ON CONCAVITY OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR THE EQUATION $(-\Delta)^{1/2}\varphi = 1$ IN CONVEX PLANAR REGIONS

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Abstract. For a sufficiently regular open bounded set $D \subset \mathbb{R}^2$ let us consider the equation $(-\Delta)^{1/2}\varphi(x) = 1$, $x \in D$ with the Dirichlet exterior condition $\varphi(x) = 0$, $x \in D^c$. Its solution, $\varphi(x)$ is the expected value of the first exit time from D of the Cauchy process in $\mathbb{R}^2$ starting from x. We prove that if $D \subset \mathbb{R}^2$ is a convex bounded domain then φ is concave on D. To show it we study the Hessian matrix of the harmonic extension of φ. The key idea of the proof is based on a deep result of Hans Lewy concerning determinants of Hessian matrices of harmonic functions.

1. Introduction

Let $D \subset \mathbb{R}^2$ be an open bounded set which satisfies a uniform exterior cone condition on ∂D and let us consider the following Dirichlet problem for the square root of the Laplacian

$$(-\Delta)^{1/2}\varphi(x) = 1, \qquad x \in D, \tag{1}$$

$$\varphi(x) = 0, \qquad x \in D^c, \tag{2}$$

where we understand that φ is a continuous function on $\mathbb{R}^2$. $(-\Delta)^{1/2}$ in $\mathbb{R}^2$ is given by $(-\Delta)^{1/2} f(x) = \frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_{|y-x|>\varepsilon} \frac{f(x)-f(y)}{|y-x|^3}\, dy$, whenever the limit exists.

It is well known that (1-2) has a unique solution, which has a natural probabilistic interpretation. Let X_t be the Cauchy process in $\mathbb{R}^2$ (that is a symmetric α-stable process in $\mathbb{R}^2$ with $\alpha = 1$) with a transition density $p_t(x) = \frac{1}{2\pi} t(t^2 + |x|^2)^{-3/2}$ and let $\tau_D = \inf\{t \geq 0 : X_t \notin D\}$ be the first exit time of X_t from D. Then $\varphi(x) = E^x(\tau_D)$, $x \in \mathbb{R}^2$, where E^x is the expected value of the process X_t starting from x, [18]. The function $E^x(\tau_D)$ plays an important role in the potential theory of symmetric stable processes (see e.g. [5], [4], [11]).

About 10 years ago R. Bañuelos posed a problem of p-concavity of $E^x(\tau_D)$ for symmetric α-stable processes. The problem was inspired by a beautiful result of Ch. Borell about $1/2$-concavity of $E^x(\tau_D)$ for the Brownian motion.

The main result of this paper is the following theorem. It solves the problem posed by R. Bañuelos for the Cauchy process in $\mathbb{R}^2$.

Theorem 1.1. *If $D \subset \mathbb{R}^2$ is a bounded convex domain then the solution of (1-2) is concave on D.*

To the best of author's knowledge this is the first result concerning concavity of solutions of equations for fractional Laplacians on general convex domains. There is a recent interesting paper of R. Bañuelos and R. D. DeBlassie [1] in which the first eigenfunction of the Dirichlet eigenvalue problem for fractional Laplacians on Lipschitz domains is studied but in that paper superharmonicity and not concavity of the first eigenfunction is proved (similar results were also obtained by M. Kaßmann and L. Silvestre [22]). In [3] concavity of the first eigenfunction for fractional Laplacians was studied but [3] concerns only boxes and not general convex domains.

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Now let $D \subset \mathbb{R}^d$, $d \geq 1$ be an open bounded set which satisfies a uniform exterior cone condition on ∂D, $\alpha \in (0, 2]$ and let us consider a more general Dirichlet problem for the fractional Laplacian

$$(-\Delta)^{\alpha/2}\varphi(x) = 1, \qquad x \in D, \tag{3}$$

$$\varphi(x) = 0, \qquad x \in D^c, \tag{4}$$

where we understand that φ is a continuous function on $\mathbb{R}^d$. $(-\Delta)^{\alpha/2}$ in $\mathbb{R}^d$ for $\alpha \in (0, 2)$ is given by $(-\Delta)^{\alpha/2} f(x) = \mathcal{A}_{d,-\alpha} \lim_{\varepsilon \to 0^+} \int_{|y-x|>\varepsilon} \frac{f(x)-f(y)}{|y-x|^{d+\alpha}} \, dy$, whenever the limit exists, $\mathcal{A}_{d,-\alpha} = 2^\alpha \Gamma((d+\alpha)/2)/(\pi^{d/2}|\Gamma(-\alpha/2)|)$. For $\alpha = 2$ the operator $(-\Delta)^{\alpha/2}$ is simply $-\Delta$.

It is well known that (3-4) has a unique solution. It is the expected value of the first exit time from D of the symmetric α-stable process in $\mathbb{R}^d$.

Remark 1.2. For $\alpha = 2$ i.e. for the Laplacian, it is well known that if $D \subset \mathbb{R}^d$ is a bounded convex domain then the solution of (3-4) is 1/2-concave, that is $\sqrt{\varphi}$ is concave. This was proved for $d = 2$ in 1969 by L. Makar-Limanov [32]. For $d \geq 3$ it was proved in 1983 by Ch. Borell [8] and independently by A. Kennington [23], [24] using ideas of N. Korevaar [25].

Remark 1.3. Let $\alpha \in (0, 2]$ and φ be a solution of (3-4) for $D = B(0, r) \subset \mathbb{R}^d$, $d \geq 1$ a ball with centre 0 and radius $r > 0$. Then φ is given by an explicit formula [18] (see also [21], [17]) $\varphi(x) = C_B(r^2 - |x|^2)^{\alpha/2}$, $x \in B(0, r)$, where $C_B = \Gamma(d/2)(2^\alpha \Gamma(1 + \alpha/2)\Gamma(d/2 + \alpha/2))^{-1}$. In particular φ is concave on $B(0, r)$.

Remark 1.4. For any $\alpha \in (1, 2)$ and $d \geq 2$ there exists a bounded convex domain $D \subset \mathbb{R}^d$ (a sufficiently narrow bounded cone) such that φ is not concave on D. The justification of this statement is in Section 7. In particular, this implies that the assertion of Theorem 1.1 is not true for the problem (3-4) for $\alpha \in (1, 2)$.

For general $\alpha \in (0, 2)$ and $d \geq 2$ we have the following regularity result.

Theorem 1.5. *Let $\alpha \in (0, 2)$, $d \geq 2$ and let φ be a solution of (3-4). If $D \subset \mathbb{R}^d$ is a bounded convex domain then we have*

a) for any $x_0 \in \partial D$, $x \in D$, $\lambda \in (0, 1)$

$$\varphi(\lambda x + (1 - \lambda)x_0) \geq \lambda^\alpha \varphi(x),$$

b) for any $x, y \in D$, $\lambda \in (0, 1)$

$$\varphi(\lambda x + (1 - \lambda)y) \geq \max\left(\lambda^\alpha \varphi(x), \, (1 - \lambda)^\alpha \varphi(y)\right).$$

The proof of this theorem is in Section 7. It is based on one tricky observation and is much easier than the proof of Theorem 1.1. Clearly, Theorem 1.5 does not imply p-concavity of φ for any $p \in [-\infty, 1]$. Some conjectures concerning p-concavity of solutions of (3-4) are presented in Section 7.

Below we present the idea of the proof of Theorem 1.1. The proof is in the spirit of papers by L. Caffarelli, A. Friedman [9] and N. Korevaar, J. Lewis [26] in which they study geometric properties of solutions of some PDEs using the constant rank theorem and the method of continuity. In the proof of Theorem 1.1 the role of the constant rank theorem is played by the following result of Hans Lewy from 1968.

Theorem 1.6 (Hans Lewy, [31])**.** *Let $u(x_1, x_2, x_3)$ be real and harmonic in a domain Ω of $\mathbb{R}^3$ and let $H(u)$ denote the determinant of the Hessian matrix of u. Suppose $H(u)$ vanishes at a point $x_0 \in \Omega$ without vanishing identically in Ω. Then $H(u)$ assumes both positive and negative values near x_0.*

The use of this result is the key element of the proof of Theorem 1.1. S. Gleason and T. Wolff [20] generalized Theorem 1.6 to higher dimensions. Their result gives some hope that it is also possible to extend Theorem 1.1 to higher dimensions, see Conjecture 7.1 in Section 7.

Let us come back to presenting the idea of the proof of Theorem 1.1. We show the theorem for a sufficiently smooth bounded convex domain $D \subset B(0,1) \subset \mathbb{R}^2$, which boundary has a strictly positive curvature (the result for an arbitrary bounded convex domain follows by an approximation argument and scaling). Let us consider the harmonic extension u of φ. Namely, let

$$K(x) = C_K \frac{x_3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}, \qquad x \in \mathbb{R}^3_+, \tag{5}$$

where $C_K = 1/(2\pi)$, $\mathbb{R}^3_+ = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$. Put $u(x_1, x_2, 0) = \varphi(x_1, x_2)$, $(x_1, x_2) \in \mathbb{R}^2$ and

$$u(x_1, x_2, x_3) = \int_D K(x_1 - y_1, x_2 - y_2, x_3)\varphi(y_1, y_2)\, dy_1\, dy_2, \qquad (x_1, x_2, x_3) \in \mathbb{R}^3_+. \tag{6}$$

Note that $K(x_1-y_1, x_2-y_2, x_3)$ is the Poisson kernel of $\mathbb{R}^3_+$ for points $x = (x_1, x_2, x_3) \in \mathbb{R}^3_+$ and $(y_1, y_2, 0) \in \partial\mathbb{R}^3_+$. By f_i we denote $\frac{\partial f}{\partial x_i}$, by f_{ij} we denote $\frac{\partial^2 f}{\partial x_i \partial x_j}$. It is well known that $u_3(x_1, x_2, 0) = -(-\Delta)^{1/2}\varphi(x_1, x_2)$, $(x_1, x_2) \in D$, so u satisfies

$$\begin{aligned} \Delta u(x) &= 0, \qquad x \in \mathbb{R}^3_+, & (7)\\ u_3(x) &= -1, \qquad x \in D \times \{0\}, & (8)\\ u(x) &= 0, \qquad x \in D^c \times \{0\}, & (9) \end{aligned}$$

where $\Delta u = u_{11} + u_{22} + u_{33}$.

The idea of studying equations for fractional Laplacians via harmonic extensions is well known. It was used for the first time by F. Spitzer in [35] and then by many other authors e.g. by S. A. Molchanov, E. Ostrovskii [34], R. D. DeBlassie [14], P. Mendez-Hernandez [33], R. Bañuelos, T. Kulczycki [2], A. El Hajj, H. Ibrahim, R. Monneau [16], L. Caffarelli, L. Silvestre [10].

In the next step of the proof we extend u to $\mathbb{R}^3_- = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < 0\}$ by putting

$$u(x_1, x_2, x_3) = u(x_1, x_2, -x_3) - 2x_3, \qquad (x_1, x_2, x_3) \in \mathbb{R}^3_-. \tag{10}$$

Note that u is continuous on $\mathbb{R}^3$ and for $(x_1, x_2) \in D$ it satisfies

$$\begin{aligned} u_{3^-}(x_1, x_2, 0) &= \lim_{h \to 0^-} \frac{u(x_1, x_2, h) - u(x_1, x_2, 0)}{h} \\ &= \lim_{h \to 0^-} \frac{u(x_1, x_2, -h) - 2h - u(x_1, x_2, 0)}{h} = -1. \end{aligned}$$

By standard arguments it follows that u is harmonic in $\mathbb{R}^3_+ \cup \mathbb{R}^3_- \cup (D \times \{0\}) = \mathbb{R}^3 \setminus (D^c \times \{0\})$.

Let $\text{Hess}(u)$ be the Hessian matrix of u and $H(u) = \det(\text{Hess}(u))$. The general strategy of the proof is as follows:

1. We show that $H(u)(x) > 0$ for every $x \in \mathbb{R}^3 \setminus (D^c \times \{0\})$.
2. We show that for $x = (x_1, x_2, 0) \in D \times \{0\}$ the Hessian matrix has the following form

$$\text{Hess}(u)(x) = \begin{pmatrix} u_{11}(x) & u_{12}(x) & 0 \\ u_{12}(x) & u_{22}(x) & 0 \\ 0 & 0 & u_{33}(x) \end{pmatrix} = \begin{pmatrix} \varphi_{11}(x_1, x_2) & \varphi_{12}(x_1, x_2) & 0 \\ \varphi_{12}(x_1, x_2) & \varphi_{22}(x_1, x_2) & 0 \\ 0 & 0 & u_{33}(x) \end{pmatrix}$$

and $u_{33}(x) > 0$.

Since $\Delta u(x) = 0$, the two assertions above immediately imply that $\varphi_{11}(x_1, x_2) < 0$, $\varphi_{22}(x_1, x_2) < 0$ for $(x_1, x_2) \in D$, so φ is strictly concave on D.

Almost entire proof is the justification of the first assertion. This is done by the continuity method i.e. by deforming the domain D to the unit disk $B(0,1)$. The continuity method requires the maximum principle for $H(u)$ (Lewy's theorem), estimates of u_{ij} near $\partial D \times \{0\}$ (see Sections 3, 4) and the result for the unit disk (see Section 5). Roughly speaking, estimates of u_{ij} justify that zeroes of $H(u)$ do not "emerge" from $\partial D \times \{0\}$ along the deformation. Lewy's theorem implies that zeroes of $H(u)$ may not appear in compact subdomains of $\mathbb{R}^3 \setminus (D^c \times \{0\})$ along the deformation.

Below, we briefly present main steps in the continuity method. It may be easily shown that $H(u)(x) \to 0$ when $x \to x_0 \in \text{int}(D^c) \times \{0\}$. This fact causes some technical difficulties in the proof. To deal with this problem we add an auxiliary harmonic function to u. Namely, for any $\varepsilon \geq 0$ we consider $v^{(\varepsilon,D)}(x) = u^{(D)}(x) + \varepsilon(-x_1^2/2 - x_2^2/2 + x_3^2)$ (where $u^{(D)}$ denotes u corresponding to D). We consider the family of domains $\{D(t)\}_{t \in [0,1]}$ such that $D(0) = D$, $D(1) = B(0,1)$, all $D(t)$ are smooth bounded convex domains which boundaries have strictly positive curvature and $\partial D(t) \to \partial D(s)$ when $t \to s$ in the appropriate sense. For large M we put (see Figure 8)

$$\Omega(M, D(t)) = \{x \in \mathbb{R}^3 : \, x_1^2 + x_2^2 < M^2, x_3 \in (-M, M)\} \setminus ((D(t))^c \times \{0\}).$$

We fix large M and a sufficiently small $\varepsilon > 0$ ($\varepsilon \in (0, C(M)]$). We define

$$T = \{t \in [0,1] : \, H(v^{(\varepsilon, D(t))})(x) > 0 \quad \text{for all} \quad x \in \Omega(M, D(t))\}.$$

Next, one can show that $1 \in T$ (the result for the unit disk). Then we prove that T is closed, which follows from Lewy's theorem applied to $v^{(\varepsilon,D(t))}$. Next, we prove that T is open (relatively in $[0,1]$), which follows from a fact that for any fixed large M and any fixed $\varepsilon \in (0, C(M)]$ and all $t \in [0,1]$ we have $H(v^{(\varepsilon,D(t))})(x) > c > 0$ near $\partial\Omega(M, D(t))$, where c does not depend on t (in the proof of this estimate the results from Section 4 are used). This implies that $T = [0,1]$. By taking $\varepsilon \to 0$ (and using again Lewy's theorem) we obtain that $H(u^{(D)})(x) > 0$ for $x \in \Omega(M, D)$. Passing $M \to \infty$ we obtain that $H(u^{(D)})(x) > 0$ for all $\mathbb{R}^3 \setminus (D^c \times \{0\})$.

The paper is organized as follows. In Section 2 we present notation and collect some known facts needed in the rest of the paper. Sections 3 and 4 are the most technical parts. In Section 3 we estimate $\varphi_{ij}^{(D)}$ near ∂D. This is done by using an explicit formula for the Poisson kernel $P_B(x,y)$ for a ball B corresponding to $(-\Delta)^{1/2}$. Note that due to the nonlocality of $(-\Delta)^{1/2}$ the corresponding harmonic measure $P_B(x,y)\, dy$ is concentrated not on ∂B but on B^c. The results for $\varphi_{ij}^{(D)}$ are obtained by estimating integrals involving the Poisson kernel and its derivatives over different subdomains of D. This method is very technical. Nevertheless this is a standard method for boundary value problems for fractional Laplacians used by many authors e.g. K. Bogdan, Z.-Q. Chen, R. Song. It seems that the reason the estimates of $\varphi_i^{(D)}$, $\varphi_{ij}^{(D)}$ are quite long and technical is just the nonlocality of the equation $(-\Delta)^{1/2}\varphi = 1$. The results from Section 3 are used only in Section 4, where estimates of $u_{ij}^{(D)}$ near $\partial D \times \{0\}$ are obtained. These estimates are also quite technical. The reason for this is that $u_{ij}^{(D)}$ are singular near $\partial D \times \{0\}$ and their behaviour is quite complicated. For example, in an appropriate coordinate system (see Figure 4) in a neighborhood of $0 \in \partial D \times \{0\}$ we have $u_{11}^{(D)}(x) \approx (\text{dist}(x, \partial D \times \{0\}))^{-3/2}$ for some points, $u_{11}^{(D)}(x)$ vanishes for some other points and $u_{11}^{(D)}(x) \approx -(\text{dist}(x, \partial D \times \{0\}))^{-3/2}$ for some other points. In order to control all 6 different $u_{ij}^{(D)}$ and ultimately control $H(v^{(\varepsilon,D)})$ we have to consider many cases. The results from Section 4 are used only in the proofs of Proposition 6.2 and Lemma 5.2. Let us point out, that the only aim of Section 3 and 4 is to obtain control of $H(v^{(\varepsilon,D)})$ and $H(u^{(D)})$ near $\partial D \times \{0\}$.

In Section 5 we prove that $H(u^{(B(0,1))})(x) > 0$ for $x \in \mathbb{R}^3 \setminus (B^c(0,1) \times \{0\})$. $u^{(B(0,1))}$ is given by an explixit formula but it seems hard to show $H(u^{(B(0,1))})(x) > 0$ using directly this explicit formula. Instead, the proof is based on an auxiliary function and Lewy's theorem.

The most important part of the paper is Section 6, which contains the proof of the main theorem. In particular, it contains the proof of positivity of $H(u^{(D)})$ via the continuity method, which was briefly described above. It is worth to emphasize that all the derivative estimates obtained in Sections 3 and 4 are used in Section 6 only in the proof of Proposition 6.2. The results from Section 5 are used only in the proof of Proposition 6.5. Corollary 6.6, in which estimates of $H(v^{(\varepsilon,D)})$ near $\partial\Omega(M,D)$ (see Figure 8) and $H(v^{(\varepsilon,B(0,1))})$ in $\Omega(M,B(0,1))$ are formulated, is a direct consequence of Propositions 6.2, 6.5. Let us point out that all the results from Sections 3, 4, 5 are invoked in the proof of the main theorem only through Corollary 6.6.

In Section 7 some extensions and conjectures are presented.

2. Preliminaries

For $x \in \mathbb{R}^d$ and $r > 0$ we let $B(x,r) = \{y \in \mathbb{R}^d : |y-x| < r\}$. By $a \wedge b$ we denote $\min(a,b)$ and by $a \vee b$ we denote $\max(a,b)$ for $a,b \in \mathbb{R}$. For $x \in \mathbb{R}^d$, $D \subset \mathbb{R}^d$ we put $\delta_D(x) = \text{dist}(x,\partial D)$. For any $\psi : \mathbb{R}^d \to \mathbb{R}$ we denote $\psi_i(x) = \frac{\partial \psi}{\partial x_i}(x)$, $\psi_{ij}(x) = \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x)$, $i,j \in \{1,\dots,d\}$. We put $\mathbb{R}^3_+ = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 > 0\}$, $\mathbb{R}^3_- = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 < 0\}$. The definition of a uniform exterior cone condition may be found e.g. in [19, page 195].

Let us define a subclass of bounded, convex $C^{2,1}$ domains in $\mathbb{R}^2$ with strictly positive curvature, which will be suitable for our purposes.

Definition 2.1. Let $C_1 > 0$, $R_1 > 0$, $\kappa_2 \ge \kappa_1 > 0$ and let us fix a Cartesian coordinate system CS in $\mathbb{R}^2$. We say that a domain $D \subset \mathbb{R}^2$ belongs to the class $F(C_1,R_1,\kappa_1,\kappa_2)$ when

1. D is convex. In CS coordinates we have
$$\{(y_1,y_2) : y_1^2 + y_2^2 < R_1^2\} \subset D \subset \{(y_1,y_2) : y_1^2 + y_2^2 < 1\}.$$
2. For any $x \in \partial D$ there exist a Cartesian coordinate system CS_x with origin at x obtained by translation and rotation of CS, there exist $R > 0$, $f : [-R,R] \to [0,\infty)$ (R, f depend on x), such that $f \in C^{2,1}[-R,R]$, $f(0) = 0$, $f'(0) = 0$ and in CS_x coordinates
$$\{(y_1,y_2) : y_2 \in [-R,R], y_1 \in (f(y_2),R]\} = D \cap \{(y_1,y_2) : y_1 \in [-R,R], y_2 \in [-R,R]\}.$$
3. For any $y \in \partial D$ we have
$$\kappa_1 \le \kappa(y) \le \kappa_2,$$
where $\kappa(y)$ denotes the curvature of ∂D at y.
4. For any $y,z \in \partial D$ we have
$$|\kappa(y) - \kappa(z)| \le C_1|y-z|.$$

For brevity, we will often use notation $\Lambda = \{C_1,R_1,\kappa_1,\kappa_2\}$ and write $D \in F(\Lambda)$.

Let $C_1 > 0$, $R_1 > 0$, $\kappa_2 \ge \kappa_1 > 0$ and put $\Lambda = \{C_1,R_1,\kappa_1,\kappa_2\}$. Let $D \in F(\Lambda)$. For any $y \in \partial D$ by $\vec{n}(y)$ we denote the normal inner unit vector at y and by $\vec{T}(y)$ we denote the tangent unit vector at y which agrees with negative (clockwise) orientation of ∂D. We put $e_1 = (1,0)$, $e_2 = (0,1)$.

It may be easily shown that there exists $\tilde{R} = \tilde{R}(\Lambda)$ such that for any $y \in D$, $\delta_D(y) \le \tilde{R}$ there exists a unique $y^* \in \partial D$ such that $|y - y^*| = \delta_D(y)$. For any $y \in D$ such that $\delta_D(y) \le \tilde{R}$ we define $\vec{n}(y) = \vec{n}(y^*)$, $\vec{T}(y) = \vec{T}(y^*)$. For any $\psi \in C^2(D)$, $y \in D$, $v_1(y), v_2(y) \in \mathbb{R}$ and $\vec{v}(y) = v_1(y)e_1 + v_2(y)e_2$ we put $\frac{\partial \psi}{\partial \vec{v}}(y) = v_1(y)\psi_1(y) + v_2(y)\psi_2(y)$, (recall that $\psi_i(y) =$

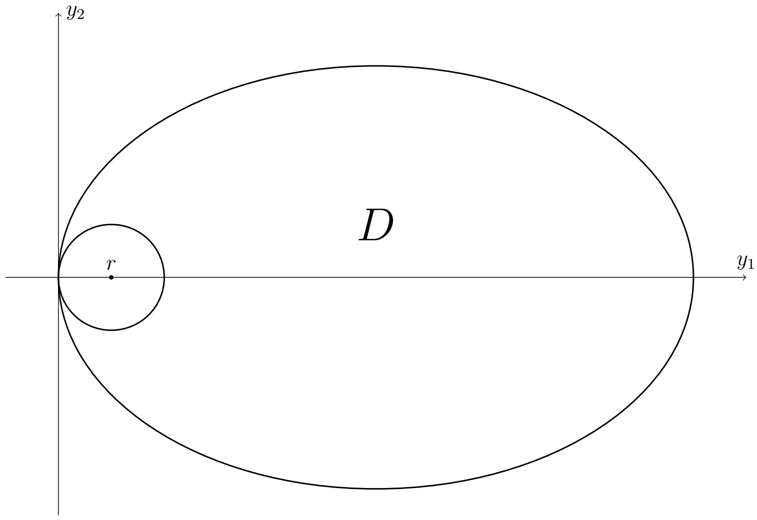


FIGURE 1

$\frac{\partial \psi}{\partial x_i}(y))$. Similarly, for any $w_1(y), w_2(y) \in \mathbb{R}$ and $\vec{w}(y) = w_1(y)e_1 + w_2(y)e_2$ we put $\frac{\partial^2 \psi}{\partial \vec{v} \partial \vec{w}}(y) = v_1(y)w_1(y)\psi_{11}(y) + v_2(y)w_2(y)\psi_{22}(y) + (v_1(y)w_2(y) + v_2(y)w_1(y))\psi_{12}(y)$.

Lemma 2.2. *Let $C_1 > 0$, $R_1 > 0$, $\kappa_2 \geq \kappa_1 > 0$ put $\Lambda = \{C_1, R_1, \kappa_1, \kappa_2\}$ and let us fix a Cartesian coordinate system CS in $\mathbb{R}^2$. Fix $D \in F(\Lambda)$ and $x_0 \in \partial D$. Choose a new Cartesian coordinate system CS_{x_0} with origin at x_0 obtained by translation and rotation of CS such that the positive coordinate halflines y_1, y_2 are in the directions $\vec{n}(x_0)$, $\vec{T}(x_0)$ respectively.*

From now on all points and vectors are in this new coordinate system CS_{x_0}, in particular $\vec{n}(0,0) = (1,0) = e_1$, $\vec{T}(0,0) = (0,1) = e_2$. For any $y \in \partial D$ define $\alpha(y) \in (-\pi, \pi]$ such that $\vec{T}(y) = \sin\alpha(y)e_1 + \cos\alpha(y)e_2$ (this is an angle between e_2 and $\vec{T}(y)$). For any $y \in D$ with $\delta_D(y) < \tilde{R}$ define $\alpha(y) = \alpha(y^*)$, where $y^* \in \partial D$ is a unique point such that $|y - y^*| = \delta_D(y)$.

