e \left(\frac{\ln x}{x}\right)^2 dx.$$

8. Calculate the area of a flat region bounded by
   (a) the graphs of parabolas $y = x^2$, $y^2 = x$,
   (b) the graph of the parabola $y = 2x - x^2$ and the line $x + y = 0$,
   (c) the curve $y = \ln x$, the axis $0x$ and the line $x = e$,

(d) the curve $y = (1 - \frac{x}{2})^5$ and the coordinate axes.

9. The parabola $y^2 = 2x$ divides the disc $x^2+y^2 \leq 8$ into two parts. What is the proportion of the areas of these parts?

10. A point of mass m is moving along the straight line with velocity $v(t) = (12t - t^2)$ m/s. The initial velocity is 36 m/s. What is the distance travelled until the point stops?

Answers

1. a) $(3/4)x^4 + (4/3)x\sqrt{x} - x + c$, b) $x^4/4 - x^3 + x^2 + c$, c) $\ln x - 3/x + c$, d) $6\sqrt[3]{x} - 3\ln x + c$; e) $x - \operatorname{arctg} x + c$, g) $(1/3)\ln(x^3 + 8) + c$, h) $81/5x^5 - 18/4x^4 + 19/3x^3 - x^2 + x + c$, i) $3/8\sqrt[3]{x^8} - 6/7\sqrt[6]{x^7}$. **2.** a) $-e^{-3x}(3x + 1)/3 + c$, b) $x\ln x - x + c$, c) $x^2(2\ln x - 1)/4 + c$, d) $-(1 + \ln x)/x + c$, e) $x \operatorname{arctg} x - (1/2)\ln(x^2+1) + c$, f) $x(\ln x)^2 - 2x\ln x + 2x + c$. **3.** a) $e^{x^2}/2 + c$, b) $-(5-3x)^{11}/33 + c$, c) $(\sqrt{x^2+1})^3/3 + c$, d) $(\ln x)^2/2 + c$, e) $(1/2)\mathrm{arctg}(x^2) + c$, f) $2(\sqrt{a+bx})^3/3b + c$. **??.** a) $(1/\sqrt{7})\mathrm{arctg}((x+1)/\sqrt{7}) + c$, b) $x - 2\mathrm{arctg}(x+1) + c$, c) $x - \ln(x^2+2x+5) - (3/2)\mathrm{arctg}((x+1)/2) + c$, d) $2\ln|x-2| + \ln|x+1| + c$, e) $\ln|x-1| - \ln|x+3| + c$, f) $2\ln|x-1| - \ln(x^2+1) - 2\mathrm{arctg}(x) + c$. **4.** a) $2 - (2\ln 7)/3$, b) 2, c) 5/2. **5.** $v(t) = at + v_0$, $s(t) = at^2/2 + v_0 t + s_0$. **6.** a) $4 - 2\ln 3$, b) $\operatorname{artctg} e - \pi/4$, c) 1/2. **7.** a) $1 - 3/e^2$, b) $\pi/12 - (1 - \ln 2)/6$, c) $2 - 5/e$. **8.** a) 1/3, b) 9/2, c) 1, d) 1/3. **10.** $s = \int_6^{12}(12t - t^2)dt = 144$ m.

Mathematics

List 7

1. Examine the graphs of sections of the function $z = z(x, y)$ and based on that draw the graphs of $z(x, y)$:

   (a) $3x + 2y + z - 6 = 0$, (b) $z^2 = x^2 + y^2$, (c) $z = x^2 + y^2$, (d) $z = xy$.

2. Calculate the first order and the second order partial derivatives of the functions:

   (a) $z = xy$, (b) $z = xe^{xy}$, (c) $z = x^2y + \ln(xy)$.

3. Find the local extrema of functions $z = z(x, y)$:

   (a) $z = x^2 + xy + y^2 - 2x - y$, (b) $z = x^3y^2(6 - x - y)$.

4. Find the maximum of the function (Cobb-Douglas production function) $u(x, y) = \sqrt{xy} = x^{1/2}y^{1/2}$ describing the production value in the case that the parameters x and y satisfy the condition $7x + 3y = 84$.

5. Determine the smallest and the largest value of $z = z(x, y)$ in the given region:
   (a) $z = x^2 + 2xy - 4x + 8y$ in the region $D: \ 0 \le x \le 1, \ 0 \le y \le 2$,
   (b) $z = x^3 + y^2 - 3x - 2y - 1$ in the region $D: \ x \ge 0, \ y \ge 0, \ x + y \le 1$,
   (c) $z = x^2 - xy + y^2$ in the region $D: \ |x| + |y| \le 1$.

6. Find the distance of the point $A = (0, 3, 0)$ from the surface $y = zx$.

7. Write a positive number a as the sum of three positive numbers in such a way that the product of these three ingredients attains the maximal value.

8. A cuboidal warehouse is supposed to have the volume $V = 64\,m^3$. One square meter of ceiling costs 20 zł, one square meter of the floor costs 40 zł and one square meter of the wall costs 30 zł. Determine the length a, width b and height c of the warehouse, minimizing the total cost.

9. The total annual income in the sale of two goods is expressed by the function $D(x, y) = 400x - 4x^2 + 1960y - 8y^2$, where x and y denote amounts of goods of respectively first and second type, sold per year. The production cost of x items of the first type and y items of the second type is: $K(x, y) = 100 + 2x^2 + 4y^2 + 2xy$. Determine the number of items of goods of the first and second type maximizing the annual profit.

10. Suppose we have the budget of 4 000 000 PLN at our disposal. What should be the amounts spent on resources x and y, so as to minimize the production cost described by the function $f(x, y) = x^2 + y^2 - xy + 3$?

11. Distribute the daily power production of 100 MWh between two power generating plants A and B in such a way so as to minimize the daily cost of fuel, given by the function:

$$f(x, y) = 2(x - 1)^2 + (y - 3)^2,$$

where x is the use of the fuel at plant A and y is the use at plant B. Moreover, 1 tone of fuel supplies 5 MWh of energy at plant A and 1 tone of fuel supplies 3 MWh of energy at plant B. Give the daily cost of fuel use at both plants.

Hints and answers

1. a) a plane; b) a cone; c) a paraboloid. **2.** a) $z_x = y$, $z_y = x$, $z_{xy} = z_{yx} = 1$, $z_{xx} = z_{yy} = 0$; b) $z_x = (xy+1)e^{xy}$, $z_y = x^2e^{xy}$, $z_{xy} = z_{yx} = (2x + x^2y)e^{xy}$, $z_{xx} = (2y + xy^2)e^{xy}$, $z_{yy} = x^3e^{xy}$; c) $z_x = 2xy + 1/x$, $z_y = x^2 + 1/y$, $z_{xy} = z_{yx} = 2x$, $z_{xx} = 2y - 1/x^2$, $z_{yy} = -1/y^2$. **3.** a) $z_{min} = z(1,0) = -1$; b) $z_{max} = z(3,2) = 72$. **5.** a) -3, 17; b) -4, -1; c) 0, 1. **6.** $\sqrt{5}$. **7.** $a/3 + a/3 + a/3$. **8.** $a = b = c = 4$. **9.** $x = 20$, $y = 80$. **10.** $x = y = 2$. **11.** $x = 11, y = 15$.