There exists $r_0 = r_0(\Lambda) \leq \tilde{R} \wedge (1/2)$, $c_1 = c_1(\Lambda)$, $c_2 = c_2(\Lambda)$, $c_3 = c_3(\Lambda)$, $c_4 = c_4(\Lambda)$, $c_5 = c_5(\Lambda)$, $c_6 = c_6(\Lambda)$, $f : [-r_0, r_0] \to [0, \infty)$ such that $f \in C^{2,1}[-r_0, r_0]$, $f(0) = 0$, $f'(0) = 0$, $c_4 r_0 \leq 1/4$ and for any fixed $r \in (0, r_0]$ we have (see Figure 1)

1. $\{(y_1, y_2) : (y_1 - r)^2 + y_2^2 < r^2\} \subset D$,

$$W := \{(y_1, y_2) : y_2 \in [-r, r], y_1 \in (f(y_2), r]\} = D \cap \{(y_1, y_2) : y_1 \in [-r, r], y_2 \in [-r, r]\}.$$

2. For any $y \in W$ we have $\alpha(y) \in [-\pi/4, \pi/4]$ and

$$c_1|y_2| \leq |\sin\alpha(y)| \leq c_2|y_2|,$$

$$\vec{T}(y) = \sin\alpha(y)e_1 + \cos\alpha(y)e_2, \quad (11)$$

$$\vec{n}(y) = \cos\alpha(y)e_1 - \sin\alpha(y)e_2. \quad (12)$$

3. For any $y_2 \in [-r, r]$ we have

$$c_3 y_2^2 \leq f(y_2) \leq c_4 y_2^2.$$

4. For any $y \in W$ we have $e_1 = \cos\alpha(y)\vec{n}(y) + \sin\alpha(y)\vec{T}(y)$, $e_2 = -\sin\alpha(y)\vec{n}(y) + \cos\alpha(y)\vec{T}(y)$. For any $\psi \in C^2(D)$ and $y \in W$ we have

$$\frac{\partial \psi}{\partial \vec{T}}(y) = \sin\alpha(y)\psi_1(y) + \cos\alpha(y)\psi_2(y), \quad (13)$$

$$\frac{\partial \psi}{\partial \vec{n}}(y) = \cos\alpha(y)\psi_1(y) - \sin\alpha(y)\psi_2(y), \quad (14)$$

$$\begin{aligned}\psi_1(y) &= \cos\alpha(y)\frac{\partial\psi}{\partial\vec{n}}(y) + \sin\alpha(y)\frac{\partial\psi}{\partial\vec{T}}(y),\\ \psi_2(y) &= -\sin\alpha(y)\frac{\partial\psi}{\partial\vec{n}}(y) + \cos\alpha(y)\frac{\partial\psi}{\partial\vec{T}}(y),\end{aligned}$$

$$\begin{aligned}\psi_{11}(y) &= \cos^2\alpha(y)\frac{\partial^2\psi}{\partial\vec{n}^2}(y) + \sin^2\alpha(y)\frac{\partial^2\psi}{\partial\vec{T}^2}(y) + 2\sin\alpha(y)\cos\alpha(y)\frac{\partial^2\psi}{\partial\vec{n}\partial\vec{T}}(y),\\ \psi_{22}(y) &= \cos^2\alpha(y)\frac{\partial^2\psi}{\partial\vec{T}^2}(y) + \sin^2\alpha(y)\frac{\partial^2\psi}{\partial\vec{n}^2}(y) - 2\sin\alpha(y)\cos\alpha(y)\frac{\partial^2\psi}{\partial\vec{n}\partial\vec{T}}(y),\\ \psi_{12}(y) &= (\cos^2\alpha(y) - \sin^2\alpha(y))\frac{\partial^2\psi}{\partial\vec{n}\partial\vec{T}}(y) - \sin\alpha(y)\cos\alpha(y)\left(\frac{\partial^2\psi}{\partial\vec{n}^2}(y) - \frac{\partial^2\psi}{\partial\vec{T}^2}(y)\right).\end{aligned}$$

5. For any $y \in \{(y_1, y_2) \in W : y_2 > 0\}$ we have

$$c_5(f^{-1}(y_1) - y_2)f^{-1}(y_1) \le \delta_D(y) \le c_6(f^{-1}(y_1) - y_2)f^{-1}(y_1),$$

where $f^{-1} : [0, f(r)] \to [0, r]$.

This lemma follows by elementary geometry and its proof is omitted.

Lemma 2.3. *Let $C_1 > 0$, $R_1 > 0$, $\kappa_2 \ge \kappa_1 > 0$ and put $\Lambda = \{C_1, R_1, \kappa_1, \kappa_2\}$. There exists a constant $c = c(\Lambda)$ such that for any $D \in F(\Lambda)$ we have*

$$\int_D \delta_D^{-1/2}(x)\,dx \le c. \tag{15}$$

Proof. By Definition 2.1 we have $B(0, R_1) \subset D \subset B(0, 1)$. Let $x_0 \in \partial D$. By convexity of D the convex hull of $B(0, R_1) \cup \{x_0\}$ is a subset of $\overline{D}$. Using this fact and $D \subset B(0, 1)$ one may easily show that for every x in the line segment between 0 and x_0 we have $|x - x_0| \le c\delta_D(x)$, where c depends only on R_1. Hence $\delta_D^{-1/2}(x) \le c^{1/2}|x - x_0|^{-1/2}$. Now (15) easily follows by using polar coordinates with centre at 0. □

In the sequel we will use the method of continuity (cf. [26, page 20], [9]). Roughly speaking, we will deform a convex bounded domain D to a ball $B(0, 1)$. To do this we will consider the following construction. Let $C_1 > 0$, $R_1 > 0$, $\kappa_2 \ge \kappa_1 > 0$. For any $D \in F(C_1, R_1, \kappa_1, \kappa_2)$ and $t \in [0, 1]$ we define

$$D(t) = \{x : \exists y \in D, z \in B(0, 1) \text{ such that } x = (1 - t)y + tz\}. \tag{16}$$

Lemma 2.4. *For any $C_1 > 0$, $R_1 > 0$, $\kappa_2 \ge \kappa_1 > 0$ there exists $C_1' > 0$, $R_1' > 0$, $\kappa_2' \ge \kappa_1' > 0$ such that for any $D \in F(C_1, R_1, \kappa_1, \kappa_2)$ and any $t \in [0, 1]$ we have $D(t) \in F(C_1', R_1', \kappa_1', \kappa_2')$.*

This lemma seems to be standard, similar results are well known (cf. Appendix in D. Gilbarg and N. Trudinger's book [19], pages 381-384 or [9, proof of Theorem 3.1]). Therefore we omit its proof.

Now we state some properties of the solution of (1-2) and its harmonic extension which will be needed in the rest of the paper.

Let $D \subset \mathbb{R}^2$ be an open bounded set and $\varphi^{(D)}$ be the solution of (1-2) for D. Then the following scaling property is well known [4, (1.61)]:

$$\varphi^{(aD)}(ax) = a\varphi^{(D)}(x), \quad x \in D,\ a > 0. \tag{17}$$

For any open bounded sets $D_1, D_2 \subset \mathbb{R}^2$ put $d(D_1, D_2) = [\sup\{\operatorname{dist}(x, \partial D_2) : x \in \partial D_1\}] \vee [\sup\{\operatorname{dist}(x, \partial D_1) : x \in \partial D_2\}]$.

Lemma 2.5. *Let $\{D_n\}_{n=0}^\infty$ be a sequence of bounded convex domains in $\mathbb{R}^2$ and $\varphi^{(D_n)}$ be the solution of (1-2) for D_n. If $d(D_n, D_0) \to 0$ as $n \to \infty$ then for any $x \in D_0$ we have $\varphi^{(D_n)}(x) \to \varphi^{(D_0)}(x)$ as $n \to \infty$.*

This lemma seems to be well known and follows easily from (17) so we omit its proof (in fact it holds not only for convex domains but we need it only in this case).

Lemma 2.6. *Let* $C_1 > 0$, $R_1 > 0$, $\kappa_2 \geq \kappa_1 > 0$ *and put* $\Lambda = \{C_1, R_1, \kappa_1, \kappa_2\}$. *There exist a constant* $c_1 = c_1(\Lambda)$ *and an absolute constant* c_2 *such that for any* $D \in F(\Lambda)$ *we have*

$$\varphi(x) \leq \frac{2}{\pi}, \quad x \in D,$$

$$c_1 \delta_D^{1/2}(x) \leq \varphi(x) \leq c_2 \delta_D^{1/2}(x), \quad x \in D,$$

where φ *is the solution of (1-2) for* D.

Proof. We have $D \subset B(0,1)$ so for any $x \in D$ we get

$$\varphi(x) = E^x(\tau_D) \leq E^x(\tau_{B(0,1)}) = \frac{2}{\pi}(1 - |x|^2)^{1/2}.$$

Let $x \in D$ and $x^* \in \partial D$ be a point such that $|x - x^*| = \delta_D(x)$. Put $z = x^* - \vec{n}(x^*)$, where $\vec{n}(x^*)$ is the normal inner unit vector at x^* (clearly $|z - x^*| = 1$). By convexity of D we get $B(z,1) \subset D^c$. Put

$$U = \{y \in \mathbb{R}^2 : 1 < |y - z| < 3\}.$$

Since $D \subset B(0,1)$ we get $\mathrm{diam}(D) \leq 2$. Clearly, $x^* \in \partial D \cap \partial U$ which implies that $D \subset U$ and $\delta_D(x) = \delta_U(x)$. By [13] there exists an absolute constant c_2 such that

$$\varphi(x) = E^x(\tau_D) \leq E^x(\tau_U) \leq c_2 \delta_U^{1/2}(x) = c_2 \delta_D^{1/2}(x).$$

Now we will show the lower bound estimate of φ. Since $D \subset B(0,1)$ we have $\delta_D(x) \leq 1$. Let $x \in D$. If $\delta_D(x) \geq r_0$, where $r_0 = r_0(\Lambda)$ is the constant from Lemma 2.2 then

$$\varphi(x) = E^x(\tau_D) \geq E^x(\tau_{B(x,r_0)}) = \frac{2}{\pi} r_0 \geq \frac{2}{\pi} r_0 \delta_D^{1/2}(x).$$

If $\delta_D(x) < r_0$ then we may choose a coordinate system as in Lemma 2.2 (see Figure 1) and assume that $x = (x_1, 0)$, $\delta_D(x) = x_1$. Put $B = B((r_0,0), r_0)$. By Lemma 2.2 we have $B \subset D$. Clearly $x \in B$ and $\delta_D(x) = \delta_B(x) = x_1$. It follows that

$$\varphi(x) = E^x(\tau_D) \geq E^x(\tau_B) = \frac{2}{\pi} \left(r_0^2 - |(r_0,0) - (x_1,0)|^2 \right)^{1/2} \geq \frac{2}{\pi} r_0^{1/2} \delta_D^{1/2}(x).$$

□

Lemma 2.7. *Let* $C_1 > 0$, $R_1 > 0$, $\kappa_2 \geq \kappa_1 > 0$, $D \in F(C_1, R_1, \kappa_1, \kappa_2)$, φ *be the solution of (1-2) for* D *and* u *the harmonic extension of* φ *given by (6-10). For any* $(x_1, x_2, x_3) \in \mathbb{R}^3_+$ *we have* $H(u)(x_1, x_2, -x_3) = H(u)(x_1, x_2, x_3)$.

Proof. For $x = (x_1, x_2, x_3)$ put $\hat{x} = (x_1, x_2, -x_3)$. For $x \in \mathbb{R}^3_+$ we have $u_{ii}(\hat{x}) = u_{ii}(x)$ for $i = 1, 2, 3$, $u_{12}(\hat{x}) = u_{12}(x)$, $u_{13}(\hat{x}) = -u_{13}(x)$, $u_{23}(\hat{x}) = -u_{23}(x)$. Hence $H(u)(\hat{x}) = H(u)(x)$. □

We recall the definition of α-harmonic function, $\alpha \in (0,2)$. A Borel function h on $\mathbb{R}^d$ is said to be *α-harmonic* on open set $D \subset \mathbb{R}^d$ if for any $x_0 \in \mathbb{R}^d$, $r > 0$ such that $\overline{B(x_0, r)} \subset D$ we have

$$h(x) = \int_{B^c(x_0,r)} P_r(x - x_0, y - x_0) h(y) \, dy,$$

where the integral is absolutely convergent and $P_r(x,y)$ is the Poisson kernel for a ball $B(0,r)$ corresponding to $(-\Delta)^{\alpha/2}$. The explicit formula for the Poisson kernel is well known, see e.g. (1.57) in [4]. For $\alpha = 1$, $d = 2$ the Poisson kernel for $B(z,s)$ is given by (19). It is well known that h is α-harmonic on open set $D \subset \mathbb{R}^d$ if and only if h is C^2 on D and $(-\Delta)^{\alpha/2} h(x) = 0$ for any $x \in D$. A Borel function h on $\mathbb{R}^d$ is said to be *singular α-harmonic* on open set $D \subset \mathbb{R}^d$ if it is α-harmonic on D and $h \equiv 0$ on D^c.

We will need the following formulas of derivatives of $K(x) = C_K x_3 (x_1^2 + x_2^2 + x_3^2)^{-3/2}$:

$$\begin{aligned} K_1(x) &= -3C_K x_3 x_1 (x_1^2 + x_2^2 + x_3^2)^{-5/2}, \\ K_2(x) &= -3C_K x_3 x_2 (x_1^2 + x_2^2 + x_3^2)^{-5/2}, \\ K_3(x) &= C_K (x_1^2 + x_2^2 - 2x_3^2)(x_1^2 + x_2^2 + x_3^2)^{-5/2}. \end{aligned}$$

$$\begin{aligned} K_{11}(x) &= C_K x_3 (12x_1^2 - 3x_2^2 - 3x_3^2)(x_1^2 + x_2^2 + x_3^2)^{-7/2}, \\ K_{22}(x) &= C_K x_3 (12x_2^2 - 3x_1^2 - 3x_3^2)(x_1^2 + x_2^2 + x_3^2)^{-7/2}, \\ K_{33}(x) &= C_K x_3 (6x_3^2 - 9x_1^2 - 9x_2^2)(x_1^2 + x_2^2 + x_3^2)^{-7/2}. \\ K_{12}(x) &= 15C_K x_3 x_1 x_2 (x_1^2 + x_2^2 + x_3^2)^{-7/2}, \\ K_{13}(x) &= C_K x_1 (12x_3^2 - 3x_1^2 - 3x_2^2)(x_1^2 + x_2^2 + x_3^2)^{-7/2}, \\ K_{23}(x) &= C_K x_2 (12x_3^2 - 3x_1^2 - 3x_2^2)(x_1^2 + x_2^2 + x_3^2)^{-7/2}. \end{aligned}$$

Remark 2.8. All constants appearing in this paper are positive and finite. We write $C = C(a, \dots, z)$ to emphasize that C depends only on $a, \dots, z$. We adopt the convention that constants denoted by c (or c_1, c_2, etc.) may change their value from one use to the next.

Remark 2.9. In Sections 3, 4 and in the proof of Proposition 6.2 we use the following convention. Constants denoted by c (or c_1, c_2, etc.) depend on $\Lambda = \{C_1, R_1, \kappa_1, \kappa_2\}$, where $\Lambda = \{C_1, R_1, \kappa_1, \kappa_2\}$ appears in Definition 2.1. We write $f(x) \approx g(x)$ for $x \in A \subset \mathbb{R}^2$ to indicate that there exist constants $c_1 = c_1(\Lambda)$, $c_2 = c_2(\Lambda)$ such that for any $x \in A$ we have $c_1 g(x) \le f(x) \le c_2 g(x)$ (in particular, it may happen that both f, g are positive on A or both f, g are negative on A).

3. Estimates of derivatives of φ near ∂D

In this section we obtain estimates of φ_i, φ_{ij} near ∂D. These results are used in this paper only in Section 4, where the behaviour of u_{ij} near $\partial D \times \{0\}$ is studied. To obtain estimates of φ_i, φ_{ij} we use the well known representation (18) formulated below. This formula involves the Poisson kernel $P(x, y)$ for a ball corresponding to $(-\Delta)^{1/2}$. Let us recall that due to nonlocality of this operator the support of the corresponding harmonic measure $P(x, y)\, dy$ for a ball B is equal to B^c. This makes proofs in this section quite long and complicated because we have to obtain estimates of integrals involving the Poisson kernel and its derivatives over different subdomains of D. Most of the technics used in this section are similar to the standard methods used in papers by Z.-Q. Chen, R. Song [12], T. Kulczycki [28] and K. Bogdan, T. Kulczycki, A. Nowak [6]. These methods were used in estimates of the Green function corresponding to $(-\Delta)^{\alpha/2}$, $\alpha \in (0, 2)$ on smooth domains [12], [28] and in estimates of gradient of α-harmonic functions [6].

It should be mentioned that similar estimates of derivatives of α-harmonic functions were simultaneously obtained by the author's student G. Żurek in his Master Thesis [36].

The most difficult part of this section is the proof of Lemma 3.7. In this lemma estimates of $\varphi_{22}(x_1, 0)$ are obtained (y_2 axis is tangent to the boundary of D at $(0, 0) \in \partial D$, see Figure 3). To the best of author knowledge the idea of the proof is new. Roughly speaking, the proof is based on the representation

$$\varphi_{22}(x_1, 0) = \int_{D \setminus B} P_2((x_1, 0), y) \varphi_2(y)\, dy$$

and the precise control of derivatives of φ in normal and tangent directions in a small neighborhood of $(0, 0)$.

In the whole section we fix $C_1 > 0$, $R_1 > 0$, $\kappa_2 \ge \kappa_1 > 0$, $D \in F(C_1, R_1, \kappa_1, \kappa_2)$ and $x_0 \in \partial D$. We put $\Lambda = \{C_1, R_1, \kappa_1, \kappa_1\}$. φ is the solution of (1-2) for D. Unless it is stated otherwise we fix the coordinate system CS_{x_0} and notation as in Lemma 2.2 (see Figure 1). In particular x_0 is $(0,0)$ in CS_{x_0} coordinates.

Let $r \in (0, r_0]$, $z = (r, 0)$, $s \in (0, r]$, $B = B(z, s)$ (where r_0 is the constant from Lemma 2.2). It is well known (see e.g. [4, (1.50), (1.56), (1.57)]) that

$$\varphi(x) = h(x) + \int_{B^c} P(x,y)\varphi(y)\,dy, \quad x \in B, \tag{18}$$

where $h(x) = C_B(s^2 - |x-z|^2)^{1/2}$, $x \in B$,

$$P(x,y) = C_P \frac{(s^2 - |x-z|^2)^{1/2}}{(|y-z|^2 - s^2)^{1/2}|x-y|^2}, \quad x \in B,\ y \in (\overline{B})^c, \tag{19}$$

$C_B = 2/\pi$, $C_P = \pi^{-2}$.

We have $h_1(x) = C_B(r - x_1)(s^2 - |x-z|^2)^{-1/2}$, $x \in B$. Put $P_i(x,y) = \frac{\partial}{\partial x_i} P(x,y)$, $i = 1, 2$. For any $x \in B$, $y \in (\overline{B})^c$ we have $P_1(x,y) = A(x,y) + E(x,y)$ where

$$A(x,y) = -C_P \frac{(s^2 - |x-z|^2)^{-1/2}(x_1 - r)}{(|y-z|^2 - s^2)^{1/2}|x-y|^2}, \tag{20}$$

$$E(x,y) = -2C_P \frac{(s^2 - |x-z|^2)^{1/2}(x_1 - y_1)}{(|y-z|^2 - s^2)^{1/2}|x-y|^4}. \tag{21}$$

In this section we use only these geometric properties of the domain D which are stated in Lemmas 2.2, 2.3 and additionally facts that $D \subset B(0,1)$ and D is convex. Let us recall that all constants in the assertions of Lemmas 2.2, 2.3 depend only on Λ. Hence all constants in estimates in this section depend also only on Λ. Let us recall that in the whole section we use convention stated in Remark 2.9.

Lemma 3.1. *There exists* $r_1 \in (0, r_0/4]$, $r_1 = r_1(\Lambda)$ *such that for any* $x_1 \in (0, r_1]$ *we have* $\varphi_1(x_1, 0) \approx x_1^{-1/2}$.

Proof. Put $r = r_0$. We will use (18) for $s = r$, in particular $B = B(z, r)$. Note that for $x = (x_1, 0)$ we have $r^2 - |x-z|^2 = x_1(r + |x_1 - r|) \le 2rx_1$. Put

$$k(x) = 1_B(x) \int_{B^c} P(x,y)\varphi(y)\,dy + 1_{B^c}(x)\varphi(x), \quad x \in \mathbb{R}^2.$$

We have $k(x) \ge 0$ on $\mathbb{R}^2$, by (18) $k(x) \le \varphi(x)$ on B and k is 1-harmonic on B. For definition and basic properties of α-harmonic functions see Section 2 and [4, pages 20-21, 61]. The fact that k is 1-harmonic follows from [4, page 61]. By [6, Lemma 3.2] (cf. also [30]) and Lemma 2.6

$$k_1(x_1, 0) \le 2\frac{k(x_1,0)}{x_1} \le 2\frac{k\varphi(x_1,0)}{x_1} \le c x_1^{-1/2}, \quad \text{for } x_1 \in (0, r].$$

By the formula for h_1 and the formula for $r^2 - |x-z|^2$ we get $h_1(x_1, 0) = C_B(r - x_1)(2r - x_1)^{-1/2} x_1^{-1/2} \le C_B r^{1/2} x_1^{-1/2}$. Hence $\varphi_1(x_1, 0) = h_1(x_1, 0) + k_1(x_1, 0) \le c x_1^{-1/2}$ for $x_1 \in (0, r/4]$.

What remains is to show that $\varphi_1(x_1, 0) \ge c x_1^{-1/2}$. For $x_1 \in (0, r]$ we have $\varphi_1(x_1, 0) = \int_{B^c} P_1((x_1, 0), y)\varphi(y)\,dy + h_1(x_1, 0)$. We will estimate $\int_{B^c} P_1 \varphi$.

Let $x_1 \in (0, f(r/2) \wedge f(-r/2)]$. By Lemma 2.2 we have $f(r/2) \le c_4(r/2)^2 \le r/16$ (because $c_4 r \le 1/4$), so $x_1 \in (0, r/16]$. Note that $f(r/2) \wedge f(-r/2) \ge c_3 r^2/4$, where c_3 and $r = r_0$ are constants from Lemma 2.2, $c_3 r^2/4$ depends only on Λ. Let $p_1 \in (0, r/2]$ be such that $f(p_1) = x_1$, $p_2 \in [-r/2, 0)$ be such that $f(p_2) = x_1$ (recall that f is defined in Lemma 2.2). By Lemma 2.2 $f(x_1) < c_4 x_1^2 \le (1/2)x_1$, $f(-x_1) \le (1/2)x_1$, so $p_1 > x_1$ and $|p_2| > x_1$. Let $f_1 : [-r, r] \to \mathbb{R}$ be defined by $f_1(y_2) = r - (r^2 - y_2^2)^{1/2}$. Put (see Figure 2)

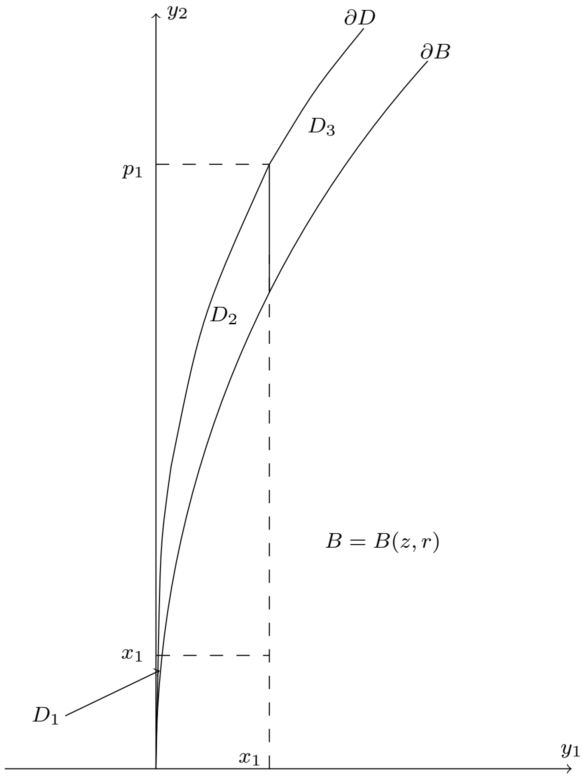


FIGURE 2

$$\begin{aligned} D_1 &= \{(y_1,y_2) : y_2 \in [-x_1,x_1], y_1 \in (f(y_2), f_1(y_2))\}, \\ D_2 &= \{(y_1,y_2) : y_2 \in (x_1,p_1] \cup [p_2,-x_1), y_1 \in (f(y_2), f_1(y_2) \wedge x_1)\}, \\ D_3 &= D \setminus (D_1 \cup D_2 \cup B). \end{aligned}$$

Note that $\int_{D\setminus B} A((x_1,0),y)\varphi(y)\,dy > 0$ and $\int_{D_3} E((x_1,0),y)\varphi(y)\,dy > 0$, because we have $A((x_1,0),y) > 0$ for $y \in D \setminus B$ and $E((x_1,0),y) > 0$ for $y \in D_3$.

Let us recall that we use (18) for $s = r$. We have $f_1(y_2) \le y_2^2/r = cy_2^2$. By Lemma 2.6 $\varphi(y) \le c\delta_D^{1/2}(y)$. For $y \in D_1 \cup D_2$ we also have $\delta_D(y) \le y_1 \le f_1(y_2) \le cy_2^2$. It follows that $\varphi(y) \le c|y_2|$ for $y \in D_1 \cup D_2$. Note that for $y \in D_1$ we have $|y_2| \le x_1$ so $\varphi(y) \le cx_1$. Note also that $|y-z|^2 - r^2 = (|y-z|+r)(|y-z|-r)$. This is bounded from above by $3r(f_1(y_2) - y_1)$ and from below by $r(f_1(y_2)-y_1)/2$. Hence for $y \in D_1 \cup D_2$ we have $|y-z|^2 - r^2 \approx f_1(y_2) - y_1$. For $y \in D_1$ we have

$$0 < y_1 \le f_1(x_1) = \frac{x_1^2}{r + (r^2 - x_1^2)^{1/2}} \le \frac{x_1^2}{r} \le \frac{x_1}{16},$$

because $x_1 \in (0, r/16]$. Hence for $y \in D_1$ we have $|x-y| \ge |x_1 - y_1| \ge 15x_1/16$ and $|x_1 - y_1| \le x_1$. It follows that

$$\begin{aligned}\left|\int_{D_1} E((x_1,0),y)\varphi(y)\,dy\right| &\leq cx_1^{-3/2}\int_{D_1}\frac{dy}{(|y-z|^2-r^2)^{1/2}}\\ &\approx x_1^{-3/2}\int_{-x_1}^{x_1} dy_2\int_{f(y_2)}^{f_1(y_2)}(f_1(y_2)-y_1)^{-1/2}\,dy_1\\ &= 2x_1^{-3/2}\int_{-x_1}^{x_1}(f_1(y_2)-f(y_2))^{1/2}\,dy_2\\ &\leq cx_1^{1/2}.\end{aligned}$$

For $y \in D_2$ we have $|x-y| = ((x_1-y_1)^2+y_2^2)^{1/2} \geq |y_2|$ and $|x_1-y_1| \leq |x_1|+|y_1| \leq 2x_1$. Note also that by Lemma 2.2 we have $p_1 \leq c\sqrt{x_1}\wedge(r/2)$, $|p_2| \leq c\sqrt{x_1}\wedge(r/2)$ so

$$\begin{aligned}\left|\int_{D_2} E((x_1,0),y)\varphi(y)\,dy\right| &\leq cx_1^{3/2}\int_{x_1}^{c\sqrt{x_1}\wedge(r/2)} dy_2 y_2^{-3}\int_{f(y_2)}^{f_1(y_2)\wedge x_1}(f_1(y_2)-y_1)^{-1/2}\,dy_1\\ &\leq cx_1^{3/2}\int_{x_1}^{c\sqrt{x_1}\wedge(r/2)} y_2^{-3}(f_1(y_2)-f(y_2))^{1/2}\,dy_2\\ &\leq cx_1^{1/2},\end{aligned}$$

(we omit here $\int_{p_2}^{-x_1}\ldots$ because it can be estimated in the same way).

We have

$$\varphi_1(x_1,0) = h_1(x_1,0)+\int_{D\setminus B} A\varphi+\int_{D_1} E\varphi+\int_{D_2} E\varphi+\int_{D_3} E\varphi.$$

By the formula for h_1 we easily get $h_1(x_1,0) \geq (2\sqrt{2})^{-1}C_B r^{1/2}x_1^{-1/2}$. It follows that

$$\varphi_1(x_1,0) \geq (2\sqrt{2})^{-1}C_B r^{1/2}x_1^{-1/2} - cx_1^{1/2} = x_1^{-1/2}\left((2\sqrt{2})^{-1}C_B r^{1/2}-cx_1\right).$$

Put $c_1 = (2\sqrt{2})^{-1}C_B r^{1/2}$. For sufficiently small x_1 we have $c_1 - cx_1 \geq c_1/2$ and $\varphi_1(x_1,0) \geq (c_1/2)x_1^{-1/2}$ (one can take $x_1 \leq r_1 := (c_1/(2c))\wedge(r/4)$). □

Lemma 3.2. *Put $r_1 = r_0/4$. For any $x_1 \in (0,r_1]$ we have $|\varphi_2(x_1,0)| \leq cx_1^{1/2}|\log x_1|$.*

Proof. Put $r = r_0$. We will use (18) for $s = r$, in particular $B = B(z,r)$. Let $x_1 \in (0,r/4]$. We have $\varphi_2(x_1,0) = \int_{B^c} P_2((x_1,0),y)\varphi(y)\,dy + h_2(x_1,0)$, $h_2(x_1,0) = 0$, $P_2((x_1,0),y) = 2C_P\frac{(r^2-|x-z|^2)^{1/2}y_2}{(|y-z|^2-r^2)^{1/2}|x-y|^4}$, $y \in (\overline{B})^c$. Let f_1 be such as in the proof of Lemma 3.1. Put

$$\begin{aligned} D_1 &= \{(y_1,y_2):\, y_2 \in [-x_1,x_1], y_1 \in (f(y_2),f_1(y_2))\},\\ D_2 &= \{(y_1,y_2):\, y_2 \in (x_1,r/2]\cup[-r/2,-x_1), y_1 \in (f(y_2),f_1(y_2))\},\\ D_3 &= D\setminus(D_1\cup D_2\cup B).\end{aligned}$$

By the same arguments as in the proof of Lemma 3.1 for $x = (x_1,0)$ we have $r^2-|x-z|^2 \leq 2rx_1$ and for $y \in D_1\cup D_2$ we have $|y-z|^2-r^2 \approx f_1(y_2)-y_1$. Note also that for $y \in D_1\cup D_2$ we have $\delta_D(y) \leq y_1 \leq f_1(y_2) \leq cy_2^2$ so (by Lemma 2.6) $\varphi(y) \leq c|y_2|$. For $y \in D_1$ we have $|y_2| \leq x_1$ so $\varphi(y) \leq cx_1$ and $|x-y| \geq 3x_1/4$. Hence

$$\left|\int_{D_1} P_2((x_1,0),y)\varphi(y)\,dy\right| \leq cx_1^{-3/2}\int_{D_1}\frac{dy}{(|y-z|^2-r^2)^{1/2}}.$$

By the same estimates as in the proof of Lemma 3.1 this is bounded by $cx_1^{1/2}$.

For $x = (x_1, 0)$ and $y \in D_2$ we have $|x - y| \geq y_2$ and $f_1(y_2) \leq c y_2^2$. It follows that

$$\begin{aligned} \left| \int_{D_2} P_2((x_1,0),y)\varphi(y)\,dy \right| &\leq c x_1^{1/2} \int_{x_1}^{r/2} dy_2 y_2^{-2} \int_{f(y_2)}^{f_1(y_2)} (f_1(y_2) - y_1)^{-1/2}\,dy_1 \\ &\leq c x_1^{1/2} \int_{x_1}^{r/2} y_2^{-2} (f_1(y_2) - f(y_2))^{1/2}\,dy_2 \\ &\leq c x_1^{1/2} |\log x_1|. \end{aligned}$$

For $x = (x_1, 0)$, $y \in D_3$ we have $|y - z|^2 - r^2 = (|y - z| + r)\delta_B(y) \geq r\delta_B(y)$ and $y_2/|x - y|^4 \leq |x - y|^{-3} \leq (r/2)^{-3}$. Put $B_1 = \{w \notin B : \delta_B(w) \leq 2\}$. Since $D \subset B(0,1)$ we have $D \setminus B \subset B_1$. Hence

$$\left| \int_{D_3} P_2((x_1,0),y)\varphi(y)\,dy \right| \leq c x_1^{1/2} \int_{B_1} \delta_B^{-1/2}(y)\,dy = c x_1^{1/2} \int_r^2 \frac{\rho}{(\rho - r)^{1/2}}\,d\rho = c x_1^{1/2}.$$

It follows that $|\varphi_2(x_1, 0)| \leq c x_1^{1/2} |\log x_1|$. □

In the following corollary we simply restate Lemmas 3.1 and 3.2 for an arbitrary point in $y \in D$ (with $\delta_D(y) \leq r_1$). Let us recall that $\vec{T}(y)$, $\vec{n}(y)$ are given by (11), (12) and $\frac{\partial\psi}{\partial\vec{T}}(y)$, $\frac{\partial\psi}{\partial\vec{n}}(y)$ are given by (13), (14).

By Lemmas 3.1, 3.2 and 2.2 we obtain

Corollary 3.3. *There exists $r_1 \in (0, r_0/4]$, $r_1 = r_1(\Lambda)$ such that for any $y \in D$, $\delta_D(y) \leq r_1$ we have*

$$\frac{\partial\varphi}{\partial\vec{n}}(y) \approx \delta_D^{-1/2}(y), \tag{22}$$

$$\left|\frac{\partial\varphi}{\partial\vec{T}}(y)\right| \leq c\delta_D^{1/2}(y)|\log\delta_D(y)|, \tag{23}$$

$$|\nabla\varphi(y)| \leq c\delta_D^{-1/2}(y). \tag{24}$$

Lemma 3.4. *For any $y \in D$ we have $|\nabla\varphi(y)| \leq c\delta_D^{-1/2}(y)$.*

Proof. Let $r_1 = r_1(\Lambda)$ be a constant from Corollary 3.3. If $y \in D$ satisfies $\delta_D(y) \leq r_1$ then the assertion follows from Corollary 3.3. Fix $y_0 \in D$ such that $\delta_D(y_0) > r_1$ and put $B = B(y_0, r_1)$. We are going to estimate $|\nabla\varphi(y_0)|$. For $y \in B$ we have $\varphi(y) = h(y) + k(y)$, where $h(y) = C_B(r_1^2 - |y - y_0|^2)^{1/2}$ and $k(y) = 1_B(y)\int_{D\setminus B} P(y - y_0, z - y_0)\varphi(z)\,dz + 1_{B^c}(y)\varphi(y)$, where P is given by (19) with $s = r_1$. Clearly $\nabla h(y_0) = 0$. k is a 1-harmonic function on B and $k(y) \leq \varphi(y) \leq 2/\pi$ (the last inequality follows from Lemma 2.6). By [6, Lemma 3.2] $|\nabla k(y_0)| \leq 2k(y_0)/r_1 \leq 4/(\pi r_1) \leq 4\delta_D^{-1/2}(y)/(\pi r_1)$. □

The definition of α-harmonic functions (see Section 2) on an open set $U \subset \mathbb{R}^d$ demands that the function is defined on the whole $\mathbb{R}^d$. φ_1, φ_2 are well defined on D and also on $D^c \setminus \partial D$. φ_1, φ_2 are not well defined on ∂D but ∂D has Lebesgue measure zero. One may formally defined $\varphi_1 = \varphi_2 = 0$ on ∂D. For the definition of singular α-harmonic functions, see Section 2.

Lemma 3.5. *φ_1, φ_2 are singular 1-harmonic on D.*

The proof of this lemma is omitted. By standard arguments (translation invariance and regularity of φ) it can be easily shown that $(-\Delta)^{1/2}\left(\frac{\partial\varphi}{\partial x_i}\right)(x) = \frac{\partial}{\partial x_i}\left((-\Delta)^{1/2}\varphi\right)(x) = 0$ for $x \in D$.

Remark 3.6. φ_{11}, φ_{22} are not 1-harmonic on D because they are not locally integrable on $\mathbb{R}^2$ (see Corollary 3.10).

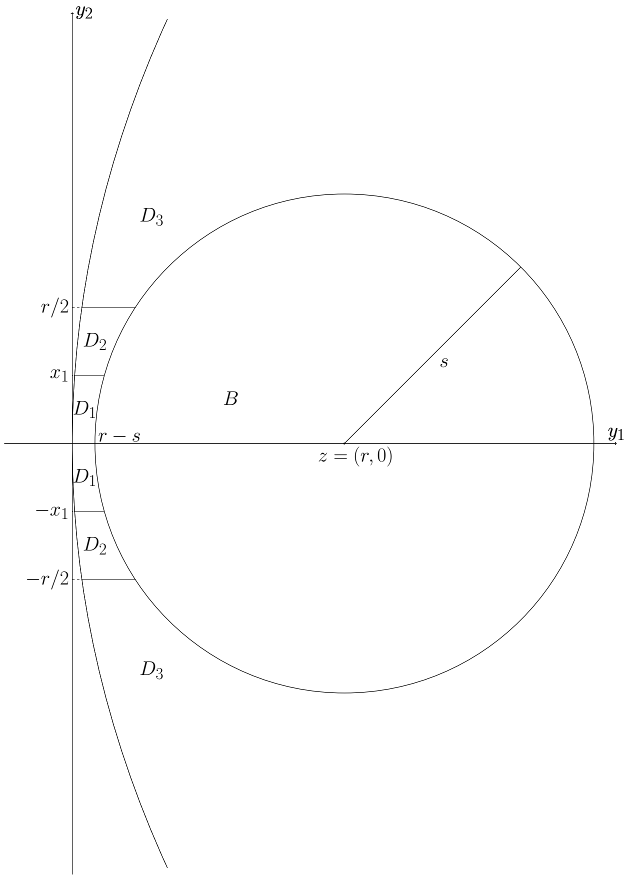


FIGURE 3

Lemma 3.7. *There exists* $r_2 \in (0, r_0/4]$, $r_2 = r_2(\Lambda)$ *such that for any* $x_1 \in (0, r_2]$ *we have* $\varphi_{22}(x_1, 0) \approx -x_1^{-1/2}$.

Proof. Put $r = r_0$. Let r_1 be the constant from Corollary 3.3. In this proof we take $s \in (r - (r_1/2)^2, r)$, i.e. $0 < r - s < (r_1/2)^2$. Recall that $z = (r, 0)$, $B = B(z, s)$ and P is given by (19). For any $x_1 \in (r - s, r]$ by Lemma 3.5 we have $\varphi_2(x_1, 0) = \int_{D \setminus B} P((x_1, 0), y)\varphi_2(y)\, dy$. It follows that $\varphi_{22}(x_1, 0) = \int_{D \setminus B} P_2((x_1, 0), y)\varphi_2(y)\, dy$. We have $P_2((x_1, 0), y) = 2C_P \frac{(s^2 - |x-z|^2)^{1/2} y_2}{(|y-z|^2 - s^2)^{1/2} |x-y|^4}$. Take $x_1 = \sqrt{r - s}$ (we have $\sqrt{r-s} < r_1/2$). Let $f_1 : [-s, s] \to \mathbb{R}$ be defined by $f_1(y_2) = r - \sqrt{s^2 - y_2^2}$. Put (see Figure 3)

$$\begin{aligned} D_1 &= \{(y_1, y_2) : \, y_2 \in [-x_1, x_1], y_1 \in (f(y_2), f_1(y_2))\}, \\ D_2 &= \{(y_1, y_2) : \, y_2 \in (x_1, r_1/2] \cup [-r_1/2, -x_1), y_1 \in (f(y_2), f_1(y_2))\}, \\ D_3 &= D \setminus (D_1 \cup D_2 \cup B). \end{aligned}$$

By Lemma 2.2 we have for $y \in D_1 \cup D_2$

$$\varphi_2(y) = \cos\alpha(y) \frac{\partial \varphi}{\partial \vec{T}}(y) - \sin\alpha(y) \frac{\partial \varphi}{\partial \vec{n}}(y).$$

Note that by definition of s we have $\delta_D(y) < r_1$ for $y \in D_1 \cup D_2$. By Corollary 3.3 we get for $y \in D_1 \cup D_2$

$$\left| \frac{\partial \varphi}{\partial \vec{T}}(y) \right| \le c(y_1 - f(y_2))^{1/2} |\log(y_1 - f(y_2))|$$

and

$$\frac{\partial \varphi}{\partial \vec{n}}(y) \approx (y_1 - f(y_2))^{-1/2}.$$

Hence

$$\left|\cos\alpha(y)\frac{\partial\varphi}{\partial\vec{T}}(y)\right| \le c(y_1 - f(y_2))^{1/2}|\log(y_1 - f(y_2))|$$

and

$$-\sin\alpha(y)\frac{\partial\varphi}{\partial\vec{n}}(y) \approx -y_2(y_1 - f(y_2))^{-1/2}.$$

Note also that for $y \in D_1 \cup D_2$ we have $(|y-z|^2 - s^2)^{1/2} \approx (-y_1 + f_1(y_2))^{1/2}$. Recall that we have chosen $x_1 = \sqrt{r-s}$. It follows that

$$\begin{aligned} & -\int_{D_1} P_2((x_1,0),y)\sin\alpha(y)\frac{\partial\varphi}{\partial\vec{n}}(y)\,dy \\ \approx\ & -x_1^{-7/2}\int_{-x_1}^{x_1} dy_2 y_2^2 \int_{f(y_2)}^{f_1(y_2)} dy_1(-y_1+f_1(y_2))^{-1/2}(y_1-f(y_2))^{-1/2} \approx -x_1^{1/2}, \end{aligned}$$

because $\int_a^b (x-a)^{-1/2}(b-x)^{-1/2}\,dx = \text{const}$.

Similarly,

$$\begin{aligned} & -\int_{D_2} P_2((x_1,0),y)\sin\alpha(y)\frac{\partial\varphi}{\partial\vec{n}}(y)\,dy \\ \approx\ & -x_1^{1/2}\int_{x_1}^{r_1/2} dy_2 y_2^{-2} \int_{f(y_2)}^{f_1(y_2)} dy_1(-y_1+f_1(y_2))^{-1/2}(y_1-f(y_2))^{-1/2} \approx -x_1^{1/2}. \end{aligned}$$

On the other hand we have

$$\begin{aligned} & \left|\int_{D_1} P_2((x_1,0),y)\cos\alpha(y)\frac{\partial\varphi}{\partial\vec{T}}(y)\,dy\right| \\ \le\ & cx_1^{-7/2}\int_{-x_1}^{x_1} dy_2 y_2 \int_{f(y_2)}^{f_1(y_2)} dy_1(-y_1+f_1(y_2))^{-1/2}(y_1-f(y_2))^{1/2}|\log(y_1-f(y_2))| \\ \le\ & cx_1^{1/2}|\log x_1|, \end{aligned}$$

$$\begin{aligned} & \left|\int_{D_2} P_2((x_1,0),y)\cos\alpha(y)\frac{\partial\varphi}{\partial\vec{T}}(y)\,dy\right| \\ \le\ & cx_1^{1/2}\int_{x_1}^{r_1/2} dy_2 y_2^{-3} \int_{f(y_2)}^{f_1(y_2)} dy_1(-y_1+f_1(y_2))^{-1/2}(y_1-f(y_2))^{1/2}|\log(y_1-f(y_2))| \\ \le\ & cx_1^{1/2}|\log x_1|^2. \end{aligned}$$

By Lemmas 2.3, 3.4 we obtain

$$\left|\int_{D_3} P_2((x_1,0),y)\varphi_2(y)\,dy\right| \le cx_1^{1/2}\int_{D_3}\delta_B^{-1/2}(y)\delta_D^{-1/2}(y)\,dy \le cx_1^{1/2}.$$

It follows that

$$-c_1x_1^{-1/2} - c_2x_1^{1/2}|\log x_1|^2 \le \varphi_{22}(x_1,0) \le -c_3x_1^{-1/2} + c_4x_1^{1/2}|\log x_1|^2,$$

where $x_1 = \sqrt{r-s}$. It is very important that c_1, c_2, c_3, c_4 do not depend on s. Hence there exists $r_2 \in (0, r/4]$, $r_2 = r_2(\Lambda)$ such that for any $x_1 \in (0, r_2]$ we have $\varphi_{22}(x_1,0) \approx -x_1^{-1/2}$. □

Lemma 3.8. *There exists $r_2 \in (0, r_0/4]$, $r_2 = r_2(\Lambda)$ such that for any $x_1 \in (0, r_2]$ we have $\varphi_{11}(x_1,0) \approx -x_1^{-3/2}$.*

Proof. First we show that $|\varphi_{11}(x_1,0)| \le c x_1^{-3/2}$, $x_1 \in (0, r_2]$. We will use similar notation as in Lemma 3.7. Put $r = r_0$. Let r_1 be the constant from Corollary 3.3. We take $s \in (r - (r_1/2)^2, r)$, $z = (r, 0)$, $B = B(z,s)$ and P is given by (19). For any $x_1 \in (r - s, r]$ by Lemma 3.5 we have $\varphi_1(x_1,0) = \int_{D \setminus B} P((x_1,0),y)\varphi_1(y)\,dy$. It follows that

$$\begin{aligned}\varphi_{11}(x_1,0) &= \int_{D\setminus B} P_1((x_1,0),y)\varphi_1(y)\,dy \\ &= \int_{D\setminus B} A((x_1,0),y)\varphi_1(y)\,dy + \int_{D\setminus B} E((x_1,0),y)\varphi_1(y)\,dy,\end{aligned}$$

where A, E are given by (20), (21).

Take $x_1 = \sqrt{r-s}$ (we have $\sqrt{r-s} < r_1/2 \le r/8$). By (24) $|\varphi_1(y)| \le c\delta_D^{-1/2}(y)$, $y \in D$. We have

$$\int_{D\setminus B} A((x_1,0),y)\varphi_1(y)\,dy = \frac{r - x_1}{s^2 - (x_1 - r)^2} \int_{D\setminus B} P((x_1,0),y)\varphi_1(y)\,dy,$$

$$\left| \int_{D\setminus B} P((x_1,0),y)\varphi_1(y)\,dy \right| = |\varphi_1(x_1,0)| \le c x_1^{-1/2}$$

and $\frac{r-x_1}{s^2-(x_1-r)^2} \approx x_1^{-1}$ so

$$\left| \int_{D\setminus B} A((x_1,0),y)\varphi_1(y)\,dy \right| \le c x_1^{-3/2}$$

for $x_1 = \sqrt{r-s}$.

Let f_1, D_1, D_2, D_3 be such as in the proof of Lemma 3.7. Using $|\varphi_1(y)| \le c\delta_D^{-1/2}(y)$ and similar arguments as in the proof of Lemma 3.7 we get the following estimates

$$\begin{aligned}&\left| \int_{D_1} E((x_1,0),y)\varphi_1(y)\,dy \right| \\ \le\ & c x_1^{-5/2} \int_{-x_1}^{x_1} dy_2 \int_{f(y_2)}^{f_1(y_2)} dy_1 (-y_1 + f_1(y_2))^{-1/2} (y_1 - f(y_2))^{-1/2} \le c x_1^{-3/2},\end{aligned} \tag{25}$$

$$\begin{aligned}&\left| \int_{D_2} E((x_1,0),y)\varphi_1(y)\,dy \right| \\ \le\ & c x_1^{1/2} \int_{x_1}^{r_1/2} dy_2\, y_2^{-4} \int_{f(y_2)}^{f_1(y_2)} dy_1 (-y_1 + f_1(y_2))^{-1/2} (y_1 - f(y_2))^{-1/2} (x_1 + y_1) \\ \le\ & c x_1^{-3/2},\end{aligned} \tag{26}$$

(here we used the estimate $y_1 \le c y_2^2$). By Lemmas 2.3, 3.4 we obtain

$$\left| \int_{D_3} E((x_1,0),y)\varphi_1(y)\,dy \right| \le c x_1^{1/2} \int_{D_3} \delta_B^{-1/2}(y)\delta_D^{-1/2}(y)\,dy \le c x_1^{1/2}.$$

It follows that $|\varphi_{11}(x_1,0)| \le c x_1^{-3/2}$, where c does not depend on s and $x_1 = \sqrt{r-s}$. Since $s \in (r - (r_1/2)^2, r)$ we get $|\varphi_{11}(x_1,0)| \le c x_1^{-3/2}$, $x_1 \in (0, r_1/2]$.

Now we will show that $\varphi_{11}(x_1,0) \le -c x_1^{-3/2}$ for $x_1 \in (0, r_2]$. Here we will use notation similar to the notation used in the proof of Lemma 3.1. We will use (18) for $s = r$, in particular $B = B(z, r)$. By (18) we get for $x_1 \in (0, r]$

$$\begin{aligned} \varphi_{11}(x_1, 0) &= h_{11}(x_1, 0) + \int_{D \setminus B} P_{11}((x_1, 0), y)\varphi(y)\, dy \\ &= h_{11}(x_1, 0) + \int_{D \setminus B} \frac{\partial A}{\partial x_1}((x_1, 0), y)\varphi(y)\, dy + \int_{D \setminus B} \frac{\partial E}{\partial x_1}((x_1, 0), y)\varphi(y)\, dy. \end{aligned}$$

One easily gets $h_{11}(x_1, 0) \approx -x_1^{-3/2}$ for $x_1 \in (0, r/4]$. For $x \in B$, $y \in (\overline{B})^c$ we have

$$\begin{aligned} \frac{\partial A}{\partial x_1}(x, y) &= \frac{-C_P(r^2 - |x - z|^2)^{-3/2}(x_1 - r)^2}{(|y - z|^2 - r^2)^{1/2}|x - y|^2} + \frac{-C_P(r^2 - |x - z|^2)^{-1/2}}{(|y - z|^2 - r^2)^{1/2}|x - y|^2} \\ &+ \frac{-2C_P(r^2 - |x - z|^2)^{-1/2}(r - x_1)(x_1 - y_1)}{(|y - z|^2 - r^2)^{1/2}|x - y|^4} \\ &= A^{(1)}(x, y) + A^{(2)}(x, y) + A^{(3)}(x, y), \end{aligned}$$

$$\begin{aligned} \frac{\partial E}{\partial x_1}(x, y) &= \frac{-2C_P(r^2 - |x - z|^2)^{-1/2}(r - x_1)(x_1 - y_1)}{(|y - z|^2 - r^2)^{1/2}|x - y|^4} + \frac{-2C_P(r^2 - |x - z|^2)^{1/2}}{(|y - z|^2 - r^2)^{1/2}|x - y|^4} \\ &+ \frac{8C_P(r^2 - |x - z|^2)^{1/2}(x_1 - y_1)^2}{(|y - z|^2 - r^2)^{1/2}|x - y|^6} \\ &= E^{(1)}(x, y) + E^{(2)}(x, y) + E^{(3)}(x, y). \end{aligned}$$

Let $x_1 \in (0, r/8]$, $y \in (\overline{B})^c$. We have $A^{(1)}(x, y) \le 0$, $A^{(2)}(x, y) \le 0$. We also have $A^{(3)}(x, y) \ge 0$ iff $y_1 \ge x_1$. Let f_1 be such as in the proof of Lemma 3.1. Let $p_1' > 0$ be such that $f_1(p_1') = x_1$, $p_2' < 0$ be such that $f_1(p_2') = x_1$ (we have $p_2' = -p_1'$). Note that $p_1' \approx \sqrt{x_1}$, $|p_2'| \approx \sqrt{x_1}$. Note also that $f_1(r/2) = r(1 - \sqrt{3}/2) > r/8$ and $f_1(p_1') = x_1 \le r/8$ so $p_1' < r/2$. Put

$$\begin{aligned} D_1' &= \{(y_1, y_2) : y_2 \in [p_2', p_1'], y_1 \in (f(y_2), f_1(y_2))\}, \\ D_2' &= \{(y_1, y_2) : y_2 \in (p_1', r/2] \cup [-r/2, p_2'), y_1 \in (f(y_2), f_1(y_2))\}, \\ D_3' &= D \setminus (D_1' \cup D_2' \cup B). \end{aligned}$$

We have $\int_{D_1'} A^{(3)}((x_1, 0), y)\varphi(y)\, dy \le 0$. Note that for $y \in D_2'$ we have $y_1 \le f_1(y_2) \le c y_2^2$, which gives $\varphi(y) \le c\delta_D^{1/2}(y) \le c(y_2^2)^{1/2} = c y_2$ by Lemma 2.6. Hence

$$\begin{aligned} \int_{D_2'} A^{(3)}((x_1, 0), y)\varphi(y)\, dy &\le c x_1^{-1/2} \int_{c\sqrt{x_1}}^{r/2} dy_2\, y_2^{-4} \int_{f(y_2)}^{f_1(y_2)} dy_1\, (y_1 - f_1(y_2))^{-1/2} y_1 \varphi(y) \\ &\le c x_1^{-1/2} \int_{c\sqrt{x_1}}^{r/2} dy_2 \le c x_1^{-1/2}, \end{aligned}$$

$$\left| \int_{D_3'} A^{(3)}((x_1, 0), y)\varphi(y)\, dy \right| \le c x_1^{-1/2} \int_{D_3'} \delta_B^{-1/2}(y)\, dy \le c x_1^{-1/2}.$$

Note that $E^{(1)}(x, y) = A^{(3)}(x, y)$ and $E^{(2)}(x, y) \le 0$. To estimate $\int_{D \setminus B} E^{(3)}\varphi$ we put

$$\begin{aligned} D_1'' &= \{(y_1, y_2) : y_2 \in [-x_1, x_1], y_1 \in (f(y_2), f_1(y_2))\}, \\ D_2'' &= \{(y_1, y_2) : y_2 \in (x_1, r/2] \cup [-r/2, -x_1), y_1 \in (f(y_2), f_1(y_2))\}, \\ D_3'' &= D \setminus (D_1'' \cup D_2'' \cup B). \end{aligned}$$

Note that for $y \in D_1''$ we have $(x_1 - y_1)^2 \le x_1^2$, which gives $\varphi(y) \le c\delta_D^{1/2}(y) \le cx_1$ by Lemma 2.6. Hence

$$\begin{aligned} \int_{D_1''} E^{(3)}((x_1,0),y)\varphi(y)\,dy &\le cx_1^{-7/2} \int_{-x_1}^{x_1} dy_2 \int_{f(y_2)}^{f_1(y_2)} dy_1\,(y_1 - f_1(y_2))^{-1/2}\varphi(y) \\ &\le cx_1^{-1/2}. \end{aligned}$$

Note that for $y \in D_2''$ we have $(x_1 - y_1)^2 \le x_1^2 + y_1^2 \le x_1^2 + cy_2^4$ and $\varphi(y) \le c\delta_D^{1/2}(y) \le cy_2$ so

$$\begin{aligned} &\int_{D_2''} E^{(3)}((x_1,0),y)\varphi(y)\,dy \\ \le\;& cx_1^{1/2} \int_{x_1}^{r/2} dy_2\, y_2^{-6}(x_1^2 + y_2^4) \int_{f(y_2)}^{f_1(y_2)} dy_1\,(y_1 - f_1(y_2))^{-1/2}\varphi(y) \\ \le\;& cx_1^{5/2} \int_{x_1}^{r/2} y_2^{-4}\,dy_2 + cx_1^{1/2} \int_{x_1}^{r/2} dy_2 \le cx_1^{-1/2}. \end{aligned}$$

We also have $\int_{D_3''} E^{(3)}((x_1,0),y)\varphi(y)\,dy \le cx_1^{1/2}$.

It follows that for sufficiently small x_1 we have $\varphi_{11}(x_1,0) \le -cx_1^{-3/2}$. □

Lemma 3.9. *There exists $r_2 \in (0, r_0/4]$, $r_2 = r_2(\Lambda)$ such that for any $x_1 \in (0, r_2]$ we have $|\varphi_{12}(x_1,0)| \le cx_1^{-1/2}|\log x_1|$.*

Proof. We will use similar notation as in Lemma 3.7. Put $r = r_0$. Let r_1 be the constant from Corollary 3.3. We take $s \in (r - (r_1/2)^2, r)$. Recall that $z = (r,0)$, $B = B(z,s)$ and P is given by (19). For any $x_1 \in (r - s, r]$ by Lemma 3.5 we have $\varphi_2(x_1,0) = \int_{D\setminus B} P((x_1,0),y)\varphi_2(y)\,dy$. It follows that

$$\begin{aligned} \varphi_{12}(x_1,0) &= \int_{D\setminus B} P_1((x_1,0),y)\varphi_2(y)\,dy \\ &= \int_{D\setminus B} A((x_1,0),y)\varphi_2(y)\,dy + \int_{D\setminus B} E((x_1,0),y)\varphi_2(y)\,dy. \end{aligned}$$

Take $x_1 = \sqrt{r-s}$ (we have $\sqrt{r-s} < r_1/2 \le r/8$). We have

$$\int_{D\setminus B} A((x_1,0),y)\varphi_2(y)\,dy = \frac{r - x_1}{(s^2 - (x_1 - r)^2)} \int_{D\setminus B} P((x_1,0),y)\varphi_2(y)\,dy.$$

By Lemma 3.2 we get

$$\left| \int_{D\setminus B} P((x_1,0),y)\varphi_2(y)\,dy \right| = |\varphi_2(x_1,0)| \le cx_1^{1/2}|\log x_1|.$$

Since $(r - x_1)(s^2 - (x_1 - r)^2)^{-1} \approx x_1^{-1}$ we obtain

$$\left| \int_{D\setminus B} A((x_1,0),y)\varphi_2(y)\,dy \right| \le cx_1^{-1/2}|\log x_1|,$$

for $x_1 = \sqrt{r-s}$.

Let f_1, D_1, D_2, D_3 be such as in the proof of Lemma 3.7. By Lemma 2.2 we have for $y \in D_1 \cup D_2$

$$\varphi_2(y) = \cos\alpha(y)\frac{\partial\varphi}{\partial\vec{T}}(y) - \sin\alpha(y)\frac{\partial\varphi}{\partial\vec{n}}(y).$$

By the arguments from the proof of Lemma 3.7 we have for $y \in D_1 \cup D_2$

$$\begin{aligned}\left|\cos\alpha(y)\frac{\partial\varphi}{\partial\vec{T}}(y)\right| &\le c(y_1 - f(y_2))^{1/2}|\log(y_1 - f(y_2))| \\ &\le c y_1^{1/2}|\log y_1|,\end{aligned}$$

$$\left|\sin\alpha(y)\frac{\partial\varphi}{\partial\vec{n}}(y)\right| \le c y_2 (y_1 - f(y_2))^{-1/2}.$$

Similarly like in the proofs of Lemmas 3.7 and 3.8 we obtain the following estimates

$$\begin{aligned}&\left|\int_{D_1} E((x_1,0),y)\cos\alpha(y)\frac{\partial\varphi}{\partial\vec{T}}(y)\,dy\right| \\ \le\ & c x_1^{-5/2}\int_{-x_1}^{x_1} dy_2 \int_{f(y_2)}^{f_1(y_2)} dy_1 (-y_1 + f_1(y_2))^{-1/2} y_1^{1/2}|\log y_1| \le c x_1^{1/2}|\log x_1|.\end{aligned}$$

Here we used the following facts $y_1^{1/2}|\log y_1| \le c y_2|\log y_2| \le c x_1|\log x_1|$, $\int_{f(y_2)}^{f_1(y_2)}(-y_1 + f_1(y_2))^{-1/2}\,dy_1 \le c f_1^{1/2}(y_2) \le c y_2 \le c x_1$.

Using similar arguments we get

$$\begin{aligned}&\left|\int_{D_2} E((x_1,0),y)\cos\alpha(y)\frac{\partial\varphi}{\partial\vec{T}}(y)\,dy\right| \\ \le\ & c x_1^{1/2}\int_{x_1}^{r/2} dy_2\, y_2^{-4}\int_{f(y_2)}^{f_1(y_2)} dy_1 (-y_1 + f_1(y_2))^{-1/2} y_1^{1/2}|\log y_1|(x_1 + y_1) \\ \le\ & c x_1^{1/2}|\log x_1|.\end{aligned}$$

By the same arguments as in (25), (26) one can easily obtain

$$\left|\int_{D_1} E((x_1,0),y) y_2 (y_1 - f(y_2))^{-1/2}\,dy\right| \le c x_1^{-1/2},$$

$$\left|\int_{D_2} E((x_1,0),y) y_2 (y_1 - f(y_2))^{-1/2}\,dy\right| \le c x_1^{-1/2} + c x_1^{1/2}|\log x_1|,$$

By Lemmas 2.3, 3.4 we obtain

$$\left|\int_{D_3} E((x_1,0),y)\varphi_2(y)\,dy\right| \le c x_1^{1/2}\int_{D_3}\delta_B^{-1/2}(y)\delta_D^{-1/2}(y)\,dy \le c x_1^{1/2}.$$

It follows that $|\varphi_{12}(x_1,0)| \le c x_1^{-1/2}|\log x_1|$, where c does not depend on s and $x_1 = \sqrt{r-s}$. Since $s \in (r - (r_1/2)^2, r)$ we get $|\varphi_{12}(x_1,0)| \le c x_1^{-1/2}|\log x_1|$, $x_1 \in (0, r_1/2]$. □

By Lemmas 2.2, 3.7, 3.8, 3.9 and Corollary 3.3 we obtain

Corollary 3.10. *There exists $r_2 \in (0, r_0/4]$, $r_2 = r_2(\Lambda)$ such that for any $y \in D$, $\delta_D(y) \le r_2$ we have (22), (23), (24) and*

$$\begin{aligned}\frac{\partial^2\varphi}{\partial\vec{n}^2}(y) &\approx -\delta_D^{-3/2}(y), \\ \frac{\partial^2\varphi}{\partial\vec{T}^2}(y) &\approx -\delta_D^{-1/2}(y), \\ \left|\frac{\partial^2\varphi}{\partial\vec{n}\partial\vec{T}}(y)\right| &\le c\delta_D^{-1/2}(y)|\log(\delta_D(y))|.\end{aligned}$$

Lemma 3.11. *There exists* $r_3 \in (0, r_0/4]$, $r_3 = r_3(\Lambda)$ *such that for any* $y = (y_1, y_2) \in B((r_3, 0), r_3)$ *we have*

$$|\varphi_2(y)| \leq c(y_1^{1/2}|\log y_1| + |y_2|y_1^{-1/2}), \tag{27}$$

$$|\varphi_{12}(y)| \leq c(y_1^{-1/2}|\log y_1| + |y_2|y_1^{-3/2}), \tag{28}$$

$$|\varphi_{22}(y)| \approx -y_1^{-1/2} \tag{29}$$

and for any $y = (y_1, y_2) \in W_{r_3}$ *we have*

$$\varphi_1(y) \approx \delta_D^{-1/2}(y), \tag{30}$$

where $W_{r_3} = \{(y_1, y_2) : y_2 \in [-r_3, r_3], y_1 \in (f(y_2), r_3]\}$.

Proof. We may assume that $y_2 > 0$. Let $r \in (0, r_2]$ where r_2 is the constant from Corollary 3.10 (recall that $r_2 \leq r_0/4$). Let $y = (y_1, y_2) \in B((r, 0), r)$ with $y_2 > 0$. By Lemma 2.2 we have $\sin\alpha(y) \approx y_2$, $\cos\alpha(y) \approx c$. We also have $\delta_D(y) \approx y_1$ and $y_2^2 \leq cy_1$.

By Corollary 3.10 we get $\frac{\partial\varphi}{\partial\vec{n}}(y) \approx -\delta_D^{-1/2}(y) \approx -y_1^{-1/2}$, $\left|\frac{\partial\varphi}{\partial\vec{T}}(y)\right| \leq c\delta_D^{1/2}(y)|\log(\delta_D(y))| \leq cy_1^{1/2}|\log y_1|$. Using this and the formula for φ_2 from Lemma 2.2 we get (27).

By Corollary 3.10 we have

$$\left|\frac{\partial^2\varphi}{\partial\vec{n}\partial\vec{T}}(y)\right| \leq c\delta_D^{-1/2}(y)|\log(\delta_D(y))| \leq cy_1^{-1/2}|\log y_1|,$$

$$\left|\frac{\partial^2\varphi}{\partial\vec{n}^2}(y) - \frac{\partial^2\varphi}{\partial\vec{T}^2}(y)\right| \leq c\delta_D^{-3/2}(y) \leq cy_1^{-3/2}.$$

Using this and the formula for φ_{12} from Lemma 2.2 we get (28).

By Corollary 3.10 we have $\frac{\partial^2\varphi}{\partial\vec{T}^2}(y) \approx -\delta_D^{-1/2}(y) \approx -y_1^{-1/2}$, $\frac{\partial^2\varphi}{\partial\vec{n}^2}(y) \approx -\delta_D^{-3/2}(y) \approx -y_1^{-3/2}$. $\sin^2\alpha(y) \approx y_2^2 \leq cy_1$,

$$\left|\sin\alpha(y)\cos\alpha(y)\frac{\partial^2\varphi}{\partial\vec{n}\partial\vec{T}}(y)\right| \leq cy_2y_1^{-1/2}|\log y_1| \leq c|\log y_1|.$$

Using this and the formula for φ_{22} from Lemma 2.2 we get (29) for sufficiently small r.

By (22), (23) and the formula for φ_1 from Lemma 2.2 we get (30) for sufficiently small r. □

We have $(-\Delta)^{1/2}\varphi(x) = 1$ for $x \in D$. We need to estimate $(-\Delta)^{1/2}\varphi(x)$ for $x \in (\overline{D})^c$. For such x we have $(-\Delta)^{1/2}\varphi(x) = -(2\pi)^{-1}\int_D \frac{\varphi(y)}{|y-x|^3}\,dy$.

Lemma 3.12. *Let* $x = (-x_1, 0)$, $x_1 > 0$. *We have*

$$\left|(-\Delta)^{1/2}\varphi(x)\right| \approx \delta_D^{-1/2}(x)(1 + |x|)^{-5/2}.$$

Proof. Put $r = r_0$. When $x_1 \in (-\infty, -r/2)$ we have

$$\int_D \frac{\varphi(y)}{|y-x|^3}\,dy \approx |x|^{-3} \approx \delta_D^{-1/2}(x)(1 + |x|)^{-5/2}.$$

When $x_1 \in [-r/2, 0)$ we obtain uisng Lemma 2.6

$$\begin{aligned}\int_D \frac{\varphi(y)}{|y-x|^3}\,dy &\approx \int_{D\cap B(0,\delta_D(x))} \delta_D^{-5/2}(x)\,dy + \int_{D\cap(B(0,r/2)\setminus B(0,\delta_D(x)))} |y|^{-5/2}\,dy \\ &+ \int_{D\cap B^c(0,r/2)} |y|^{-5/2}\,dy \approx \delta_D^{-1/2}(x).\end{aligned}$$

□

By Lemma 3.12 we obtain immediately

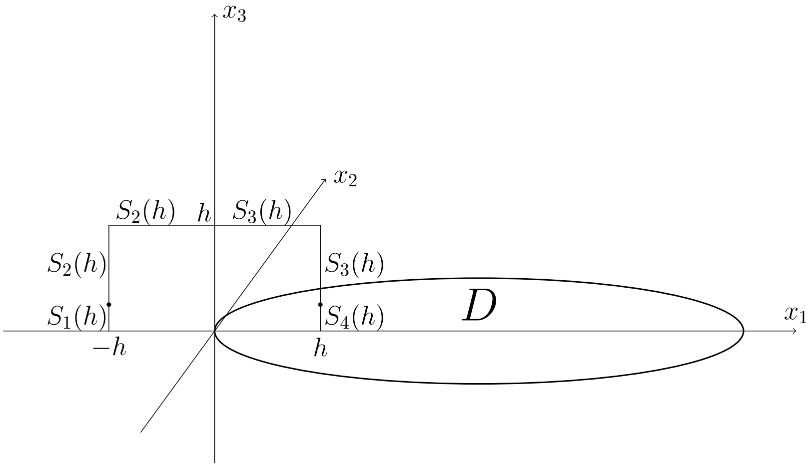


FIGURE 4

Corollary 3.13. *For any $x \in (\overline{D})^c$ we have*

$$\left|(-\Delta)^{1/2}\varphi(x)\right| \approx \delta_D^{-1/2}(x)(1+|x|)^{-5/2}.$$

4. Estimates of derivatives of u near $\partial D \times \{0\}$

In this section we study the behaviour of u_{ij} near $\partial D \times \{0\}$. The ultimate aim of these estimates is to control determinants of Hessian matrices of the function u and the function $v^{(\varepsilon,D)}$ (which is equal to u plus a small auxiliary harmonic function, for a precise definition see Section 6) near $\partial D \times \{0\}$. The estimates are quite long and technical because u_{ij} are singular near $\partial D \times \{0\}$ and their behaviour is quite complicated.

In the whole section we fix $C_1 > 0$, $R_1 > 0$, $\kappa_2 \geq \kappa_1 > 0$, $D \in F(C_1, R_1, \kappa_1, \kappa_2)$ and $x_0 \in \partial D$. We put $\Lambda = \{C_1, R_1, \kappa_1, \kappa_1\}$. φ is the solution of (1-2) for D and u is the harmonic extension of φ given by (6-10). Unless it is otherwise stated we fix a 2-dimensional coordinate system CS_{x_0} and notation as in Lemma 2.2 (see Figure 1). In particular x_0 is $(0,0)$ in CS_{x_0} coordinates. To study u we also use a 3-dimensional Cartesian coordinate system $0x_1x_2x_3$, see Figure 4, which is formed (roughly speaking) by adding $0x_3$ axis to the above 2-dimensional coordinate system. Let us recall that in the whole section we use convention stated in Remark 2.9.

Put $r = r_1 \wedge r_2 \wedge r_3 \wedge f(r_0/4) \wedge f(-r_0/4)$, where r_0, r_1, r_2, r_3 are the constants from Lemma 2.2, Corollary 3.3, Corollary 3.10 and Lemma 3.11. Note that $f(r_0/4) \wedge f(-r_0/4) \geq c_3 r_0^2/16$, where c_3 is a constant from Lemma 2.2, $c_3 r_0^2/16$ depends only on Λ. Let us define $f_1 : [-r, r] \to \mathbb{R}$ by $f_1(y_2) = r - \sqrt{r^2 - y_2^2}$ and $g_1 : [0, r] \to \mathbb{R}$ by $g_1(y_1) = \sqrt{r^2 - (y_1 - r)^2}$ (the graphs of f_1, g_1 are parts of the circle $\{(y_1, y_2) : (y_1 - r)^2 + y_2^2 = r^2\}$). For any $h \in (0, r]$ we put (see Figure 4):

$$\begin{aligned}
S_1(h) &= \{(x_1, x_2, x_3) : x_1 = -h, x_2 = 0, x_3 \in (0, h/4]\}, \\
S_2(h) &= \{(x_1, x_2, x_3) : x_1 = -h, x_2 = 0, x_3 \in (h/4, h]\} \\
&\quad \cup \{(x_1, x_2, x_3) : x_1 \in (-h, 0], x_2 = 0, x_3 = h\}, \\
S_3(h) &= \{(x_1, x_2, x_3) : x_1 \in (0, h], x_2 = 0, x_3 = h\} \\
&\quad \cup \{(x_1, x_2, x_3) : x_1 = h, x_2 = 0, x_3 \in (h/4, h]\}, \\
S_4(h) &= \{(x_1, x_2, x_3) : x_1 = h, x_2 = 0, x_3 \in (0, h/4]\}.
\end{aligned}$$

The main tool which we use in this section is the following formula

$$u(x) = \int_D K(x_1 - y_1, x_2 - y_2, x_3)\varphi(y_1, y_2)\, dy_1\, dy_2.$$

To obtain estimates of u_{ij} we differentiate under the integral sign in the above formula. The results concerning estimates of u_{ij} are divided into 6 propositions. In the proof of Proposition 4.1 we use the formula

$$u_{22}(x) = \int_D K_2(x_1 - y_1, x_2 - y_2, x_3)\varphi_2(y_1, y_2)\, dy_1\, dy_2,$$

(for brevity we simply write $u_{22} = \int_D K_2\varphi_2$), the estimates of $\frac{\partial\varphi}{\partial\vec{n}}$, $\frac{\partial\varphi}{\partial\vec{T}}$ from Corollary 3.3 and the estimate of $|\nabla\varphi|$ from Lemma 3.4. In this proof we use also the formula $\varphi_2(y_1, y_2) - \varphi_2(y_1, -y_2) = 2y_2\varphi_{22}(y_1, \xi)$ and the estimate of φ_{22} from Lemma 3.11. In the proof of Proposition 4.2 (which is the easiest result of this section) we use formulas $u_{11} = \int_D K_{11}\varphi$, $u_{13} = \int_D K_{13}\varphi$ and the estimate $\varphi(x) \le c\delta_D^{1/2}(x)$. In the proof of Proposition 4.3 we use formulas $u_{11} = \int_D K_1\varphi_1$, $u_{13} = \int_D K_3\varphi_1$, the estimate of φ_1 from Lemma 3.11 and the estimate of $|\nabla\varphi|$ from Lemma 3.4. The proof of Proposition 4.4 is based on a different idea than the proofs of previous propositions. Namely, we use the fact that $u_3(y_1, y_2, 0) = -(-\Delta)^{1/2}\varphi(y_1, y_2)$, for $(y_1, y_2) \notin \partial D$. We use also formulas $u_{13} = \int_{\mathbb{R}^2} K_1 u_3$, $u_{33} = \int_{\mathbb{R}^2} K_3 u_3$ and the estimate of $|(-\Delta)^{1/2}\varphi|$ from Corollary 3.13. In the proof of Proposition 4.5 we use formulas $u_{12} = \int_D K_{12}\varphi$, $u_{23} = \int_D K_{23}\varphi$, $\varphi(y_1, y_2) - \varphi(y_1, -y_2) = 2y_2\varphi_2(y_1, \xi)$, the estimate of $\varphi(x)$ from Lemma 2.6 and the estimate of φ_2 from Lemma 3.11. In the proof we use also the formula $\varphi(z_1 + h, z_2) - \varphi(-z_1 + h, z_2) - \varphi(z_1 + h, -z_2) + \varphi(-z_1 + h, -z_2) = 4z_1 z_2\varphi_{12}(\xi_1 + h, \xi_2)$ and the estimate of φ_{12} from Lemma 3.11. The most difficult result of this section is Proposition 4.6. In this proposition we study u_{23} on $S_4(h)$ using two different formulas: $u_{23} = \int_{\mathbb{R}^2} K_2 u_3$ and $u_{23} = \int_D K_{23}\varphi$. We use the estimate of $|(-\Delta)^{1/2}\varphi|$ from Corollary 3.13, estimates of φ_2, φ_{12}, φ_{22} from Lemma 3.11 and the estimate of $\varphi(x)$ from Lemma 2.6. In Lemma 4.7 we obtain results concerning $u_{i3}(x_1, x_2, 0)$ for $i = 1, 2, 3$ and $(x_1, x_2) \in D$.

In this section we use only these geometric properties of a domain D, which are stated in Lemmas 2.2, 2.3 (and additionally the fact that D is convex and $D \subset B(0, 1)$). Let us recall that all constants in Lemmas 2.2, 2.3 depend only on Λ. We use only these inequalities of φ, φ_i, φ_{ij} which are stated in Section 3 and in Lemma 2.6. The constants in these inequalities depend only on Λ. Therefore all constants in estimates of u_{ij} obtained in Section 4 depend only on Λ.

Proposition 4.1. *There exists $h_0 \in (0, r/8]$, $h_0 = h_0(\Lambda)$ such that for any $h \in (0, h_0]$ we have $u_{22}(x) \approx -x_3 h^{-3/2}$ for $x \in S_1(h) \cup S_2(h) \cup S_3(h)$, $u_{22}(x) \approx -h^{-1/2}$ for $x \in S_4(h)$.*

Proof. Let $h \in (0, r/8]$. We have

$$u_{22}(x) = \int_D K_2(x_1 - y_1, -y_2, x_3)\varphi_2(y_1, y_2)\, dy_1\, dy_2. \tag{31}$$

Put (see Figure 5)

$$\begin{aligned}
D_1 &= \{(y_1, y_2) : y_1 \in [f_1(h), h], y_2 \in [-g_1(y_1), g_1(y_1)]\},\\
D_2 &= \{(y_1, y_2) : y_1 \in (h, r], y_2 \in [-g_1(y_1), g_1(y_1)]\},\\
D_3 &= \{(y_1, y_2) : y_2 \in [-h, h], y_1 \in (f(y_2), f_1(h))\},\\
D_4 &= \{(y_1, y_2) : y_2 \in [-r/2, -h] \cup [h, r/2], y_1 \in (f(y_2), f_1(y_2))\},\\
D_5 &= D \setminus (D_1 \cup D_2 \cup D_3 \cup D_4).
\end{aligned}$$

For $i = 1, 2, 3, 4$ we also put $D_{i+} = \{(y_1, y_2) \in D_i : y_2 > 0\}$, $D_{i-} = \{(y_1, y_2) \in D_i : y_2 < 0\}$.

Note that $f_1(h) \le h^2/r \le h/4$.

We will estimate (31). The most important is $\int_{D_1 \cup D_2} K_2\varphi_2$. By Lemma 3.11 for $y \in D_{1+} \cup D_{2+}$ we have $\varphi_2(y_1, y_2) - \varphi_2(y_1, -y_2) = 2y_2\varphi_{22}(y_1, \xi) \approx -y_2 y_1^{-1/2}$, where $\xi \in$

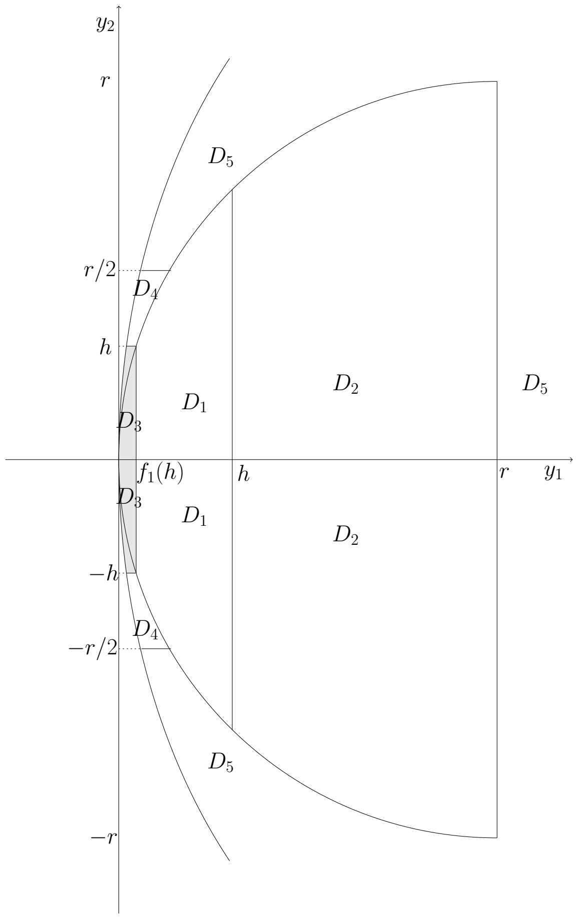


FIGURE 5

$(-y_2, y_2)$. It follows that

$$\begin{aligned}
&\int_{D_1\cup D_2} K_2(x_1 - y_1, -y_2, x_3)\varphi_2(y_1, y_2)\, dy_1\, dy_2\\
= \ & cx_3 \int_{D_{1+}\cup D_{2+}} \frac{y_2}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{5/2}} (\varphi_2(y_1, y_2) - \varphi_2(y_1, -y_2))\, dy_1\, dy_2\\
\approx \ & cx_3 \int_{D_{1+}\cup D_{2+}} \frac{-y_2^2 y_1^{-1/2}}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{5/2}}\, dy_1\, dy_2.
\end{aligned}$$

We have

$$\begin{aligned}
&\int_{D_{1+}} \frac{-y_2^2 y_1^{-1/2}}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{5/2}}\, dy_1\, dy_2\\
\approx \ & \frac{1}{h^5}\int_{f_1(h)}^{h} dy_1\, y_1^{-1/2} \int_0^h dy_2\, (-y_2^2) + \int_{f_1(h)}^{h} dy_1\, y_1^{-1/2} \int_h^{g_1(y_1)} dy_2\, \frac{-y_2^2}{y_2^5}.
\end{aligned}$$

We have $f_1(y_2) = y_2^2(r + (r^2 - y_2^2)^{1/2})^{-1}$ and $g_1(y_1) = y_1^{1/2}(2r - y_1)^{1/2}$, so $c_1 y_2^2 \le f_1(y_2) \le c_2 y_2^2$ and $c_3 y_1^{1/2} \le g_1(y_1) \le c_4 y_1^{1/2}$ and constants c_1, c_2, c_3, c_4 depend only on Λ. Hence the last expression is comparable to $-h^{-3/2}$ (with constants depending only on Λ).

By similar arguments we have

$$\begin{aligned}
&\int_{D_{2+}} \frac{-y_2^2 y_1^{-1/2}}{((x_1-y_1)^2+y_2^2+x_3^2)^{5/2}}\, dy_1\, dy_2 \\
\approx\ & \int_h^r dy_1 \int_0^{y_1} dy_2\, \frac{-y_2^2 y_1^{-1/2}}{y_1^5} + \int_h^r dy_1 \int_{y_1}^{g_1(y_1)} dy_2\, \frac{-y_2^2 y_1^{-1/2}}{y_2^5} \\
\approx\ & -h^{-3/2}.
\end{aligned}$$

It follows that $\int_{D_1 \cup D_2} K_2 \varphi_2 \approx -x_3 h^{-3/2}$.

Now we will estimate $\int_{D_3 \cup D_4} K_2 \varphi_2$. It is sufficient to estimate $\int_{D_{3+} \cup D_{4+}} K_2 \varphi_2$. The estimate $\int_{D_{3-} \cup D_{4-}} K_2 \varphi_2$ is the same. By Lemma 2.2 and Corollary 3.3 we get for $y \in D_{3+} \cup D_{4+}$

$$\begin{aligned}
|\varphi_2(y)| &= \left| \cos\alpha(y) \frac{\partial \varphi}{\partial \vec{T}}(y) - \sin\alpha(y) \frac{\partial \varphi}{\partial \vec{n}}(y) \right| \\
&\le c\delta_D^{1/2}(y) |\log \delta_D(y)| + c y_2 \delta_D^{-1/2}(y) \\
&\le c(f^{-1}(y_1) - y_2)^{1/2} (f^{-1}(y_1))^{1/2} |\log((f^{-1}(y_1) - y_2) f^{-1}(y_1))| \\
&+ c y_2 (f^{-1}(y_1) - y_2)^{-1/2} (f^{-1}(y_1))^{-1/2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left| \int_{D_{3+}} K_2(x_1 - y_1, -y_2, x_3) \varphi_2(y_1, y_2)\, dy_1\, dy_2 \right| \\
\le\ & \frac{c x_3}{h^5} \int_0^{f_1(h)} dy_1 \int_0^{f^{-1}(y_1)} dy_2 y_2 |\varphi_2(y_1, y_2)| \\
\le\ & \frac{c x_3}{h^5} \int_0^{f_1(h)} dy_1 \int_0^{f^{-1}(y_1)} dy_2 (f^{-1}(y_1) - y_2)^{1/2} (f^{-1}(y_1))^{1/2} \\
& \qquad \times |\log((f^{-1}(y_1) - y_2) f^{-1}(y_1))| y_2 \\
+\ & \frac{c x_3}{h^5} \int_0^{f_1(h)} dy_1 \int_0^{f^{-1}(y_1)} dy_2 (f^{-1}(y_1) - y_2)^{-1/2} (f^{-1}(y_1))^{-1/2} y_2^2.
\end{aligned}$$

By substituting $w = f^{-1}(y_1) - y_2$ and using $y_2 = f^{-1}(y_1) - w \le f^{-1}(y_1)$, $f^{-1}(y_1) \approx y_1^{1/2}$, $f_1(h) \le ch^2$ this is bounded from above by

$$\begin{aligned}
& \frac{c x_3}{h^5} \int_0^{f_1(h)} dy_1 \int_0^{f^{-1}(y_1)} dw w^{1/2} (f^{-1}(y_1))^{3/2} |\log(w f^{-1}(y_1))| \\
+\ & \frac{c x_3}{h^5} \int_0^{f_1(h)} dy_1 \int_0^{f^{-1}(y_1)} dw w^{-1/2} (f^{-1}(y_1))^{3/2} \\
\le\ & c x_3 |\log h| + c x_3 h^{-1}.
\end{aligned}$$

In the above estimate we used the inequality $f^{-1}(y_1) \le c y_1^{1/2}$. This follows from Lemma 2.2, property 3, so the constant c depends only on Λ.

In the same way we get

$$\begin{aligned}
&\left|\int_{D_{4+}} K_2(x_1-y_1,-y_2,x_3)\varphi_2(y_1,y_2)\,dy_1\,dy_2\right| \\
\le\ & cx_3\int_h^{r/2} dy_2 \int_{f(y_2)}^{f_1(y_2)} dy_1 \frac{y_2}{y_2^5}|\varphi_2(y_1,y_2)| \\
\le\ & cx_3\int_{f(h)}^{f_1(r/2)} dy_1 \int_{g_1(y_1)}^{f^{-1}(y_1)} dy_2 y_2^{-4}(f^{-1}(y_1)-y_2)^{1/2}(f^{-1}(y_1))^{1/2} \\
& \qquad\qquad \times|\log((f^{-1}(y_1)-y_2)f^{-1}(y_1))| \\
+\ & cx_3\int_{f(h)}^{f_1(r/2)} dy_1 \int_{g_1(y_1)}^{f^{-1}(y_1)} dy_2 y_2^{-3}(f^{-1}(y_1)-y_2)^{-1/2}(f^{-1}(y_1))^{-1/2}.
\end{aligned}$$

Similarly like in the estimate $\int_{D_{3+}} K_2\varphi_2$ using substitution $w = f^{-1}(y_1) - y_2$ we obtain that it is bounded from above by $cx_3|\log h|^2 + cx_3 h^{-1}$. By Lemma 3.4 we get

$$\left|\int_{D_5} K_2(x_1-y_1,-y_2,x_3)\varphi_2(y_1,y_2)\,dy_1\,dy_2\right| \le cx_3\int_{D_5} \delta_D^{-1/2}(y)\,dy.$$

By Lemma 2.3 this is bounded from above by cx_3. We finally obtained $\int_{D_1\cup D_2} K_2\varphi_2 \approx -x_3h^{-3/2}$ and $\left|\int_{D_3\cup D_4\cup d_5} K_2\varphi_2\right| \le cx_3h^{-1}$, where all constants depend only on Λ. It is clear that one can choose $h_0 = h_0(\Lambda)$ such that for any $h \in (0,h_0]$ we have $u_{22}(x) = \int_{D_1\cup\ldots\cup D_5} K_2\varphi_2 \approx -x_3h^{-3/2}$ for $x \in S_1(h)\cup S_2(h)\cup S_3(h)$.

Now we estimate $u_{22}(x)$ for $x \in S_4(h)$. Put $A = B((h,0),h/2)$, $A_+ = \{y \in A : y_2 > 0\}$, $A_{1+} = \{y \in B((h,0),x_3) : y_2 > 0\}$, $A_{2+} = A_+ \setminus A_{1+}$. By similar arguments as above we obtain $\int_{D\setminus A} K_2\varphi_2 \approx -x_3h^{-3/2}$ and for $y \in A$ we get $\varphi_2(y_1,y_2)-\varphi_2(y_1,-y_2) \approx -y_2y_1^{-1/2} \approx -y_2h^{-1/2}$. Note that for $x \in S_4(h)$ we have $x = (h,0,x_3)$, where $x_3 \in (0,h/4]$. It follows that

$$\begin{aligned}
&\int_A K_2(x_1-y_1,-y_2,x_3)\varphi_2(y_1,y_2)\,dy_1\,dy_2 \\
=\ & \int_{A_+} K_2(x_1-y_1,-y_2,x_3)(\varphi_2(y_1,y_2)-\varphi_2(y_1,-y_2))\,dy_1\,dy_2 \\
\approx\ & -x_3h^{-1/2}\int_{A_{1+}\cup A_{2+}} \frac{y_2^2}{((h-y_1)^2+y_2^2+x_3^2)^{5/2}}\,dy_1\,dy_2 \\
\approx\ & \frac{-h^{-1/2}}{x_3^4}\int_0^{x_3}\rho^3\,d\rho - x_3h^{-1/2}\int_{x_3}^{h/2}\rho^{-2}\,d\rho \approx -h^{-1/2}.
\end{aligned}$$

□

Proposition 4.2. *There exists $h_0 \in (0,r/8]$, $h_0 = h_0(\Lambda)$ such that for any $h \in (0,h_0]$ we have $|u_{11}(x)| \le cx_3h^{-5/2}$, $|u_{33}(x)| \le cx_3h^{-5/2}$, $|u_{13}(x)| \le ch^{-3/2}$ for $x \in S_1(h)\cup S_2(h)\cup S_3(h)$.*

Proof. Let $h \in (0,r/8]$.

We have

$$u_{11}(x) = \int_D K_{11}(x_1-y_1,-y_2,x_3)\varphi(y_1,y_2)\,dy_1\,dy_2,$$

Put $D_1 = D \cap B(0,h)$. By Lemma 2.6 for $y \in D_1$ we have $\varphi(y) \le ch^{1/2}$, for $y \in D \setminus D_1$ we have $\varphi(y) \le c(\text{dist}(0,y))^{1/2}$. It follows that

$$\left|\int_{D_1} K_{11}\varphi\right| \le cx_3 \frac{h^2}{h^7} h^{1/2} \int_{D_1} dy \approx cx_3 h^{-5/2},$$

$$\left|\int_{D\setminus D_1} K_{11}\varphi\right| \le cx_3 \int_h^\infty \frac{\rho^2}{\rho^7}\rho^{1/2}\rho\, d\rho \approx cx_3 h^{-5/2}.$$

Since $u_{11}(x) + u_{22}(x) + u_{33}(x) = 0$ and by Lemma 4.1 $u_{22}(x) \approx -x_3 h^{-3/2}$ for $x \in S_1(h) \cup S_2(h) \cup S_3(h)$ we get $|u_{33}(x)| \le cx_3 h^{-5/2}$.

Similarly we have

$$u_{13}(x) = \int_D K_{13}(x_1 - y_1, -y_2, x_3)\varphi(y_1,y_2)\, dy_1\, dy_2,$$

$$\left|\int_{D_1} K_{13}\varphi\right| \le ch \frac{h^2}{h^7} h^{1/2} \int_{D_1} dy \approx ch^{-3/2},$$

$$\left|\int_{D\setminus D_1} K_{13}\varphi\right| \le c \int_h^\infty \frac{\rho^3}{\rho^7}\rho^{1/2}\rho\, d\rho \approx ch^{-3/2}.$$

□

Proposition 4.3. *There exists $h_0 \in (0, r/8]$, $h_0 = h_0(\Lambda)$ such that for any $h \in (0, h_0]$ we have $u_{13}(x) \approx h^{-3/2}$ for $x \in S_1(h)$, $u_{11}(x) \approx h^{-3/2}$, $u_{33}(x) \approx -h^{-3/2}$ for $x \in S_2(h)$.*

Proof. Let $h \in (0, r/8]$.

We have

$$u_{13}(x) = \int_D K_3(x_1 - y_1, -y_2, x_3)\varphi_1(y_1,y_2)\, dy_1\, dy_2,$$

$$K_3(x_1 - y_1, -y_2, x_3) = C_K \frac{(x_1 - y_1)^2 + y_2^2 - 2x_3^2}{((x_1-y_1)^2 + y_2^2 + x_3^2)^{5/2}}.$$

Put $D_1 = \{(y_1,y_2) : y_2 \in (-r,r), y_1 \in (f(y_2), r)\}$. By Lemma 3.11 we get $\varphi_1(y) \approx \delta_D^{-1/2}(y)$ for $y \in D_1$. We also have $K_3(x_1 - y_1, -y_2, x_3) \ge 0$ for $y \in D_1$ and $x \in S_1(h)$. Let $\beta(y)$ be the acute angle between $0y$ and y_1 axis. Put $D_2 = \{(y_1,y_2) : |y| \in (h,r), \beta(y) \in [0, \pi/6)\}$. Clearly, $D_2 \subset D_1$. For $y \in D_2$ we have $\varphi_1(y) \approx \delta_D^{-1/2}(y) \approx |y|^{-1/2}$ and $K_3(x_1 - y_1, -y_2, x_3) \ge c|y|^{-3}$. It follows that

$$\int_{D_1} K_3\varphi_1 \ge \int_{D_2} |y|^{-7/2}\, dy \approx h^{-3/2}.$$

By Lemmas 3.4 and 2.3 we get

$$\left|\int_{D\setminus D_1} K_3\varphi_1\right| \le c\int_{D\setminus D_1} \delta_D^{-1/2}(y)\, dy \le c.$$

Hence $u_{13}(x) \ge ch^{-3/2}$ for $x \in S_1(h)$ and sufficiently small h. By Proposition 4.2 $|u_{13}(x)| \le ch^{-3/2}$ so $u_{13}(x) \approx h^{-3/2}$.

We have

$$u_{11}(x) = \int_D K_1(x_1 - y_1, -y_2, x_3)\varphi_1(y_1,y_2)\, dy_1\, dy_2,$$

$$K_1(x_1 - y_1, -y_2, x_3) = 3C_K \frac{x_3(y_1 - x_1)}{((x_1-y_1)^2 + y_2^2 + x_3^2)^{5/2}}.$$

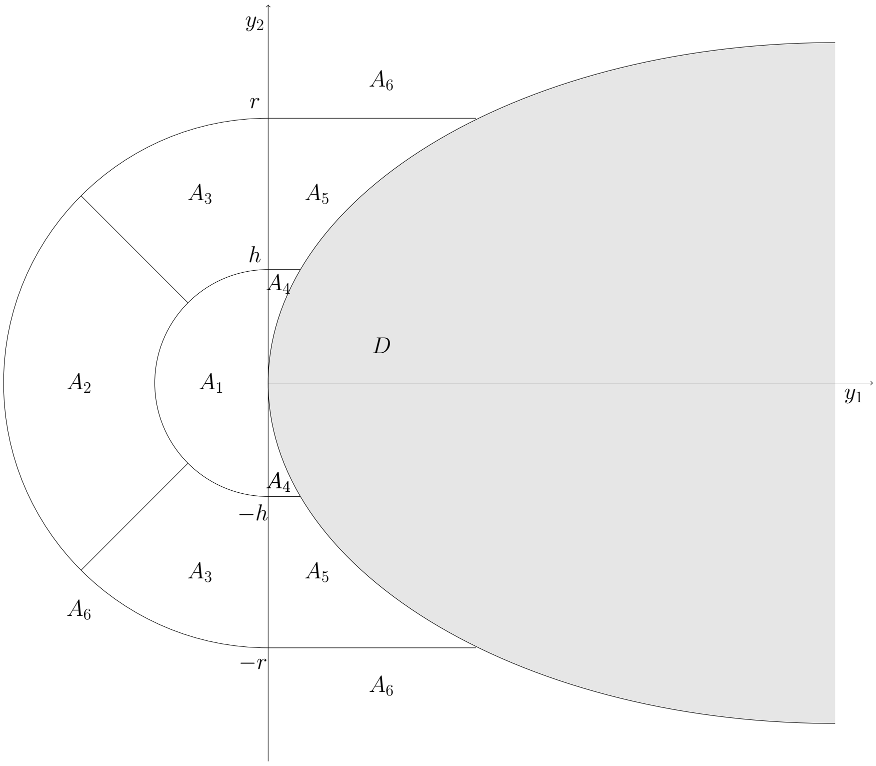


FIGURE 6

We have $K_1(x_1 - y_1, -y_2, x_3) \geq 0$ for $y \in D_1$ and $x \in S_2(h)$. For $y \in D_2$ and $x \in S_2(h)$ we have $K_1(x_1 - y_1, -y_2, x_3) \geq ch|y|^{-4}$. It follows that

$$\int_{D_1} K_1\varphi_1 \geq ch \int_{D_2} |y|^{-9/2}\, dy \approx h^{-3/2}.$$

By Lemmas 3.4 and 2.3 we get $\left|\int_{D\setminus D_1} K_1\varphi_1\right| \leq c$. Hence $u_{11}(x) \geq ch^{-3/2}$ for $x \in S_2(h)$ and sufficiently small h. By Proposition 4.2 $|u_{11}(x)| \leq ch^{-3/2}$ so $u_{11}(x) \approx h^{-3/2}$. Since $u_{11}(x) + u_{22}(x) + u_{33}(x) = 0$ and by Proposition 4.1 $u_{22}(x) \approx -h^{-1/2}$ for $x \in S_2(h)$ we get $u_{33}(x) \approx -h^{-3/2}$. □

Proposition 4.4. *There exists $h_0 \in (0, r/8]$, $h_0 = h_0(\Lambda)$ such that for any $h \in (0, h_0]$ we have $|u_{13}(x)| \leq ch^{-3/2}$ for $x \in S_4(h)$, $u_{13}(x) \approx -h^{-3/2}$ for $x \in S_3(h)$, $u_{13}(x) \leq -cx_3h^{-5/2}$ for $x \in S_4(h)$, $u_{33}(x) \approx h^{-3/2}$, $u_{11}(x) \approx -h^{-3/2}$ for $x \in S_4(h)$.*

Proof. Let $h \in (0, r/8]$.

We have

$$u_{13}(x) = \int_{\mathbb{R}^2} K_1(x_1 - y_1, -y_2, x_3) u_3(y_1, y_2, 0)\, dy_1\, dy_2,$$

$$K_1(x_1 - y_1, -y_2, x_3) = 3C_K \frac{x_3(y_1 - x_1)}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{5/2}}.$$

For $y \in D$ we have $u_3(y_1, y_2, 0) = -1$ and for $y \in (\overline{D})^c$ by Corollary 3.13

$$u_3(y_1, y_2, 0) = -(-\Delta)^{1/2}\varphi(y) \approx (1 + |y|^{-5/2})\delta_D^{-1/2}(y).$$

Put (see Figure 6)

$$\begin{aligned} A_1 &= \{y \in B(0,h) : \, y_1 \le 0\}, \\ A_2 &= \{y \in B(0,r) \setminus B(0,h) : \, y_1 < 0, |y_2| \le |y_1|\}, \\ A_3 &= \{y \in B(0,r) \setminus B(0,h) : \, y_1 \le 0, |y_2| \ge |y_1|\}, \\ A_4 &= \{y : \, y_2 \in [-h,h], y_1 \in (0, f(y_2)]\}, \\ A_5 &= \{y : \, y_2 \in (h,r] \cup [-r,-h), y_1 \in (0, f(y_2)]\}, \\ A_6 &= D^c \setminus (A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5). \end{aligned}$$

Clearly $A_1, A_2, A_3, A_4, A_5, A_6 \subset D^c$. We also put $D_1 = B((h,0),h/2)$.

Let $x \in S_3(h) \cup S_4(h)$. We have

$$\left| \int_{A_1} K_1 u_3 \right| \le c h^{-3} \int_{A_1} \delta_D^{-1/2}(y) \, dy \le c h^{-3/2},$$

$$\int_{A_2} K_1 u_3 \approx -x_3 \int_{A_2} |y|^{-9/2} \, dy \approx -x_3 h^{-5/2},$$

$$\left| \int_{A_3} K_1 u_3 \right| \le ch \int_{h/\sqrt{2}}^{r} dy_2 \int_{-y_2}^{0} dy_1 \, |y_1|^{-1/2} y_2^{-4} \le c h^{-3/2}.$$

For $x \in S_3(h) \cup S_4(h)$ and $y \in A_4$ we estimate $|y_1 - x_1| \le y_1 + h \le ch$, $f(y_2) \le c y_2^2$. Hence

$$\left| \int_{A_4} K_1 u_3 \right| \le c x_3 h^{-4} \int_{-h}^{h} dy_2 \int_{0}^{f(y_2)} dy_1 \, (-y_1 + f(y_2))^{-1/2} \le c x_3 h^{-2}.$$

For $x \in S_3(h) \cup S_4(h)$ and $y \in A_5$ we estimate $|y_1 - x_1| \le y_1 + h \le c|y_2|$, $f(y_2) \le c y_2^2$. Hence

$$\left| \int_{A_5} K_1 u_3 \right| \le c x_3 \int_{h}^{r} dy_2 \int_{0}^{f(y_2)} dy_1 \, (-y_1 + f(y_2))^{-1/2} y_2^{-4} \le c x_3 h^{-2}.$$

We also have

$$\left| \int_{A_6} K_1 u_3 \right| \le c x_3 \int_{A_6} |y|^{-13/2} \delta_D^{-1/2}(y) \, dy \le c x_3.$$

For $x \in S_3(h)$ we have

$$\left| \int_{D_1} K_1 u_3 \right| = \left| \int_{D_1} K_1 \right| \le c x_3 h^{-4} \int_{D_1} dy \approx x_3 h^{-2}.$$

For $x \in S_4(h)$ we have

$$\left| \int_{D_1} K_1 u_3 \right| = c x_3 \int_{D_1} \frac{y_1 - h}{((y_1 - h)^2 + y_2^2 + x_3^2)^{5/2}} \, dy_1 \, dy_2 = 0.$$

For $x \in S_3(h) \cup S_4(h)$ we also have

$$\left| \int_{D \setminus D_1} K_1 u_3 \right| \le c x_3 \int_{D \setminus D_1} ((y_1 - h)^2 + y_2^2)^{-2} \, dy \le c x_3 h^{-2}.$$

It follows that for $x \in S_3(h) \cup S_4(h)$

$$|u_{13}(x)| = \left| \int_{\mathbb{R}^2} K_1 u_3 \right| \le c h^{-3/2}, \tag{32}$$

(for $x \in S_3(h)$ such estimate follows also from Proposition 4.2).

Now note that $K_1(x_1 - y_1, -y_2, x_3) \le 0$ and $u_3(y_1, y_2, 0) \ge 0$ for $x \in S_3(h) \cup S_4(h)$ and $y \in A_1 \cup A_3$. So $\int_{A_1 \cup A_3} K_1 u_3 \le 0$. It follows that for $x \in S_3(h) \cup S_4(h)$ we have

$$u_{13}(x) = \int_{\mathbb{R}^2} K_1 u_3 \le \int_{A_2 \cup A_4 \cup A_5 \cup A_6 \cup D} K_1 u_3 \le -c x_3 h^{-5/2} + c_1 x_3 h^{-2}.$$

It is clear that one can choose sufficiently small $h_0 = h_0(\Lambda)$ such that for any $h \in (0, h_0]$ and $x \in S_3(h) \cup S_4(h)$ we have $u_{13}(x) \le -c_2 x_3 h^{-5/2}$. Using this and (32) we also obtain $u_{13}(x) \approx -h^{-3/2}$ for any $h \in (0, h_0]$ and $x \in S_3(h)$.

Now we will estimate $u_{33}(x)$ for $x \in S_4(h)$. We have

$$u_{33}(x) = \int_{\mathbb{R}^2} K_3(x_1 - y_1, -y_2, x_3) u_3(y_1, y_2, 0)\, dy_1\, dy_2,$$

$$K_3(x_1 - y_1, -y_2, x_3) = C_K \frac{(x_1 - y_1)^2 + y_2^2 - 2x_3^2}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{5/2}}.$$

For $x \in S_4(h)$ and $y \in D^c$ we have $K_3(x_1 - y_1, -y_2, x_3) > 0$, $u_3(y_1, y_2, 0) \approx (1 + |y|^{-5/2})\delta_D^{-1/2}(y)$. For $y \in D$ we have $u_3(y_1, y_2, 0) = -1$. We have

$$\begin{aligned} \left| \int_{A_1 \cup A_4} K_3 u_3 \right| &\le \frac{c}{h^3} \int_{A_1 \cup A_4} \delta_D^{-1/2}(y)\, dy \\ &\le \frac{c}{h^3} \int_0^h dy_2 \int_{-h}^{f(y_2)} dy_1\, (-y_1 + f(y_2))^{-1/2} \approx h^{-3/2}, \end{aligned}$$

$$\int_{A_2} K_3 u_3 \approx \int_{A_2} |y|^{-7/2}\, dy \approx h^{-3/2},$$

$$\left| \int_{A_3 \cup A_5} K_3 u_3 \right| \le c \int_{h/\sqrt{2}}^r dy_2 \int_{-y_2}^{f(y_2)} dy_1 \, \frac{(-y_1 + f(y_2))^{-1/2}}{y_2^3} \approx h^{-3/2},$$

$$\left| \int_{A_6} K_3 u_3 \right| \le c \int_{A_6} |y|^{-11/2} \delta_D^{-1/2}(y)\, dy \le c,$$

$$\left| \int_{D \setminus D_1} K_3 u_3 \right| \le c \int_{D \setminus D_1} ((y_1 - h)^2 + y_2^2)^{-3/2}\, dy \le c h^{-1}.$$

The integral over D_1 we compute directly. Recall that $D_1 = B((h, 0), h/2)$ and $x = (x_1, x_2, x_3) \in S_4(h)$ so $x_1 = h$, $x_2 = 0$, $x_3 \in (0, h/4]$. We have

$$\int_{D_1} K_3(x_1 - y_1, -y_2, x_3) u_3(y_1, y_2, 0)\, dy_1\, dy_2 = C_K \int_{D_1} \frac{(h - y_1)^2 + y_2^2 - 2x_3^2}{((h - y_1)^2 + y_2^2 + x_3^2)^{5/2}}\, dy_1\, dy_2. \tag{33}$$

Let us introduce polar coordinates $h - y_1 = \rho \cos\theta$, $y_2 = \rho \sin\theta$. Then (33) equals $2\pi C_K \int_0^{h/2} \frac{\rho^2 - 2x_3^2}{(\rho^2 + x_3^2)^{5/2}} \rho\, d\rho$. By substitution $t = \rho^2$ this is equal to $\pi C_K \int_0^{h^2/4} \frac{t - 2x_3^2}{(t + x_3^2)^{5/2}}\, dt$. By elementary calculations this is equal to $\frac{-\pi C_K h^2}{2(h^2/4 + x_3^2)^{3/2}}$. Hence $\left|\int_{D_1} K_3 u_3\right| \le c/h$.

It follows that $|u_{33}(x)| \le c h^{-3/2}$. Since for $x \in S_4(h)$ and $y \in (\overline{D})^c$ we have $K_3(x_1 - y_1, -y_2, x_3) > 0$ and $u_3(y_1, y_2, 0) > 0$ we get

$$u_{33}(x) = \int_{\mathbb{R}^2} K_3 u_3 \ge \int_{A_2 \cup D} K_3 u_3 \ge \int_{A_2} K_3 u_3 - \left| \int_D K_3 u_3 \right| \ge c h^{-3/2} - c_1 h^{-1}.$$

It follows that $u_{33}(x) \approx h^{-3/2}$ for $x \in S_4(h)$ and sufficiently small h. Since $u_{11}(x) + u_{22}(x) + u_{33}(x) = 0$ and by Proposition 4.1 $u_{22}(x) \approx -h^{-1/2}$ for $x \in S_4(h)$ we get $u_{11}(x) \approx -h^{-3/2}$. □

Proposition 4.5. *There exists $h_0 \in (0, r/8]$, $h_0 = h_0(\Lambda)$ such that for any $h \in (0, h_0]$ we have $|u_{12}(x)| \le c x_3 h^{-3/2} |\log h|$ for $x \in S_1(h) \cup S_2(h) \cup S_3(h)$, $|u_{12}(x)| \le c h^{-1/2} |\log h|$ for $x \in S_4(h)$, $|u_{23}(x)| \le c h^{-1/2} |\log h|$ for $x \in S_1(h) \cup S_2(h) \cup S_3(h)$.*

Proof. Let $h \in (0, r/8]$.

We have

$$u_{12}(x) = \int_D K_{12}(x_1 - y_1, -y_2, x_3)\varphi(y_1, y_2)\, dy_1\, dy_2, \tag{34}$$

$$K_{12}(x_1 - y_1, -y_2, x_3) = -15 C_K \frac{x_3(x_1 - y_1)y_2}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{7/2}}.$$

Let D_1, D_2, D_3, D_4, D_5 and D_{i+}, D_{i-} for $i = 1, 2, 3, 4$ be such as in the proof of Proposition 4.2. We have

$$\int_{D_1 \cup D_2} K_{12}\varphi = -cx_3 \int_{D_{1+} \cup D_{2+}} \frac{(x_1 - y_1)y_2}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{7/2}} (\varphi(y_1, y_2) - \varphi(y_1, -y_2))\, dy_1\, dy_2.$$

For $y \in D_{1+} \cup D_{2+}$ by Lemma 3.11 we get $|\varphi(y_1, y_2) - \varphi(y_1, -y_2)| = |2y_2\varphi_2(y_1, \xi)| \leq cy_2(y_2 y_1^{-1/2} + y_1^{1/2}|\log y_1|)$, where $\xi \in (-y_2, y_2)$. Hence

$$\begin{aligned}
\left|\int_{D_1} K_{12}\varphi\right| &\leq cx_3 \int_{D_{1+}} \frac{|x_1 - y_1|}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{7/2}} (y_2^3 y_1^{-1/2} + y_2^2 y_1^{1/2}|\log y_1|)\, dy_1\, dy_2 \\
&\leq cx_3 h^{-6} \int_0^h dy_1 \int_0^h dy_2 (y_2^3 y_1^{-1/2} + y_2^2 y_1^{1/2}|\log y_1|) \\
&+ cx_3 h \int_0^h dy_1 \int_h^{c_1 y_1^{1/2}} dy_2 (y_2^{-4} y_1^{-1/2} + y_2^{-5} y_1^{1/2}|\log y_1|) \\
&\leq cx_3 h^{-3/2}|\log h|.
\end{aligned}$$

Note that for $y \in D_2$ we have $|x_1 - y_1| \leq cy_1$. We obtain

$$\begin{aligned}
\left|\int_{D_2} K_{12}\varphi\right| &\leq cx_3 \int_{D_{2+}} \frac{|x_1 - y_1|}{((x_1 - y_1)^2 + y_2^2 + x_3^2)^{7/2}} (y_2^3 y_1^{-1/2} + y_2^2 y_1^{1/2}|\log y_1|)\, dy_1\, dy_2 \\
&\leq cx_3 \int_h^r dy_1 \int_0^{y_1} dy_2 (y_2^3 y_1^{-13/2} + y_2^2 y_1^{-11/2}|\log y_1|) \\
&+ cx_3 \int_h^r dy_1 \int_{y_1}^r dy_2 (y_2^{-4} y_1^{1/2} + y_2^{-5} y_1^{3/2}|\log y_1|) \\
&\leq cx_3 h^{-3/2}|\log h|.
\end{aligned}$$

By Lemma 2.6 for $y \in D_3 \cup D_4$ we have $\varphi(y) \leq c\delta_D^{1/2}(y) \leq cy_2$. Note also that $|x_1 - y_1| \leq 2h$ for $y \in D_3$ and $|x_1 - y_1| \leq h + y_1$ for $y \in D_4$. We get

$$\left|\int_{D_3} K_{12}\varphi\right| \leq cx_3 h^{-5} \int_0^h dy_2 \int_0^{f_1(h)} dy_1 y_2 \leq cx_3 h^{-1},$$

$$\left|\int_{D_4+} K_{12}\varphi\right| \leq cx_3 \int_h^r dy_2 \int_0^{c_1 y_2^2} dy_1 (h + y_1) y_2^{-5} \leq cx_3 h^{-1}.$$

The estimate of $\left|\int_{D_{4-}} K_{12}\varphi\right|$ is the same so $\left|\int_{D_4} K_{12}\varphi\right| \leq cx_3 h^{-1}$. Note that for $y \in D_5$ we have $|x_1 - y_1| \leq cy_1$ and $\varphi(y) \leq c$. Hence

$$\left|\int_{D_5} K_{12}\varphi\right| \leq cx_3 \int_{B^c(0, c_1 r^2)} \frac{y_1 |y_2|}{(y_1^2 + y_2^2)^{7/2}}\, dy_1\, dy_2 \leq cx_3.$$

For $x \in S_1(h) \cup S_2(h) \cup S_3(h)$ we have

$$u_{23}(x) = \int_D K_{23}(x_1 - y_1, -y_2, x_3)\varphi(y_1, y_2)\, dy_1\, dy_2.$$

The proof of the estimate $\left|\int_D K_{23}\varphi\right| \leq ch^{-1/2}|\log h|$ is very similar to the proof of the estimate $\left|\int_D K_{12}\varphi\right| \leq cx_3 h^{-3/2}|\log h|$ and it is omitted.

We have

$$u_{12}(x) = \int_D K_{12}(x_1 - y_1, -y_2, x_3)\varphi(y_1, y_2)\, dy_1\, dy_2.$$

Put $A = B((h,0), h/2)$. By the same argument as above we obtain $\left|\int_{D\setminus A} K_{12}\varphi\right| \leq cx_3 h^{-3/2}|\log h|$. We have

$$\left|\int_A K_{12}\varphi\right| = \left|cx_3 \int_A \frac{(y_1 - h)y_2}{((y_1 - h)^2 + y_2^2 + x_3^2)^{7/2}}\varphi(y_1, y_2)\, dy_1\, dy_2\right|.$$

By substitution $z_1 = y_1 - h$, $z_2 = y_2$ this is equal to

$$\left|cx_3 \int_{B(0,h/2)} \frac{z_1 z_2}{(z_1^2 + z_2^2 + x_3^2)^{7/2}}\varphi(z_1 + h, z_2)\, dz_1\, dz_2\right| = \left|cx_3 \int_W \frac{z_1 z_2 g(z_1, z_2)}{(z_1^2 + z_2^2 + x_3^2)^{7/2}}\, dz_1\, dz_2\right|, \tag{35}$$

where $g(z_1, z_2) = \varphi(z_1 + h, z_2) - \varphi(-z_1 + h, z_2) - \varphi(z_1 + h, -z_2) + \varphi(-z_1 + h, -z_2)$ and $W = \{z \in B(0, h/2) : z_1 \geq 0, z_2 \geq 0\}$. Note that for $z \in W$ we have $g(z_1, z_2) = 4z_1 z_2 \varphi_{12}(\xi_1 + h, \xi_2)$, where $\xi_1 \in (-z_1, z_1)$, $\xi_2 \in (-z_2, z_2)$. By Lemma 3.11 we have for $z \in W$ and ξ_1, ξ_2 as above

$$|\varphi_{12}(\xi_1 + h, \xi_2)| \leq ch^{-1/2}|\log h| + cz_2 h^{-3/2}.$$

It follows that (35) is bounded from above by

$$cx_3 \int_W \frac{z_1^2 z_2^2 (h^{-1/2}|\log h| + z_2 h^{-3/2})}{(z_1^2 + z_2^2 + x_3^2)^{7/2}}\, dz_1\, dz_2. \tag{36}$$

Put $W_1 = \{z : z_1 \in [0, x_3], z_2 \in [0, x_3]\}$, $W_2 = \{z \in B(0, h/2) \setminus B(0, x_3) : z_1 \geq 0,\ z_2 \geq 0\}$. We have $W \subset W_1 \cup W_2$. (36) is bounded from above by

$$\begin{aligned} & cx_3 \int_{W_1} \frac{z_1^2 z_2^2 (h^{-1/2}|\log h| + z_2 h^{-3/2})}{x_3^7}\, dz_1\, dz_2 \\ + \quad & cx_3 \int_{W_2} \frac{z_1^2 z_2^2 (h^{-1/2}|\log h| + z_2 h^{-3/2})}{(z_1^2 + z_2^2)^{7/2}}\, dz_1\, dz_2 \\ \leq \quad & ch^{-1/2}|\log h|. \end{aligned}$$

□

Proposition 4.6. *There exists $h_0 \in (0, r/8]$, $h_0 = h_0(\Lambda)$ such that for any $h \in (0, h_0]$ we have $|u_{23}(x)| \leq ch^{-3/4}|\log h|$ for $x \in S_4(h)$.*

Proof. Let $h \in (0, r/8]$. Put $p = (-r, 0)$, recall that $z = (r, 0)$. We have

$$\begin{aligned} u_{23}(x) &= \int_{\mathbb{R}^2} K_2(x_1 - y_1, -y_2, x_3) u_3(y_1, y_2, 0)\, dy_1\, dy_2 \\ &= \int_{B(0,r/4)\cap B(p,r)} K_2 u_3 + \int_{(D\cap B(0,r/4))\setminus(B(p,r)\cup B(z,r))} K_2 u_3 \\ &+ \int_{(D^c\cap B(0,r/4))\setminus(B(p,r)\cup B(z,r))} K_2 u_3 + \int_{B(0,r/4)\cap B(z,r)} K_2 u_3 \\ &+ \int_{B^c(0,r/4)} K_2 u_3 = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}. \end{aligned}$$

Note that $u_3(y_1, y_2, 0) = -(-\Delta)^{1/2}\varphi(y_1, y_2)$ for $(y_1, y_2) \in \mathbb{R}^2 \setminus \partial D$.

Put $A = B(0, r/4) \cap B(p, r)$. For $y \in A$ by Corollary 3.13 we get $|(-\Delta)^{1/2}\varphi(y)| \le c\delta_D^{-1/2}(y) \le c|y_1|^{-1/2}$. It follows that

$$\begin{aligned}|\mathrm{I}| &\le cx_3 \int_A \frac{y_2|y_1|^{-1/2}}{((h-y_1)^2+y_2^2+x_3^2)^{5/2}}\,dy_1\,dy_2 \\ &\le cx_3 \int_0^h dy_2 \int_{-r/4}^{-f_1(y_2)} dy_1 \frac{y_2|y_1|^{-1/2}}{h^5} + cx_3 \int_h^{r/4} dy_2 \int_{-r/2}^{-f_1(y_2)} dy_1 \frac{y_2|y_1|^{-1/2}}{y_2^5} \\ &\le cx_3 h^{-3}.\end{aligned}$$

We also have

$$|\mathrm{II}| \le cx_3 \int_0^h dy_2 \int_0^{f_1(y_2)} dy_1\, y_2 h^{-5} + cx_3 \int_h^{r/2} dy_2 \int_0^{f_1(y_2)} dy_1\, y_2 y_2^{-5} \le cx_3 h^{-1}.$$

For $y \in (D^c \cap B(0,r/4)) \setminus (B(p,r) \cup B(z,r))$ by Corollary 3.13 we get $|(-\Delta)^{1/2}\varphi(y)| \le c\delta_D^{-1/2}(y) \approx (f(y_2)-y_1)^{-1/2}$. Hence

$$|\mathrm{III}| \le cx_3 \int_0^{r/4} dy_2 \int_{-f_1(y_2)}^{f(y_2)} dy_1\, (f(y_2)-y_1)^{-1/2} \frac{y_2}{h^5 \vee y_2^5}.$$

For $y_2 \in (0, r/4)$ we have

$$\int_{-f_1(y_2)}^{f(y_2)} (f(y_2)-y_1)^{-1/2}\,dy_1 = \int_0^{f_1(y_2)+f(y_2)} z^{-1/2}\,dz \le cy_2.$$

It follows that

$$|\mathrm{III}| \le cx_3 \int_0^h \frac{y_2^2}{h^5}\,dy_2 + cx_3 \int_h^{r/4} \frac{y_2^2}{y_2^5}\,dy_2 \le \frac{cx_3}{h^2}.$$

Clearly

$$\mathrm{IV} = \int_{B(0,r/4)\cap B(z,r)} \frac{-cx_3 y_2}{((h-y_1)^2+y_2^2+x_3^2)^{5/2}}\,dy_1\,dy_2 = 0.$$

Using Corollary 3.13 we get

$$|\mathrm{V}| \le cx_3 \int_D dy + cx_3 \int_{D^c} \frac{\delta_D(y)^{-1/2}}{(1+|y|)^{5/2}}\,dy \le cx_3.$$

It follows that for $x \in S_4(h)$ we have

$$|u_{23}(x)| \le |\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}| \le \frac{cx_3}{h^3}. \tag{37}$$

On the other hand we have for $x \in S_4(h)$

$$u_{23}(x) = \int_D K_{23}(x_1-y_1, -y_2, x_3)\varphi(y_1,y_2)\,dy_1\,dy_2.$$

Put $W = B((h,0),h/2)$, $W_+ = \{y \in W : y_2 > 0\}$. For $x \in S_4(h)$ one may show $\left|\int_{D\setminus W} K_{23}\varphi\right| \le ch^{-1/2}|\log h|$. The proof of this inequality is omitted. It is very similar to the proof of the estimate $\left|\int_{D\setminus W} K_{12}\varphi\right| \le cx_3 h^{-3/2}|\log h|$ see the proof of Proposition 4.5.

We have

$$\begin{aligned}\int_W K_{23}\varphi &= -c\int_W \frac{12x_3^2 - 3(y_1-h)^2 - 3y_2^2}{((y_1-h)^2+y_2^2+x_3^2)^{7/2}} y_2\varphi(y_1,y_2)\,dy_1\,dy_2 \\ &= -c\int_{W_+} \frac{12x_3^2 - 3(y_1-h)^2 - 3y_2^2}{((y_1-h)^2+y_2^2+x_3^2)^{7/2}} y_2(\varphi(y_1,y_2)-\varphi(y_1,-y_2))\,dy_1\,dy_2. \quad (38)\end{aligned}$$

For $y \in W_+$ we have $\varphi(y_1,y_2) - \varphi(y_1,-y_2) = 2y_2\varphi_2(y_1,\xi_2)$ where $\xi_2 \in (-y_2,y_2)$ and $\varphi_2(y_1,\xi_2) = \varphi_2(h,0) + (y_1-h,\xi_2)\circ\nabla\varphi_2(\xi')$, where ξ' is a point between $(h,0)$ and (y_1,ξ_2). It follows that (38) equals

$$\begin{aligned} &- c\varphi_2(h,0)\int_{W_+} \frac{12x_3^2 - 3(y_1-h)^2 - 3y_2^2}{((y_1-h)^2+y_2^2+x_3^2)^{7/2}} 2y_2^2\,dy_1\,dy_2 \\ &- c\int_{W_+} \frac{12x_3^2 - 3(y_1-h)^2 - 3y_2^2}{((y_1-h)^2+y_2^2+x_3^2)^{7/2}} 2y_2^2(y_1-h,\xi_2)\circ\nabla\varphi_2(\xi')\,dy_1\,dy_2 = \mathrm{I} + \mathrm{II}. \end{aligned}$$

Put $V = B(0,h/2)$, $V_+ = \{z \in V : z_2 > 0\}$. By substitution $z_1 = y_1 - h$, $z_2 = y_2$ we obtain

$$\begin{aligned} \mathrm{I} &= -c\varphi_2(h,0)\int_{V_+} \frac{12x_3^2 - 3z_1^2 - 3z_2^2}{(z_1^2+z_2^2+x_3^2)^{7/2}} 2z_2^2\,dy_1\,dy_2 \\ &= -c\varphi_2(h,0)\int_{V} \frac{12x_3^2 - 3z_1^2 - 3z_2^2}{(z_1^2+z_2^2+x_3^2)^{7/2}} z_2^2\,dy_1\,dy_2. \end{aligned}$$

By symmetry of z_1, z_2 the above integral equals

$$\frac{1}{2}\int_V \frac{12x_3^2 - 3z_1^2 - 3z_2^2}{(z_1^2+z_2^2+x_3^2)^{7/2}}(z_1^2+z_2^2)\,dy_1\,dy_2.$$

Let us introduce polar coordinates $z_1 = \rho\cos\theta$, $z_2 = \rho\sin\theta$. Then the above expression equals $\pi\int_0^{h/2} \frac{12x_3^2-3\rho^2}{(\rho^2+x_3^2)^{7/2}}\rho^3\,d\rho$. By elementary calculation this is equal to $(3\pi/16)h^4(x_3^2 + h^2/4)^{-5/2}$. By Lemma 3.11 $\varphi_2(h,0) \le ch^{1/2}|\log h|$. Hence $|\mathrm{I}| \le ch^{-1/2}|\log h|$.

Now we estimate II. For $y \in W_+$ and ξ_2, ξ' as above we have

$$(y_1-h,\xi_2)\circ\nabla\varphi_2(\xi') = (y_1-h)\varphi_{12}(\xi') + \xi_2\varphi_{22}(\xi'). \tag{39}$$

For any $w \in W$ by Lemma 3.11 we get $|\varphi_{12}(w)| \le ch^{-1/2}|\log h|$, $|\varphi_{22}(w)| \le ch^{-1/2}$ so (39) is bounded from above by $c|y_1-h|h^{-1/2}|\log h| + c|y_2|h^{-1/2}$. Put $B_+((h,0),x_3) = \{y \in B((h,0),x_3) : y_2 > 0\}$. It follows that

$$\begin{aligned} |\mathrm{II}| &\le \frac{c}{x_3^5}\int_{B_+((h,0),x_3)} |y-(h,0)|^3 h^{-1/2}|\log h|\,dy \\ &+ c\int_{W_+\setminus B_+((h,0),x_3)} |y-(h,0)|^{-2}h^{-1/2}|\log h|\,dy \le ch^{-1/2}|\log h||\log x_3|. \end{aligned}$$

Hence for $x \in S_4(h)$ we have

$$|u_{23}(x)| \le \left|\int_{D\setminus W} K_{23}\varphi\right| + |\mathrm{I}| + |\mathrm{II}| \le ch^{-1/2}|\log h||\log x_3|. \tag{40}$$

For any $\beta > 0$ and $x \in S_4(h)$ we get by (37) $|u_{23}(x)|^\beta \le c_1^\beta x_3^\beta h^{-3\beta}$. Using this and (40) we get $|u_{23}(x)|^{1+\beta} \le cc_1^\beta x_3^\beta|\log x_3|h^{-3\beta-1/2}|\log h|$. Putting $\beta = 1/9$ we obtain $|u_{23}(x)| \le ch^{-3/4}|\log h|^{9/10} \le ch^{-3/4}|\log h|$. □

Lemma 4.7. *For any $(x_1,x_2) \in D$ we have $u_{13}(x_1,x_2,0) = u_{23}(x_1,x_2,0) = 0$ and $u_{33}(x_1,x_2,0) > 0$.*

Proof. The equalities $u_{13}(x_1,x_2,0) = u_{23}(x_1,x_2,0) = 0$ for $(x_1,x_2) \in D$ follows easily from (8). For $(x_1,x_2) \in \mathrm{int}(D^c)$ we have

$$u_3(x_1,x_2,0) = -(-\Delta)^{1/2}\varphi(x) = \frac{1}{2\pi}\int_D \frac{\varphi(y)}{|y-x|^3}\,dy > 0.$$

By Corollary 3.13 we have $f(x_1,x_2) = u_3(x_1,x_2,0) \in L^1(\mathbb{R}^2)$. By the normal derivative lemma ([15, Lemma 2.33]) we get $u_{33}(x_1,x_2,0) > 0$ for $(x_1,x_2) \in D$. □

5. Harmonic extension for a ball

The aim of this section is to show the following result.

Proposition 5.1. *Let φ be the solution of (1-2) for the ball $B(0,1) \subset \mathbb{R}^2$ and u be the harmonic extension of φ given by (6-10). We have*

$$H(u)(x) > 0, \quad x \in \mathbb{R}^3 \setminus \{B^c(0,1) \times \{0\}\}. \tag{41}$$

Let us recall that $H(u)(x)$ is the determinant of the Hessian matrix of u in x. Recall also that the solution of (1-2) for the ball $B(0,1)$ is given by an explicit formula $\varphi(x) = C_B(1-|x|)^{1/2}$, $C_B = 2/\pi$. Hence for $x = (x_1, x_2, x_3)$, where $x_3 > 0$ the function u is given by an explicit formula $u(x) = \int_{B(0,1)} K(x_1 - y_1, x_2 - y_2, x_3)\varphi(y_1, y_2)\, dy_1\, dy_2$. Applying this it is easy to check numerically that (41) holds (e.g. using Mathematica). Unfortunately, it seems very hard to prove formally (41) using directly the explicit formula for u.

Instead, to show (41) we use a "trick": we add an auxiliary function w to the function u and we use Lewy's Theorem 1.6. First, we briefly present the idea of the proof. We define

$$\Psi^{(b)}(x) = (1-b)u(x) + bw(x), \quad b \in [0,1],$$

where w is an appropriately chosen auxiliary function given by

$$w(x) = K(x_1, x_2, x_3 + \sqrt{3/2}). \tag{42}$$

Note that for any $q \geq 0$ the set $\{(x_1, x_2, x_3) : K_{33}(x_1, x_2, x_3 + q) = 0, x_3 > -q\} = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = (2/3)(x_3 + q)^2, x_3 > -q\}$. The function w is chosen so that $w_{33}(x) = 0$ for $x \in \partial B(0,1) \times \{0\}$ i.e. for $x = (x_1, x_2, 0)$ where $x_1^2 + x_2^2 = 1$. Such a choice helps to control $H(\Psi^{(b)})(x)$ near $\partial B(0,1) \times \{0\}$. One can directly check that $\Psi^{(1)} = w$ satisfies $H(\Psi^{(1)})(x) > 0$ for $x \in \mathbb{R}^3_+ \cup B(0,1) \times \{0\}$ (recall that $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) : x_3 > 0\}$). If $\Psi^{(0)} = u$ does not satisfy $H(\Psi^{(0)})(x) > 0$ for $x \in \mathbb{R}^3_+ \cup B(0,1) \times \{0\}$ one can show that there exists $b \in [0,1)$ for which $H(\Psi^{(b)})(x) \geq 0$ for $x \in \mathbb{R}^3_+ \cup B(0,1) \times \{0\}$ and such that there exists $x_0 \in \mathbb{R}^3_+$ for which $H(\Psi^{(b)})(x_0) = 0$. This gives contradiction with Theorem 1.6. If $\Psi^{(0)} = u$ does not satisfy $H(\Psi^{(0)})(x) > 0$ for $x \in \mathbb{R}^3_-$ one can use Lemma 2.7 and again obtain contradiction. This finishes the presentation of the idea of the proof.

Lemma 5.2. *Let w be given by (42) and $v = u + aw$, $a \geq 0$. There exists $M_1 \geq 10$ and $h_1 \in (0, 1/2]$ such that for any $a \geq 0$ we have*

$$H(v)(x) > 0, \qquad x \in A_1 \cup A_2 \cup A_3 \cup A_4,$$

where

$$\begin{aligned}
A_1 &= \{(x_1, x_2, x_3) : x_1^2 + x_2^2 \in [(1-h_1)^2, (1+h_1)^2], x_3 \in (0, h_1]\}, \\
A_2 &= \{(x_1, x_2, x_3) : x_1^2 + x_2^2 \in [(1+h_1)^2, M_1^2], x_3 \in (0, h_1]\}, \\
A_3 &= \{(x_1, x_2, 0) : x_1^2 + x_2^2 < 1\}, \\
A_4 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3_+ : x_1^2 + x_2^2 \geq M_1^2 \text{ or } x_3 \geq M_1\}.
\end{aligned}$$

Proof. First note that for any fixed $x_3 > 0$ the function $(x_1, x_2) \to v(x_1, x_2, x_3)$ is radial so it is enough to show the assertion for $x \in (A_1 \cup A_2 \cup A_3 \cup A_4) \cap L$, where $L = \{(x_1, x_2, x_3) : x_2 = 0, x_1 \leq 0\}$. Put $A_i' = A_i \cap L$, $i = 1, 2, 3, 4$. For $x \in A_1' \cup A_2' \cup A_3' \cup A_4'$ we have $v_{12}(x) = v_{23}(x) = 0$ and $v_{22}(x) < 0$. Hence $H(v)(x) = v_{22}(x) f(a, x)$, where

$$f(a,x) = \begin{vmatrix} v_{11} & v_{13} \\ v_{13} & v_{33} \end{vmatrix} = \begin{vmatrix} u_{11} + aw_{11} & u_{13} + aw_{13} \\ u_{13} + aw_{13} & u_{33} + aw_{33} \end{vmatrix} \tag{43}$$

and it is enough to show $f(a, x) < 0$ for $x \in A_1' \cup A_2' \cup A_3' \cup A_4'$.

We will consider 4 cases: $x \in A_1'$, $x \in A_2'$, $x \in A_3'$, $x \in A_4'$.

Case 1. $x \in A_1'$.

Put $q_0 = \sqrt{3/2}$ and $z_0 = (-1, 0, 0)$. Note that $w_{33}(z_0) = 0$, $w_{11}(z_0) = C_K q_0 (12 - 3q_0^2)(1+q_0^2)^{-7/2} \approx 9.185 C_K (1+q_0^2)^{-7/2}$, $w_{13}(z_0) = -C_K(12q_0^2 - 3)(1+q_0^2)^{-7/2} = -15C_K(1+q_0^2)^{-7/2}$. Let us denote $w_{11}(x) = p_1(x)$, $w_{13}(x) = p_2(x)$. It is clear that for sufficiently small h_1 and $x \in A_1'$ we have

$$\sqrt{\frac{9}{10}} |p_2(x)| > |p_1(x)|. \tag{44}$$

Let h_0 denote the minimum of constants h_0 from Propositions 4.1-4.6. For any $h \in (0, h_0]$ put

$$\begin{aligned}
T_1(h) &= \{(-1+h, 0, x_3) : x_3 \in (0, h/4]\}, \\
T_2(h) &= \{(-1+h, 0, x_3) : x_3 \in (h/4, h]\} \cup \{(x_1, 0, h) : x_1 \in [-1, -1+h)\}, \\
T_3(h) &= \{(x_1, 0, h) : x_1 \in [-\sqrt{2/3}h - 1, -1]\}, \\
T_4(h) &= \{(x_1, 0, h) : x_1 \in [-1-h, -\sqrt{2/3}h - 1)\} \cup \{(-1-h, 0, x_3) : x_3 \in (0, h)\}.
\end{aligned}$$

Note that the value $-\sqrt{2/3}h - 1$ in the definition of $T_3(h)$, $T_4(h)$ is chosen so that $w_{33}(-\sqrt{2/3}h - 1, 0, h) = 0$. Note also that $w_{33}(x) \geq 0$ for $x \in T_1(h) \cup T_2(h) \cup T_3(h)$ and $w_{33}(x) < 0$ for $x \in T_4(h)$.

We will consider 4 subcases: $x \in T_1(h)$, $x \in T_2(h)$, $x \in T_3(h)$, $x \in T_4(h)$.

Subcase 1a. $x \in T_1(h)$.

By (43), Propositions 4.1, 4.4 and definition of w we have

$$f(a,x) = \begin{vmatrix} -b_1(x)h^{-3/2} + p_1(x)a & -b_2(x)h^{-3/2} - p_2(x)a \\ -b_2(x)h^{-3/2} - p_2(x)a & \varepsilon(x)a + b_1(x)h^{-3/2} + b_3(x)h^{-1/2} \end{vmatrix},$$

where $0 < B_1' \leq b_1(x) \leq B_1$, $0 \leq b_2(x) \leq B_2$, $0 < B_3' \leq b_3(x) \leq B_3$, $0 < P_1' \leq p_1(x) \leq P_1$, $0 < P_2' \leq p_2(x) \leq P_2$, $0 \leq \varepsilon(x) \leq E(h) \leq E(h_0)$, $\lim_{h\to 0^+} E(h) = 0$. More precisely, estimates of $b_1(x)$, $b_2(x)$ follow from estimates of $u_{11}(x)$, $u_{13}(x)$ for $S_4(h)$ in Proposition 4.4, estimates of $b_3(x)$ follow from $u_{33}(x) = -u_{11}(x) - u_{22}(x)$ and estimates of $u_{11}(x)$, $u_{22}(x)$ for $S_4(h)$ in Propositions 4.1, 4.4. Estimates of $p_1(x)$, $p_2(x)$ follow from formulas of $w_{11}(z_0)$, $w_{13}(z_0)$ and continuity of $w_{11}(x)$, $w_{13}(x)$ near z_0. Estimates of $\varepsilon(x)$ and $\lim_{h\to 0^+} E(h) = 0$ follow from equality $w_{33}(z_0) = 0$ and continuity of $w_{33}(x)$ near z_0.

Hence

$$\begin{aligned}
&f(a,x) = -\varepsilon(x)b_1(x)ah^{-3/2} - b_1^2(x)h^{-3} - b_1(x)b_3(x)h^{-2} + \varepsilon(x)p_1(x)a^2 \\
&+ b_1(x)p_1(x)ah^{-3/2} + p_1(x)b_3(x)ah^{-1/2} - b_2^2(x)h^{-3} - p_2^2(x)a^2 - 2b_2(x)p_2(x)ah^{-3/2}.
\end{aligned}$$

Note that for sufficiently small h we have

$$p_1(x)b_3(x)ah^{-1/2} < p_1(x)b_1(x)ah^{-3/2}.$$

For sufficiently small h, using this and (44) we get

$$\begin{aligned}
(9/10)p_2^2(x)a^2 + b_1^2(x)h^{-3} &> p_1^2(x)a^2 + b_1^2(x)h^{-3} \\
&\geq 2b_1(x)p_1(x)ah^{-3/2} \\
&> b_1(x)p_1(x)ah^{-3/2} + b_3(x)p_1(x)ah^{-1/2}.
\end{aligned}$$

For sufficiently small h we also have $p_1(x)\varepsilon(x)a^2 < (1/10)p_2^2(x)a^2$. It follows that for sufficiently small $h_1 > 0$ and for all $0 < h \leq h_1$, $a \geq 0$, $x \in T_1(h)$ we have $f(a,x) < 0$.

Subcase 1b. $x \in T_2(h)$.

By (43), Propositions 4.1, 4.2, 4.4 and definition of w we have

$$f(a,x) = \begin{vmatrix} b_1(x)h^{-3/2} + p_1(x)a & -b_2(x)h^{-3/2} - p_2(x)a \\ -b_2(x)h^{-3/2} - p_2(x)a & \varepsilon(x)a - b_1(x)h^{-3/2} + b_3(x)h^{-1/2} \end{vmatrix},$$

where $-B_1 \leq b_1(x) \leq B_1$, $0 < B_2' \leq b_2(x) \leq B_2$, $0 < B_3' \leq b_3(x) \leq B_3$, $0 < P_1' \leq p_1(x) \leq P_1$, $0 < P_2' \leq p_2(x) \leq P_2$, $0 \leq \varepsilon(x) \leq E(h) \leq E(h_0)$, $\lim_{h\to 0^+} E(h) = 0$.

More precisely, estimates of $b_1(x)$, $b_2(x)$ follow from estimates of $u_{11}(x)$, $u_{13}(x)$ for $S_3(h)$ in Propositions 4.2, 4.4 estimates of $b_3(x)$ follow from $u_{33}(x) = -u_{11}(x) - u_{22}(x)$ and estimates of $u_{11}(x)$, $u_{22}(x)$ for $S_3(h)$ in Propositions 4.1, 4.2. Estimates of $p_1(x)$, $p_2(x)$, $\varepsilon(x)$ and $\lim_{h\to 0^+} E(h) = 0$ follow by the same arguments as in Subcase 1a. Hence

$$\begin{aligned} f(a,x) &= \varepsilon(x)b_1(x)ah^{-3/2} - b_1^2(x)h^{-3} + b_1(x)b_3(x)h^{-2} + \varepsilon(x)p_1(x)a^2 \\ &-b_1(x)p_1(x)ah^{-3/2} + p_1(x)b_3(x)ah^{-1/2} - b_2^2(x)h^{-3} - p_2^2(x)a^2 - 2b_2(x)p_2(x)ah^{-3/2}. \end{aligned}$$

Let us first assume that $b_1(x) \geq 0$. Then for sufficiently small h we have

$$\begin{aligned} \varepsilon(x)b_1(x)ah^{-3/2} &< b_2(x)p_2(x)ah^{-3/2}, \\ p_1(x)b_3(x)ah^{-1/2} &< b_2(x)p_2(x)ah^{-3/2}, \\ b_1(x)b_3(x)h^{-2} &< b_2^2(x)h^{-3}, \\ \varepsilon(x)p_1(x)a^2 &< p_2^2(x)a^2, \end{aligned}$$

which implies $f(a,x) < 0$.

Now let us assume that $b_1(x) < 0$. By (44) for sufficiently small h we get

$$\begin{aligned} &(9/10)p_2^2(x)a^2 + b_1^2(x)h^{-3} > p_1^2(x)a^2 + b_1^2(x)h^{-3} \geq |2b_1(x)p_1(x)ah^{-3/2}|, \\ &p_1(x)\varepsilon(x)a^2 < (1/10)p_2^2(x)a^2, \\ &p_1(x)b_3(x)ah^{-1/2} < 2b_2(x)p_2(x)ah^{-3/2}, \end{aligned}$$

which implies $f(a,x) < 0$.

It follows that for sufficiently small $h_1 > 0$ and for all $0 < h \leq h_1$, $a \geq 0$, $x \in T_2(h)$ we have $f(a,x) < 0$.

Subcase 1c. $x \in T_3(h)$.

By (43), Propositions 4.1, 4.2, 4.3 and definition of w we have

$$f(a,x) = \begin{vmatrix} b_1(x)h^{-3/2} + p_1(x)a & -b_2(x)h^{-3/2} - p_2(x)a \\ -b_2(x)h^{-3/2} - p_2(x)a & \varepsilon(x)a - b_1(x)h^{-3/2} + b_3(x)h^{-1/2} \end{vmatrix},$$

where $0 < B_1' \leq b_1(x) \leq B_1$, $-B_2 \leq b_2(x) \leq B_2$, $0 < B_3' \leq b_3(x) \leq B_3$, $0 < P_1' \leq p_1(x) \leq P_1$, $0 < P_2' \leq p_2(x) \leq P_2$, $0 \leq \varepsilon(x) \leq E(h) \leq E(h_0)$, $\lim_{h\to 0^+} E(h) = 0$. More precisely, estimates of $b_1(x)$, $b_2(x)$ follow from estimates of $u_{11}(x)$, $u_{13}(x)$ for $S_2(h)$ in Propositions 4.2, 4.3 estimates of $b_3(x)$ follow from $u_{33}(x) = -u_{11}(x) - u_{22}(x)$ and estimates of $u_{11}(x)$, $u_{22}(x)$ for $S_2(h)$ in Propositions 4.1, 4.2, 4.3. Estimates of $p_1(x)$, $p_2(x)$, $\varepsilon(x)$ and $\lim_{h\to 0^+} E(h) = 0$ follow by the same arguments as in Subcase 1a.

For sufficiently small h we have

$$b_3(x)h^{-1/2} < b_1(x)h^{-3/2}/2, \tag{45}$$

$$\frac{2B_2}{B_1'}\varepsilon(x) < \frac{P_2'}{2} \tag{46}$$

$$\varepsilon(x)(p_1(x) + 2\varepsilon(x)) < \frac{p_2^2(x)}{4}. \tag{47}$$

If $\varepsilon(x)a - b_1(x)h^{-3/2} + b_3(x)h^{-1/2} < 0$ then clearly $f(a,x) < 0$. So we may assume $\varepsilon(x)a - b_1(x)h^{-3/2} + b_3(x)h^{-1/2} \geq 0$ which implies (see (45))

$$\varepsilon(x)a \geq b_1(x)h^{-3/2} - b_3(x)h^{-1/2} > (b_1(x)h^{-3/2})/2, \tag{48}$$

$$\varepsilon(x)a > \varepsilon(x)a - b_1(x)h^{-3/2} + b_3(x)h^{-1/2} \geq 0. \tag{49}$$

By (46) and (48) we get

$$|b_2(x)|h^{-3/2} = \frac{2|b_2(x)|}{b_1(x)}\frac{b_1(x)h^{-3/2}}{2} < \frac{2B_2}{B_1'}\varepsilon(x)a < \frac{P_2'a}{2} < \frac{p_2(x)a}{2}. \tag{50}$$

By (48), (49), (50), (47) we get

$$\begin{aligned} f(a,x) &\le (p_1(x)a + b_1(x)h^{-3/2})\varepsilon(x)a - \left(\frac{p_2(x)a}{2}\right)^2 \\ &\le (p_1(x)a + 2\varepsilon(x)a)\varepsilon(x)a - \frac{p_2^2(x)a^2}{4} < 0. \end{aligned}$$

It follows that for sufficiently small $h_1 > 0$ and for all $0 < h \le h_1$, $a \ge 0$, $x \in T_3(h)$ we have $f(a,x) < 0$.

Subcase 1d. $x \in T_4(h)$.

Note that for $x = (x_1, 0, x_3) \in T_4(h)$ we have $w_{33}(x) < 0$. We also have

$$u_{33}(x) = \int_{B(0,1)} K_{33}(x_1 - y_1, x_2 - y_2, x_3)\varphi(y_1, y_2)\, dy_1\, dy_2.$$

Recall that $K_{33}(x_1-y_1, x_2-y_2, x_3) = C_K x_3((x_1-y_1)^2+(x_2-y_2)^2+x_3^2)^{-7/2}(6x_3^2 - 9(x_1 - y_1)^2 - 9(x_2-y_2)^2)$. Hence to have $K_{33}(x_1 - y_1, -y_2, x_3) < 0$ for all $(y_1, y_2) \in B(0,1)$ and $x_1 \le -1$ it is sufficient to have $6x_3^2 - 9(x_1+1)^2 < 0$. Note that for $x = (x_1, 0, x_3) \in T_4(h)$ we have $0 < x_3 < -\sqrt{3/2}(x_1 + 1)$, $x_1 < -1$. It follows that $6x_3^2 - 9(x_1 + 1)^2 < 0$ and $u_{33}(x) < 0$. Hence $u_{33}(x) + aw_{33}(x) < 0$. Note that $u_{22}(x) + aw_{22}(x) < 0$ so $u_{11}(x) + aw_{11}(x) = -u_{22}(x) - aw_{22}(x) - u_{33}(x) - aw_{33}(x) > 0$. This and (43) implies that $f(a,x) < 0$ for any $a \ge 0$ and $x \in T_4(h)$.

Case 2. $x \in A_2'$.

This case follows from the same arguments as in subcase 1d.

Case 3. $x \in A_3'$.

Note that $w_{33}(x) > 0$ for $x \in A_3'$. Put $\overline{x}_3 = x_3 + \sqrt{3/2}$. We have

$$w_{11}(x) = C_K \overline{x}_3 (x_1^2 + \overline{x}_3^2)^{-7/2}(12x_1^2 - 3\overline{x}_3^2).$$

Note that

$$\{(x_1, 0, x_3) : w_{11}(x_1, 0, x_3) = 0, x_1 \le 0, x_3 > -\sqrt{3/2}\} = \{(x_1, 0, x_3) : x_3 + \sqrt{3/2} = -2x_1\}.$$

Put $T_1 = \left\{(x_1, 0, 0) : x_1 \in \left[\frac{-\sqrt{3}}{2\sqrt{2}}, 0\right]\right\}$, $T_2 = \left\{(x_1, 0, 0) : x_1 \in \left(-1, \frac{-\sqrt{3}}{2\sqrt{2}}\right)\right\}$. We have $A_2' = T_1 \cup T_2$. Note that $w_{11}(-\sqrt{3}/(2\sqrt{2}), 0, 0)) = 0$, $w_{11}(x) \le 0$ for $x \in T_1$ and $w_{11}(x) > 0$ for $x \in T_2$. Note also that for $x = (x_1, 0, 0) \in A_3'$ we have $u(x) = \varphi(x_1, 0) = C_B(1 - x_1^2)^{1/2}$ so $u_{11}(x) < 0$.

We will consider 2 subcases: $x \in T_1$, $x \in T_2$.

Subcase 3a. $x \in T_1$.

Note that $w_{11}(x) \le 0$, $u_{11}(x) < 0$ so $u_{11}(x) + aw_{11}(x) < 0$ for $a \ge 0$. It follows that $u_{33}(x) + aw_{33}(x) > 0$ (because $u_{33} + aw_{33} = -(u_{11} + aw_{11} + u_{22} + aw_{22})$). Hence $f(a,x) < 0$.

Subcase 3b. $x \in T_2$.

For $(y_1, y_2) \in B(0,1)$ and $y = (y_1, y_2, 0)$ we have $u(y) = \varphi(y_1, y_2) = C_B(1 - y_1^2 - y_2^2)^{1/2}$. Therefore for $x \in T_2$ we obtain $u_{11}(x) = \varphi_{11}(x_1, 0) = -C_B(1 - x_1^2)^{-3/2}$, $u_{33}(x) = -\varphi_{11}(x_1, 0) - \varphi_{22}(x_1, 0) = C_B(1 - x_1^2)^{-3/2}(2 - x_1^2)$. Hence

$$u_{33}(x) < 2|u_{11}(x)|. \tag{51}$$

For $x \in T_2$ we also have $-w_{22}(x) - w_{11}(x) = w_{33}(x) > 0$ so

$$|w_{22}(x)| > |w_{11}(x)|. \tag{52}$$

Note that for $x = (x_1, x_2, x_3) = (x_1, 0, 0) \in T_2$ we have $\frac{\overline{x}_3}{|x_1|} = \frac{\sqrt{3/2}}{|x_1|}$ and $\frac{\overline{x}_3}{|x_1|} \in \left(\sqrt{\frac{3}{2}}, 2\right)$.

For $x \in T_2$ we have

$$\frac{|w_{13}(x)|}{|w_{22}(x)|} = \frac{|x_1|}{\overline{x}_3}\frac{(12\overline{x}_3^2 - 3x_1^2)}{(3x_1^2 + 3\overline{x}_3^2)} = \frac{|x_1|}{\overline{x}_3}\left(4 - \frac{5}{\left(\frac{\overline{x}_3}{|x_1|}\right)^2 + 1}\right) > \frac{2|x_1|}{\overline{x}_3} > 1,$$

so

$$|w_{13}(x)| > |w_{22}(x)|. \tag{53}$$

If $a = 0$ then by explicit formulas $f(a, x) < 0$. If $a > 0$ and $u_{11}(x) + aw_{11}(x) \le 0$ then $u_{33}(x) + aw_{33}(x) = -(u_{11}(x) + aw_{11}(x) + u_{22}(x) + aw_{22}(x)) > 0$ and $u_{13}(x) + aw_{13}(x) = aw_{13}(x) \neq 0$ (see (53)) so $f(a, x) < 0$. So we may assume $a > 0$ and $u_{11}(x) + aw_{11}(x) > 0$.

Again by (43) and (51), (53) we get

$$f(a,x) < \begin{vmatrix} u_{11}(x) + aw_{11}(x) & a|w_{22}(x)| \\ a|w_{22}(x)| & 2|u_{11}(x)| - aw_{11}(x) - aw_{22}(x) \end{vmatrix}.$$

Hence

$$\begin{aligned} f(a,x) &< -2|u_{11}(x)|^2 + 3|u_{11}(x)|w_{11}(x)a - |u_{11}(x)||w_{22}(x)|a \\ &\quad -w_{11}^2(x)a^2 + w_{11}(x)|w_{22}(x)|a^2 - |w_{22}(x)|^2a^2. \end{aligned}$$

By (52) this is bounded from above by

$$\begin{aligned} & -2|u_{11}(x)|^2 + 2|u_{11}(x)||w_{11}(x)|a - w_{11}^2(x)a^2 + w_{11}(x)|w_{22}(x)|a^2 - |w_{22}(x)|^2a^2 \\ = & -\left(\sqrt{2}|u_{11}(x)| - \frac{w_{11}(x)a}{\sqrt{2}}\right)^2 - \left(\frac{w_{11}(x)a}{\sqrt{2}} - \frac{|w_{22}(x)|a}{\sqrt{2}}\right)^2 - \left(\frac{|w_{22}(x)|a}{\sqrt{2}}\right)^2 \\ < & \ 0. \end{aligned}$$

Case 4. $x \in A_4'$.

Recall that $\overline{x}_3 = x_3 + \sqrt{3/2}$ and put $\overline{x} = (x_1, x_2, \overline{x}_3)$. Recall also that $w(x) = K(\overline{x})$. We have

$$\begin{aligned} K_{11}(\overline{x}) &= C_K\overline{x}_3(x_1^2 + x_2^2 + \overline{x}_3^2)^{-7/2}(12x_1^2 - 3x_2^2 - 3\overline{x}_3^2), \\ K_{13}(\overline{x}) &= C_K x_1(x_1^2 + x_2^2 + \overline{x}_3^2)^{-7/2}(12\overline{x}_3^2 - 3x_1^2 - 3x_2^2), \\ K_{33}(\overline{x}) &= C_K\overline{x}_3(x_1^2 + x_2^2 + \overline{x}_3^2)^{-7/2}(6\overline{x}_3^2 - 9x_1^2 - 9x_2^2). \end{aligned}$$

For any $M \ge 10$ put

$$\begin{aligned} T_1(M) &= \{(x_1, 0, x_3) : \overline{x}_3 = M, x_1 \le 0, \overline{x}_3 \ge 3|x_1|\}, \\ T_2(M) &= \{(x_1, 0, x_3) : \overline{x}_3 = M, x_1 \le 0, \sqrt{3/2}|x_1| \le \overline{x}_3 < 3|x_1|\}, \\ T_3(M) &= \{(x_1, 0, x_3) : \overline{x}_3 = M, x_1 \le 0, |x_1| \le \overline{x}_3 < \sqrt{3/2}|x_1|\} \\ &\quad \cup \{(x_1, 0, x_3) : x_1 = -M, 0 < \overline{x}_3 < M\}. \end{aligned}$$

We will consider 3 subcases: $x \in T_1(M)$, $x \in T_2(M)$, $x \in T_3(M)$.

Subcase 4a. $x \in T_1(M)$.

Put $B = B(0, 1) \subset \mathbb{R}^2$. We have

$$\begin{aligned} u_{11}(x) &= \int_B (K_{11}(x_1 - y_1, -y_2, x_3) - K_{11}(\overline{x}))\varphi(y_1, y_2)\, dy_1\, dy_2 \\ &\quad + K_{11}(\overline{x}) \int_B \varphi(y_1, y_2)\, dy_1\, dy_2, \end{aligned}$$

$$K_{11}(\overline{x}) = \frac{C_K\overline{x}_3(12x_1^2 - 3\overline{x}_3^2)}{(x_1^2 + \overline{x}_3^2)^{7/2}} < \frac{C_K\overline{x}_3^3\left(\frac{12}{9} - 3\right)}{(x_1^2 + \overline{x}_3^2)^{7/2}} < \frac{-c}{\overline{x}_3^4}. \tag{54}$$

For $(y_1, y_2) \in B$ we also have

$$|K_{11}(x_1 - y_1, -y_2, x_3) - K_{11}(\overline{x})| \le (|y_1| + |y_2| + |x_3 - \overline{x}_3|)|\nabla K_{11}(\xi)| \le 4|\nabla K_{11}(\xi)|,$$

where ξ is a point between $(x_1 - y_1, -y_2, x_3)$ and $\overline{x} = (x_1, 0, \overline{x}_3)$. For such ξ we have

$$|\nabla K_{11}(\xi)| \leq \frac{c}{x_3^5}. \tag{55}$$

By (54), (55) for sufficiently large M and all $x \in T_1(M)$ we have $u_{11}(x) < 0$. We also have $aw_{11}(x) = aK_{11}(\overline{x}) < 0$ for $a \geq 0$, $x \in T_1(M)$. Hence $u_{11}(x) + aw_{11}(x) < 0$ which implies $f(a, x) < 0$. It follows that for sufficiently large $M_1 \geq 10$ and for all $M \geq M_1$, $a \geq 0$, $x \in T_1(M)$ we have $f(a, x) < 0$.

Subcase 4b. $x \in T_2(M)$.

First we need the following auxiliary lemma.

Lemma 5.3. *Let $f(y_1, y_3) = -6y_1^3 - 3y_1^2 y_3 + 24y_1 y_3^2 - 3y_3^3$. For any $y_3 > 0$ and $y_1 \in [y_3/3, y_3]$ we have $f(y_1, y_3) > 4y_3^3$.*

Proof. The proof is elementary. Fix $y_3 > 0$ and put $g(y_1) = f(y_1, y_3)$. We have $g'(y_1) = -18y_1^2 - 6y_1 y_3 + 24y_3^2$, $g'(y_1) = 0$ for $y_1 = (-8/6)y_3$ and $y_1 = y_3$ so g is increasing for $y_1 \in [(-8/6)y_3, y_3]$. We also have $g(y_3/3) = (40/9)y_3^3$ so for any $y_1 \in [y_3/3, y_3]$ we have $g(y_1) > 4y_3^3$. □

Put $b = \int_B \varphi(y_1, y_2)\, dy_1\, dy_2$. For $x \in T_2(M)$ we have

$$f(a, x) = \begin{vmatrix} K_{11}(\overline{x})(a + b) + \varepsilon_{11}(x) & K_{13}(\overline{x})(a + b) + \varepsilon_{13}(x) \\ K_{13}(\overline{x})(a + b) + \varepsilon_{13}(x) & K_{33}(\overline{x})(a + b) + \varepsilon_{33}(x) \end{vmatrix},$$

where

$$\varepsilon_{ij}(x) = \int_B (K_{ij}(x_1 - y_1, -y_2, x_3) - K_{ij}(\overline{x}))\varphi(y_1, y_2)\, dy_1\, dy_2$$

for $(i, j) = (1, 1)$ or $(1, 3)$ or $(3, 3)$. For $(y_1, y_2) \in B$ we have

$$|K_{ij}(x_1 - y_1, -y_2, x_3) - K_{ij}(\overline{x})| \leq (|y_1| + |y_2| + |x_3 - \overline{x}_3|)|\nabla K_{ij}(\xi)| \leq 4|\nabla K_{ij}(\xi)|,$$

where ξ is a point between $(x_1 - y_1, -y_2, x_3)$ and $\overline{x} = (x_1, 0, \overline{x}_3)$. We have $|\nabla K_{ij}(\xi)| \leq c x_3^{-5}$, so

$$|\varepsilon_{ij}(x)| \leq \frac{cb}{x_3^5}. \tag{56}$$

Put

$$f_1(a, x) = \begin{vmatrix} K_{11}(\overline{x})(a + b) & K_{13}(\overline{x})(a + b) \\ K_{13}(\overline{x})(a + b) & K_{33}(\overline{x})(a + b) \end{vmatrix}.$$

We have $|K_{ij}(\overline{x})| \leq c x_3^{-4}$ so by (56) we obtain

$$|f(a, x) - f_1(a, x)| \leq c(a + b) b x_3^{-9}. \tag{57}$$

On the other hand we have

$$\begin{aligned} |f_1(a, x)| &\geq (a + b)^2 \left(K_{13}^2(\overline{x}) - K_{11}(\overline{x}) K_{33}(\overline{x})\right) \\ &\geq (a + b)^2 \left(K_{13}^2(\overline{x}) - \left(\frac{K_{11}(\overline{x}) + K_{33}(\overline{x})}{2}\right)^2\right) \\ &= (a + b)^2 \left(|K_{13}(\overline{x})|^2 - \left(\frac{|K_{22}(\overline{x})|}{2}\right)^2\right). \end{aligned} \tag{58}$$

We have

$$|K_{13}(\overline{x})| - \frac{|K_{22}(\overline{x})|}{2} = \frac{1}{2} C_K (|x_1|^2 + \overline{x}_3^2)^{-7/2} (-6|x_1|^3 - 3|x_1|^2 \overline{x}_3 + 24|x_1| \overline{x}_3^2 - 3\overline{x}_3^3).$$

By Lemma 5.3 we obtain

$$|K_{13}(\overline{x})| - \frac{|K_{22}(\overline{x})|}{2} \geq \frac{1}{2} C_K (|x_1|^2 + \overline{x}_3^2)^{-7/2} 4\overline{x}_3^3 \geq c x_3^{-4}.$$

Using this and (58) we obtain

$$|f_1(a,x)| \geq (a+b)^2 \left(|K_{13}(\overline{x})| - \frac{|K_{22}(\overline{x})|}{2} \right)^2 \geq c(a+b)^2 x_3^{-8}.$$

It follows that $f_1(a,x) < -c(a+b)^2 x_3^{-8}$. Using this and (57) we obtain that for sufficiently large $M_1 \geq 10$ and for all $M \geq M_1$, $a \geq 0$, $x \in T_2(M)$ we have $f(a,x) < 0$.

Subcase 4c. $x \in T_3(M)$.

This subcase follows from the same arguments as in subcase 1d. □

proof of Proposition 5.1. On the contrary assume that there exists $z = (z_1, z_2, z_3) \in \mathbb{R}^3 \setminus (B^c(0,1) \times \{0\})$ such that $H(u)(z) \leq 0$. By Lemma 2.7 we may assume that $z_1 \geq 0$. By an explicit formula for φ and Lemma 4.7 we may assume that $z_1 > 0$. Define

$$\Psi^{(b)}(x) = (1-b)u(x) + bw(x), \qquad b \in [0,1],$$

where w is given by (42). By direct computation for any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ with $x_3 > -\sqrt{3/2}$ we have

$$H(w)(x) = C_K^3 \frac{27(x_3 + \sqrt{3/2})(x_1^2 + x_2^2 + 2(x_3 + \sqrt{3/2})^2)}{(x_1^2 + x_2^2 + (x_3 + \sqrt{3/2})^2)^{15/2}} > 0.$$

Recall that $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ and put $\Omega = \mathbb{R}^3_+ \setminus (A_1 \cup A_2 \cup A_4)$, where A_1, A_2, A_4 are sets from Lemma 5.2. By this lemma we obtain that $z \in \Omega$ and $H(\Psi^{(b)})(x) > 0$ for all $b \in [0,1]$ and $x \in \partial\Omega$. Note that $\Psi^{(0)} = u$ and $\Psi^{(1)} = w$, $H(\Psi^{(0)})(z) < 0$, $H(\Psi^{(1)})(x) > 0$ for all $x \in \overline{\Omega}$. Clearly, all second partial derivatives of $\Psi^{(b)}$ are uniformly Lipschitz continuos on $\overline{\Omega}$ that is

$$\exists c \ \forall b \in [0,1] \ \forall x, y \in \overline{\Omega} \ \forall i, j \in \{1,2,3\} \qquad \left| \Psi^{(b)}_{ij}(x) - \Psi^{(b)}_{ij}(x) \right| \leq c|x - y|.$$

It follows that there exists $b_0 \in [0,1)$ such that $H(\Psi^{(b_0)})(z_0) = 0$ for some $z_0 \in \Omega$ and $H(\Psi^{(b_0)})(x) \geq 0$ for all $x \in \overline{\Omega}$. This gives contradiction with Theorem 1.6. □

6. Concavity of φ

In this section we prove the main result of this paper Theorem 1.1. This is done by using the method of continuity, Lewy's Theorem 1.6 and results from Sections 3, 4, 5.

For any $\varepsilon \geq 0$ we define

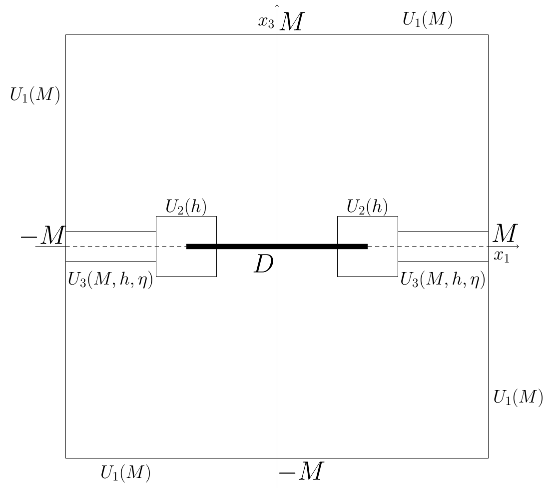
$$v^{(\varepsilon)}(x) = u(x) + \varepsilon \left(-\frac{x_1^2}{2} - \frac{x_2^2}{2} + x_3^2 \right), \qquad x \in \mathbb{R}^3 \setminus (D^c \times \{0\}), \tag{59}$$

where u is the harmonic extension of φ given by (6-10) and φ is the solution of (1-2) for an open bounded set $D \subset \mathbb{R}^2$. When D is not fixed we will sometimes write $v^{(\varepsilon, D)}$ instead of $v^{(\varepsilon)}$.

Lemma 6.1. *Let $C_1 > 0$, $R_1 > 0$, $\kappa_2 \geq \kappa_1 > 0$, $D \in F(C_1, R_1, \kappa_1, \kappa_2)$, φ be the solution of (1-2) for D and u the harmonic extension of φ given by (6-10). For any $\varepsilon \geq 0$ let $v^{(\varepsilon)}$ be given by (59). For any $(x_1, x_2, x_3) \in \mathbb{R}^3_+$ we have $H(v^{(\varepsilon)})(x_1, x_2, -x_3) = H(v^{(\varepsilon)})(x_1, x_2, x_3)$.*

The proof of this lemma is similar to the proof of Lemma 2.7 and it is omitted.

Proposition 6.2. *Fix $C_1 > 0$, $R_1 > 0$, $\kappa_2 \geq \kappa_1 > 0$ and $D \in F(C_1, R_1, \kappa_1, \kappa_2)$. Denote $\Lambda = \{C_1, R_1, \kappa_1, \kappa_1\}$. Let φ be the solution of (1-2) for D, u the harmonic extension of φ*



A cross section parallel to the x_1x_3 plane

FIGURE 7

and $v^{(\varepsilon)}$ *given by (59). For* $M \geq 10$, $h \in (0, 1/2]$, $\eta \in (0, 1/2]$ *we define (see Figure 7)*

$$\begin{aligned}
U_1(M) &= \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq M^2, x_3 = M \text{ or } x_3 = -M\} \\
&\cup \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = M^2, x_3 \in [-M, M] \setminus \{0\}\}, \\
U_2(h) &= \{x \in \mathbb{R}^3 : (x_1, x_2) \in D, \delta_D((x_1, x_2)) \leq h, x_3 \in [-h, h]\} \\
&\cup \{x \in \mathbb{R}^3 : (x_1, x_2) \notin D, \delta_D((x_1, x_2)) \leq h, x_3 \in [-h, h] \setminus \{0\}\}, \\
U_3(M, h, \eta) &= \{x \in \mathbb{R}^3 : (x_1, x_2) \notin D, \delta_D((x_1, x_2)) \geq h, x_1^2 + x_2^2 \leq M^2, \\
&\qquad x_3 \in [-\eta, \eta] \setminus \{0\}\}, \\
U_4(h) &= \{x \in \mathbb{R}^3 : (x_1, x_2) \in D, \delta_D((x_1, x_2)) \leq h, x_3 = 0\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\exists c_1 = c_1(\Lambda) \in (0, 1] \; \exists M_0 \geq 10 \; \exists h_1 = h_1(\Lambda) \in (0, 1/2] \; \forall M \geq M_0 \; \forall \varepsilon \in (0, c_1 M^{-7}] \\
&\exists \eta = \eta(\Lambda, M, \varepsilon) \in (0, 1/2] \; \exists C = C(\Lambda, M, \varepsilon) > 0 \; \forall x \in U_1(M) \cup U_2(h_1) \cup U_3(M, h_1, \eta) \\
&\qquad H(v^{(\varepsilon)})(x) \geq C.
\end{aligned}$$

We also have

$$\exists \tilde{h} = \tilde{h}(\Lambda) \in (0, 1/2] \; \exists \tilde{C} = \tilde{C}(\Lambda) > 0 \; \forall x \in U_4(\tilde{h}) \qquad H(u)(x) \geq \tilde{C}. \tag{60}$$

Proof. In the whole proof we use convention stated in Remark 2.9. We have $H(v^{(\varepsilon)})(x) = W_1(x) + W_2(x) + W_3(x)$, where

$$\begin{aligned}
W_1(x) &= v_{12}^{(\varepsilon)}(x) \left(v_{13}^{(\varepsilon)}(x) v_{23}^{(\varepsilon)}(x) - v_{12}^{(\varepsilon)}(x) v_{33}^{(\varepsilon)}(x) \right), \\
W_2(x) &= -v_{23}^{(\varepsilon)}(x) \left(v_{11}^{(\varepsilon)}(x) v_{23}^{(\varepsilon)}(x) - v_{13}^{(\varepsilon)}(x) v_{12}^{(\varepsilon)}(x) \right), \\
W_3(x) &= v_{22}^{(\varepsilon)}(x) f(\varepsilon, x), \\
f(\varepsilon, x) &= v_{11}^{(\varepsilon)}(x) v_{33}^{(\varepsilon)}(x) - (v_{13}^{(\varepsilon)}(x))^2.
\end{aligned}$$

The proof consists of 3 parts.

Part 1. Estimates on $U_1(M)$.

We may assume in this part that $x_2 = 0$, $x_3 > 0$, $x_1 \leq 0$.

By formulas $u_{ij}(x) = \int_D K_{ij}(x_1-y_1, x_2-y_2, x_3)\varphi(y_1, y_2)\, dy_1\, dy_2$ and explicit formulas for K_{ij} (see Section 2), there exist $M_1 \geq 10$ and c such that for any $M \geq M_1$ and $x \in U_1(M)$ we have $|u_{11}(x)| \leq cx_3M^{-5}$, $u_{22}(x) \approx -x_3M^{-5}$, $|u_{33}(x)| \leq cx_3M^{-5}$, $|u_{13}(x)| \leq cM^{-4}$, $|u_{23}(x)| \leq cM^{-5}$, $|u_{12}(x)| \leq cx_3M^{-6}$.

Let us fix arbitrary $M \geq M_1$.

Let $x \in U_1(M)$ (recall that we assume that $x_2 = 0$, $x_3 > 0$, $x_1 \leq 0$). We have

$$|W_1(x)| \leq cx_3M^{-6}(M^{-4}M^{-5} + x_3M^{-6}(x_3M^{-5} + 2\varepsilon)) \leq cx_3M^{-15} + c\varepsilon M^{-10}, \quad (61)$$

$$|W_2(x)| \leq cM^{-5}((x_3M^{-5} + \varepsilon)M^{-5} + M^{-4}x_3M^{-6}) \leq cx_3M^{-15} + c\varepsilon M^{-10}. \quad (62)$$

Now we estimate $W_3(x)$. We have

$$v_{22}^{(\varepsilon)}(x) = u_{22}(x) - \varepsilon \approx -cx_3M^{-5} - \varepsilon. \quad (63)$$

The most important is the estimate of $f(\varepsilon, x)$. To obtain this estimate we will consider 6 cases.

Case 1.1. $x_3 = M$, $|x_1| < x_3/3$.

Put $m(x) = C_K(x_1^2 + x_3^2)^{-7/2}$. We have

$$u_{11}(x) \approx K_{11}(x) = m(x)x_3(12x_1^2 - 3x_3^2) < cM^{-7}x_3\left(12\left(\frac{x_3}{3}\right)^2 - 3x_3^2\right),$$

so $u_{11}(x) \leq -cM^{-4}$. We also have

$$u_{33}(x) \approx K_{33}(x) = m(x)x_3(6x_3^2 - 9x_1^2) \geq cM^{-7}x_3\left(6x_3^2 - 9\left(\frac{x_3}{3}\right)^2\right),$$

so $u_{33}(x) \geq cM^{-4}$. Therefore for any $\varepsilon \geq 0$ we have $v_{11}^{(\varepsilon)}(x) \leq -cM^{-4}$, $v_{33}^{(\varepsilon)}(x) \geq cM^{-4}$. Hence $f(\varepsilon, x) \leq -cM^{-8}$.

Case 1.2. $x_3 = M$, $|x_1| \in [x_3/3, x_3/\sqrt{3/2}]$.

By the arguments from Subcase 4b in the proof of Lemma 5.2 we have $u_{11}(x)u_{33}(x) - (u_{13}(x))^2 < -cM^{-8}$ for sufficiently large M. For any $\varepsilon \geq 0$ we have

$$\left|f(\varepsilon, x) - \left(u_{11}(x)u_{33}(x) - (u_{13}(x))^2\right)\right| \leq 2\varepsilon^2 + 2\varepsilon|u_{11}(x)| + \varepsilon|u_{33}(x)|.$$

For any $c_1 \in (0, 1]$ and all $\varepsilon \in (0, c_1M^{-7}]$ this is bounded from above by cc_1M^{-11}. It follows that for sufficiently small $c_1 \in (0, 1]$, for sufficiently large M and all $\varepsilon \in (0, c_1M^{-7}]$ we have $f(\varepsilon, x) < -cM^{-8}$.

Case 1.3. $x_3 = M$, $|x_1| \in [x_3/\sqrt{3/2}, x_3]$.

We have

$$u_{11}(x) \approx K_{11}(x) = m(x)x_3(12x_1^2 - 3x_3^2) \approx M^{-7}x_3\left(12\frac{x_3^2}{3/2} - 3x_3^2\right) \approx M^{-4}.$$

For $y \in D \subset B(0, 1)$ we also have

$$\begin{aligned} & K_{33}(x_1 - y_1, -y_2, x_3) \leq C_Kx_3((x_1 - y_1)^2 + y_2^2 + x_3^2)^{-7/2}(6x_3^2 - 9(x_1 - y_1)^2) \\ = \; & C_Kx_3((x_1 - y_1)^2 + y_2^2 + x_3^2)^{-7/2}(6x_3^2 - 9x_1^2 + 18x_1y_1 - 9y_1^2) \leq cM^{-5}, \end{aligned}$$

so $u_{33}(x) \leq cM^{-5}$. For sufficiently small $c_1 \in (0, 1]$ and all $\varepsilon \in (0, c_1M^{-7}]$ we obtain $v_{11}^{(\varepsilon)}(x) \approx M^{-4}$, $v_{33}^{(\varepsilon)}(x) \leq cM^{-5}$. We also have $u_{13}(x) \approx K_{13}(x) = m(x)x_1(12x_3^2 - 3x_1^2) \geq cM^{-4}$. It follows that for sufficiently small c_1, for sufficiently large M and all $\varepsilon \in (0, c_1M^{-7}]$ we have $f(\varepsilon, x) < -cM^{-8}$.

Case 1.4. $x_3 \in [M/4, M]$, $x_1 = -M$.

We have

$$u_{11}(x) \approx K_{11}(x) = m(x)x_3(12x_1^2 - 3x_3^2),$$

so $u_{11}(x) \geq cM^{-4}$. We also have

$$u_{33}(x) \approx K_{33}(x) = m(x)x_3(6x_3^2 - 9x_1^2),$$

so $u_{33}(x) \le -cM^{-4}$. Therefore for sufficiently small $c_1 \in (0,1]$ and all $\varepsilon \in (0, c_1 M^{-7}]$ we have $v_{11}^{(\varepsilon)}(x) \ge cM^{-4}$, $v_{33}^{(\varepsilon)}(x) \le -cM^{-4}$. Hence $f(\varepsilon, x) \le -cM^{-8}$.

Case 1.5. $x_3 \in [1, M/4]$, $x_1 = -M$.

We have

$$u_{13}(x) \approx K_{13}(x) = m(x) x_1 (12 x_3^2 - 3 x_1^2),$$

so $u_{13}(x) \le -cM^{-4}$. We also have

$$u_{11}(x) \approx K_{11}(x) = m(x) x_3 (12 x_1^2 - 3 x_3^2),$$

$$u_{33}(x) \approx K_{33}(x) = m(x) x_3 (6 x_3^2 - 9 x_1^2),$$

so $u_{11}(x) \ge cM^{-5}$, $u_{33}(x) \le -cM^{-5}$. Therefore for sufficiently small $c_1 \in (0,1]$ and all $\varepsilon \in (0, c_1 M^{-7}]$ we have $v_{11}^{(\varepsilon)}(x) \ge cM^{-5}$, $v_{33}^{(\varepsilon)}(x) \le -cM^{-5}$. Hence $f(\varepsilon, x) \le -cM^{-8}$.

Case 1.6. $x_3 \in (0, 1]$, $x_1 = -M$.

By similar arguments as in Case 1.5 we get $u_{13}(x) \le -cM^{-4}$, $|u_{11}(x)| \le cM^{-5}$, $|u_{33}(x)| \le cM^{-5}$. Therefore for sufficiently small $c_1 \in (0,1]$ and all $\varepsilon \in (0, c_1 M^{-7}]$ we have $|v_{11}^{(\varepsilon)}(x)| \le cM^{-5}$, $|v_{33}^{(\varepsilon)}(x)| \le cM^{-5}$. Hence for sufficiently small $c_1 \in (0,1]$, for sufficiently large M and all $\varepsilon \in (0, c_1 M^{-7}]$ we have $f(\varepsilon, x) \le -cM^{-8}$.

Finally in all 6 cases we get that for sufficiently small $c_1 \in (0,1]$, for sufficiently large M and all $\varepsilon \in (0, c_1 M^{-7}]$ we have $f(\varepsilon, x) \le -cM^{-8}$. By (63) we get $W_3(x) = v_{22}^{(\varepsilon)}(x) f(\varepsilon, x) \ge c x_3 M^{-13} + c\varepsilon M^{-8}$. By (61), (62) we have $|W_1(x) + W_2(x)| \le c x_3 M^{-15} + c\varepsilon M^{-10}$. Recall that $H(v^{(\varepsilon)})(x) = W_1(x) + W_2(x) + W_3(x)$. It follows that there exists sufficiently small $c_1' = c_1'(\Lambda) \in (0,1]$ and sufficiently large $M_0 \ge M_1 \ge 10$ such that for any $M \ge M_0$ and $\varepsilon \in (0, c_1' M^{-7}]$ and all $x \in U_1(M)$ we have $H(v^{(\varepsilon)})(x) \ge c\varepsilon M^{-8}$.

Let us fix the above M_0 and $M \ge M_0$ in the rest of the proof of this proposition.

Part 2. Estimates on $U_2(h)$.

We will use notation and results from Section 4 (Propositions 4.1-4.6). In particular we choose a point on ∂D and choose a Cartesian coordinate system with origin at that point in the same way as in Section 4 (see Figures 1, 4). Let $h \in (0, h_0]$, where h_0 denotes the minimum of constants h_0 from Propositions 4.1-4.6. By Lemma 6.1 we may assume $x_3 \ge 0$, by continuity we may assume $x_3 > 0$. It follows that it is enough to estimate $H(v^{(\varepsilon)})(x)$ for $x \in S_1(h) \cup S_2(h) \cup S_3(h) \cup S_4(h)$. We will consider 2 cases. Assume that $\varepsilon \in (0,1]$.

Case 2.1. $x \in S_1(h) \cup S_2(h) \cup S_3(h)$.

If $x \in S_1(h) \cup S_3(h)$ we have $(v_{13}^{(\varepsilon)}(x))^2 = u_{13}^2(x) \ge ch^{-3}$, $v_{11}^{(\varepsilon)}(x) v_{33}^{(\varepsilon)}(x) = u_{11}(x) u_{33}(x) + 2\varepsilon u_{11}(x) - \varepsilon u_{33}(x) - 2\varepsilon^2$, $|2\varepsilon u_{11}(x)| \le c\varepsilon h^{-3/2}$, $|-\varepsilon u_{33}(x)| \le c\varepsilon h^{-3/2}$.

If $u_{11}(x) \le 0$ or $u_{33}(x) \le 0$ then $u_{11}(x) u_{33}(x) \le 0$ (recall that $u_{11}(x) + u_{33}(x) = -u_{22}(x) > 0$). If $u_{11}(x) > 0$ and $u_{33}(x) > 0$ then

$$u_{11}(x) u_{33}(x) \le \left(\frac{u_{11}(x) + u_{33}(x)}{2} \right)^2 = \left(\frac{u_{22}(x)}{2} \right)^2 \le ch^{-1}.$$

Hence $f(\varepsilon, x) = -(v_{13}^{(\varepsilon)}(x))^2 + v_{11}^{(\varepsilon)}(x) v_{33}^{(\varepsilon)}(x) \le -ch^{-3}$ for sufficiently small h and all $\varepsilon \in (0,1]$.

If $x \in S_2(h)$ we have $u_{11}(x) \approx h^{-3/2}$, $u_{33}(x) \approx -h^{-3/2}$. Hence for sufficiently small h and all $\varepsilon \in (0,1]$ we have $v_{11}^{(\varepsilon)}(x) \approx h^{-3/2}$, $v_{33}^{(\varepsilon)}(x) \approx -h^{-3/2}$ and $f(\varepsilon, x) \le -ch^{-3}$.

Hence for any $x \in S_1(h) \cup S_2(h) \cup S_3(h)$ for sufficiently small h and all $\varepsilon \in (0,1]$ we have $f(\varepsilon, x) \le -ch^{-3}$. We have $v_{22}^{(\varepsilon)}(x) \approx -x_3 h^{-3/2} - \varepsilon$. It follows that $W_3(x) =$

$v_{22}^{(\varepsilon)}(x)f(\varepsilon,x) \geq cx_3h^{-9/2} + c\varepsilon h^{-3}$. We also have

$$\begin{aligned} |W_1(x)| &\leq cx_3h^{-3/2}|\log h| \left(h^{-3/2}h^{-1/2}|\log h| + (2\varepsilon + x_3h^{-5/2})x_3h^{-3/2}|\log h|\right) \\ &\leq cx_3h^{-7/2}|\log h|^2 + c\varepsilon h^{-1}|\log h|^2, \end{aligned}$$

$$\begin{aligned} |W_2(x)| &\leq ch^{-1/2}|\log h| \left((\varepsilon + x_3h^{-5/2})h^{-1/2}|\log h| + h^{-3/2}x_3h^{-3/2}|\log h|\right) \\ &\leq cx_3h^{-7/2}|\log h|^2 + c\varepsilon h^{-1}|\log h|^2. \end{aligned}$$

Hence there exists sufficiently small h_1' such that for all $h \in (0, h_1']$ and $\varepsilon \in (0,1]$ we have $H(v^{(\varepsilon)})(x) \geq cx_3h^{-9/2} + c\varepsilon h^{-3}$.

Case 2.2. $x \in S_4(h)$.

For sufficiently small h and all $\varepsilon \in [0,1]$ we have $W_3(x) \geq ch^{-1/2}h^{-3} = ch^{-14/4}$,

$$\begin{aligned} |W_1(x)| &\leq ch^{-1/2}|\log h| \left(h^{-3/2}h^{-3/4}|\log h| + h^{-3/2}h^{-1/2}|\log h|\right) \\ &\leq ch^{-11/4}|\log h|^2, \end{aligned}$$

$$\begin{aligned} |W_2(x)| &\leq ch^{-3/4}|\log h| \left(h^{-3/2}h^{-3/4}|\log h| + h^{-1/2}|\log h|h^{-3/2}\right) \\ &\leq ch^{-12/4}|\log h|^2. \end{aligned}$$

So there exists sufficiently small h_1'' such that for all $h \in (0, h_1'']$ and $\varepsilon \in [0,1]$ we have $H(v^{(\varepsilon)})(x) \geq ch^{-14/4}$.

Since $u = v^{(0)}$ is continuous in a neighbourhood of any $x \in D \times \{0\}$ we obtain (60).

Let us fix $h_1 = h_1' \wedge h_1''$ in the rest of the proof of this proposition.

Part 3. Estimates on $U_3(M, h_1, \eta)$.

Let us choose arbitrary point on ∂D and choose a Cartesian coordinate system in the same way as in Part 2. Note that it is enough to estimate $H(v^{(\varepsilon)})(x)$ for $x \in U_3'(M, h_1, \eta) = \{(x_1, x_2, x_3) : x_2 = 0, x_1 \in [-M, -h_1], x_3 \in (0, \eta]\}$ and sufficiently small $\eta = \eta(\Lambda, M, \varepsilon)$.

Let $x \in U_3'(M, h_1, 1/2)$. Note that $\text{dist}(x, \partial D) \geq h_1$. By formulas $u_{ij}(x) = \int_D K_{ij}(x_1 - y_1, x_2 - y_2, x_3)\varphi(y_1, y_2)\, dy_1\, dy_2$ and explicit formulas for K_{ij} (see Section 2) we have $|u_{11}(x)| \leq cx_3h_1^{-5}$, $|u_{22}(x)| \leq cx_3h_1^{-5}$, $|u_{33}(x)| \leq cx_3h_1^{-5}$, $|u_{13}(x)| \leq ch_1^{-4}$, $|u_{23}(x)| \leq ch_1^{-4}$, $|u_{12}(x)| \leq cx_3h_1^{-5}$. Note also that by our choice of coordinate system for any $y = (y_1, y_2) \in D$ we have $y_1 > 0$. From now on let us assume additionally that $x = (x_1, x_2, x_3) \in U_3'(M, h_1, 1/2)$ is such that $x_3 \leq |x_1|/\sqrt{6}$ (this condition implies $12x_3^2 \leq 2x_1^2$). For such $x = (x_1, x_2, x_3)$ and any $y = (y_1, y_2) \in D$ we have $12x_3^2 - 3(x_1 - y_1)^2 - 3(x_2 - y_2)^2 \leq -(x_1 - y_1)^2 \leq -x_1^2 \leq -h_1^2$.

It follows that

$$\begin{aligned} |u_{13}(x)| &= \left| C_K \int_D \frac{(x_1 - y_1)(12x_3^2 - 3(x_1 - y_1)^2 - 3(x_2 - y_2)^2)}{((x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2)^{7/2}} \varphi(y_1, y_2)\, dy_1\, dy_2 \right| \\ &\geq \frac{\tilde{C}h_1^3}{M^7}. \end{aligned} \tag{64}$$

The constant $\tilde{C}$ will play an important role in the rest of the proof and this is the reason why it is not as usual denoted by c. Clearly, $\tilde{C}$ depends only on Λ.

Let us recall that in Parts 1 and 2 of this proof we have fixed constants M_0, $M \geq M_0$, h_1. At the end of Part 1 we have chosen a constant $c_1' \in (0,1]$. Let us choose a constant c_1 to be

$$c_1 = c_1' \wedge \frac{1}{4}\tilde{C}h_1^3, \tag{65}$$

where $\tilde{C}$ is a constant from (64). In the rest of the proof let us fix this constant c_1 and $\varepsilon \in (0, c_1 M^{-7}]$. The reason to define c_1 by (65) is so that $2\varepsilon^2 \le 2c_1^2 M^{-14} \le \frac{1}{8}\tilde{C}^2 h_1^6 M^{-14}$ which implies

$$2\varepsilon^3 \le \frac{1}{4}\frac{\varepsilon}{2}\tilde{C}^2 h_1^6 M^{-14}, \tag{66}$$

which will be crucial in the sequel.

Note that for sufficiently small $\eta = \eta(\Lambda, M, \varepsilon)$ and $x \in U_3'(M, h_1, \eta)$ we have $x_3 \le |x_1|/\sqrt{6}$ and

$$\begin{aligned} v_{22}^{(\varepsilon)}(x) &= -\varepsilon + u_{22}(x) \le -\varepsilon + cx_3 h_1^{-5} \le -\frac{\varepsilon}{2}, \\ v_{11}^{(\varepsilon)}(x) &= -\varepsilon + u_{11}(x) \le -\varepsilon + cx_3 h_1^{-5} \le -\frac{\varepsilon}{2}. \end{aligned}$$

We have

$$H(v^{(\varepsilon)})(x) = v_{11}^{(\varepsilon)}(x)v_{22}^{(\varepsilon)}(x)v_{33}^{(\varepsilon)}(x) + 2v_{12}^{(\varepsilon)}(x)v_{23}^{(\varepsilon)}(x)v_{13}^{(\varepsilon)}(x)$$
$$-v_{22}^{(\varepsilon)}(x)\left(v_{13}^{(\varepsilon)}(x)\right)^2 - v_{11}^{(\varepsilon)}(x)\left(v_{23}^{(\varepsilon)}(x)\right)^2 - v_{33}^{(\varepsilon)}(x)\left(v_{12}^{(\varepsilon)}(x)\right)^2,$$

$$-v_{22}^{(\varepsilon)}(x)\left(v_{13}^{(\varepsilon)}(x)\right)^2 \ge \frac{\varepsilon}{2}\frac{\tilde{C}^2 h_1^6}{M^{14}}, \tag{67}$$
$$-v_{11}^{(\varepsilon)}(x)\left(v_{23}^{(\varepsilon)}(x)\right)^2 \ge 0,$$

$$\left| v_{33}^{(\varepsilon)}(x)\left(v_{12}^{(\varepsilon)}(x)\right)^2 \right| \le (cx_3 h_1^{-5})^2(2\varepsilon + cx_3 h_1^{-5}), \tag{68}$$
$$|v_{12}^{(\varepsilon)}(x)v_{23}^{(\varepsilon)}(x)v_{13}^{(\varepsilon)}(x)| \le cx_3 h_1^{-5} h_1^{-4} h_1^{-4}, \tag{69}$$
$$|v_{11}^{(\varepsilon)}(x)v_{22}^{(\varepsilon)}(x)v_{33}^{(\varepsilon)}(x)| \le (\varepsilon + cx_3 h_1^{-5})^2(2\varepsilon + cx_3 h_1^{-5}). \tag{70}$$

Note that the right hand sides of (68), (69), (70) are bounded by $2\varepsilon^3 + x_3 C(\Lambda, h_1)$ (note that h_1 depends only on Λ so $C(\Lambda, h_1) = C(\Lambda)$). By (66) and (67) we have $2\varepsilon^3 \le -\frac{1}{4}v_{22}^{(\varepsilon)}(x)\left(v_{13}^{(\varepsilon)}(x)\right)^2$. We also have $x_3 C(\Lambda, h_1) < -\frac{1}{4}v_{22}^{(\varepsilon)}(x)\left(v_{13}^{(\varepsilon)}(x)\right)^2$ for sufficiently small $\eta = \eta(\Lambda, M, \varepsilon)$ and $x \in U_3'(M, h_1, \eta)$. For such η and x we have

$$H(v^{(\varepsilon)})(x) \ge -\frac{1}{2}v_{22}^{(\varepsilon)}(x)\left(v_{13}^{(\varepsilon)}(x)\right)^2 \ge \frac{\varepsilon}{4}\frac{\tilde{C}^2 h_1^6}{M^{14}}.$$

□

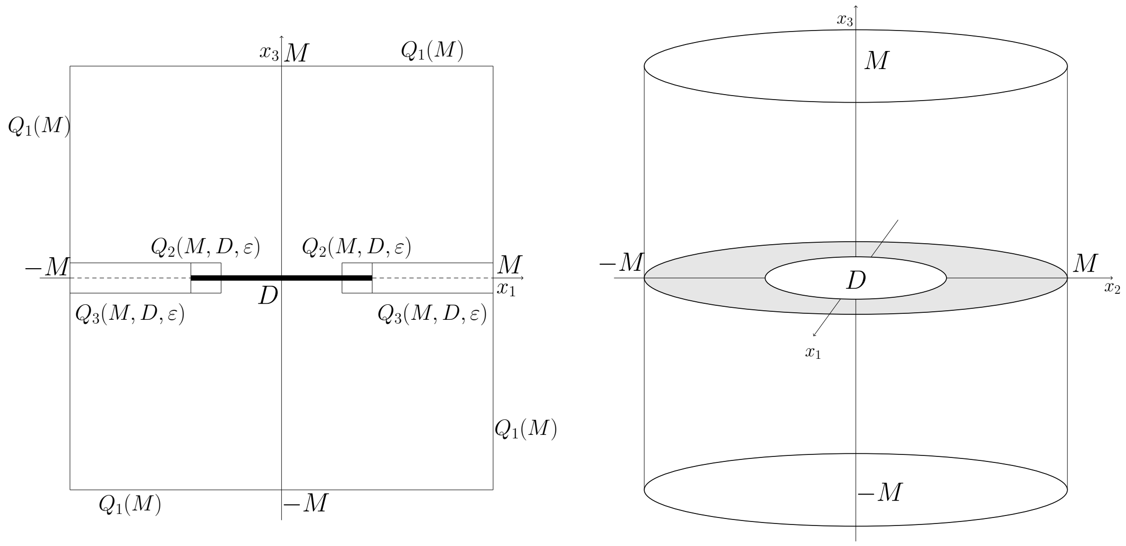
Lemma 6.3. *Let φ be the solution of (1-2) for $B(0,1)$, u the harmonic extension of φ and $v^{(\varepsilon)}$ given by (59). For $M \ge 10$, $h \in (0, 1/2]$, $\eta \in (0, 1/2]$ we define*

$$\begin{aligned} U_1(M) &= \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \le M^2, x_3 = M \text{ or } x_3 = -M\} \\ &\quad \cup \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = M^2, x_3 \in [-M, M] \setminus \{0\}\}, \\ U_2(h) &= \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \in [(1-h)^2, 1), x_3 \in [-h, h]\} \\ &\quad \cup \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \in [1, (1+h)^2], x_3 \in [-h, h] \setminus \{0\}\}, \\ U_3(M, h, \eta) &= \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \in [(1+h)^2, M^2], x_1^2 + x_2^2 \le M^2, x_3 \in [-\eta, \eta] \setminus \{0\}\}. \end{aligned}$$

Then we have

$$\exists c_1 \in (0,1]\ \exists M_0 \ge 10\ \exists h_1 \in (0, 1/2]\ \forall M \ge M_0\ \exists \eta = \eta(M) \in (0, 1/2]$$
$$\forall \varepsilon \in (0, c_1 M^{-7}]\ \forall x \in U_1(M) \cup U_2(h_1) \cup U_3(M, h_1, \eta)$$
$$H(v^{(\varepsilon)})(x) > 0.$$

Remark 6.4. It is important here that η does not depend on ε.



A cross section of $\Omega(M, D)$ $\qquad$ $\Omega(M, D)$

FIGURE 8

Proof. Existence of c_1, M_0, h_1 and the estimate $H(v^{(\varepsilon)})(x) > 0$ for $x \in U_1(M) \cup U_2(h_1)$ (where $M \geq M_0$, $\varepsilon \in (0, c_1 M^{-7}]$) follow from the arguments from the proof of Proposition 6.2.

Let $\varepsilon \in (0, 1]$. Fix $M \geq M_0$ and let $x \in U_3(M, h_1, 1/2)$. We may assume that $x_2 = 0$, $x_3 > 0$, $x_1 < 0$. We have $H(v^{(\varepsilon)})(x) = v_{22}^{(\varepsilon)}(x) f(\varepsilon, x)$, where $f(\varepsilon, x) = v_{11}^{(\varepsilon)}(x) v_{33}^{(\varepsilon)}(x) - (v_{13}^{(\varepsilon)}(x))^2$. We have $u_{22}(x) < 0$ so $v_{22}^{(\varepsilon)}(x) = u_{22}(x) - \varepsilon < 0$. We also have $|u_{11}(x)| \leq c x_3 h_1^{-5}$, $|u_{33}(x)| \leq c x_3 h_1^{-5}$ which gives

$$v_{11}^{(\varepsilon)}(x) v_{33}^{(\varepsilon)}(x) = (u_{11}(x) - \varepsilon)(u_{33}(x) + 2\varepsilon) < c x_3 h_1^{-10} + c x_3 h_1^{-5}.$$

Let us additionally assume that x_3 is sufficiently small so that $x_3 \leq \frac{|x_1| - 1}{\sqrt{6}}$. For such x by the arguments from the proof of Proposition 6.2 we have $|u_{13}(x)| \geq c h_1^3 M^{-7}$ so $|v_{13}^{(\varepsilon)}(x)|^2 = |u_{13}(x)|^2 \geq c h_1^6 M^{-14}$. Hence for sufficiently small $\eta = \eta(M)$ and $x \in U_3(M, h_1, \eta)$ we have $f(\varepsilon, x) < 0$, which implies $H(v^{(\varepsilon)})(x) > 0$. $\square$

Proposition 6.5. *Let φ be the solution of (1-2) for $B(0,1)$, u the harmonic extension of φ and $v^{(\varepsilon)}$ given by (59). For $M \geq 10$ put*

$$\Omega_M = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq M^2, x_3 \in [-M, M]\} \setminus \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \in [1, M^2], x_3 = 0\}.$$

Let c_1 and M_0 be the constants from Lemma 6.3. Then we have

$$\forall M \geq M_0 \ \forall \varepsilon \in (0, c_1 M^{-7}] \ \forall x \in \Omega_M \qquad H(v^{(\varepsilon)})(x) > 0.$$

Proof. On the contrary assume that there exists $M_1 \geq M_0$, $\varepsilon_1 \in (0, c_1 M_1^{-7}]$, $z \in \Omega_{M_1}$ such that $H(v^{(\varepsilon_1)})(z) \leq 0$. By Lemma 6.3 there exists $h_1 \in (0, 1/2]$ and $\eta_1 = \eta_1(M_1) \in (0, 1/2]$ such that $\forall \varepsilon \in (0, c_1 M_1^{-7}]$, $\forall x \in U_1(M_1) \cup U_2(h_1) \cup U_3(M_1, h_1, \eta_1)$ $H(v^{(\varepsilon)})(x) > 0$.

Note that by $v^{(0)} = u$ and by Proposition 5.1 we have $H(v^{(0)})(x) > 0$ for all $x \in \Omega_{M_1}$. It follows that there exists $\varepsilon_2 \in (0, \varepsilon_1]$ and $\tilde{z} \in \Omega_{M_1} \setminus (U_1(M_1) \cup U_2(h_1) \cup U_3(M_1, h_1, \eta_1))$ such that $H(v^{(\varepsilon_2)})(\tilde{z}) = 0$ and $H(v^{(\varepsilon_2)})(x) \geq 0$ for all $x \in \Omega_{M_1}$. This gives contradiction with Theorem 1.6. $\square$

As a direct conclusion of Propositions 6.2 and 6.5 we obtain

Corollary 6.6. *Fix $C_1 > 0$, $R_1 > 0$, $\kappa_2 \ge \kappa_1 > 0$ and $D \in F(C_1, R_1, \kappa_1, \kappa_2)$. Denote $\Lambda = \{C_1, R_1, \kappa_1, \kappa_1\}$. Let $\varphi^{(D)}$ be the solution of (1-2) for D, $u^{(D)}$ the harmonic extension of $\varphi^{(D)}$ given by (6-10) and $v^{(\varepsilon,D)}$ given by (59). Then we have*

$$\begin{aligned}
&\exists c_1 = c_1(\Lambda) \in (0,1]\ \exists c_2 = c_2(\Lambda) > 0\ \exists M_0 \ge 10\ \exists h_1 = h_1(\Lambda) \in (0,1/2]\ \forall M \ge M_0\\
&\forall \varepsilon \in (0, c_1 M^{-7}]\ \exists \eta = \eta(\Lambda, M, \varepsilon) \in (0,(1/2)\wedge\varepsilon]\ \exists c_3 = c_3(\Lambda, M, \varepsilon) > 0,\\
&\forall x \in Q(M,D,\varepsilon) \qquad H(v^{(\varepsilon,D)})(x) \ge c_3,\\
&\forall x \in \Omega(M, B(0,1)) \qquad H(v^{(\varepsilon,B(0,1))})(x) \ge c_3,\\
&\forall x \in Q_4(D) \qquad H(u^{(D)})(x) \ge c_2,
\end{aligned}$$

where (see Figure 8) $Q(M,D,\varepsilon) = Q_1(M) \cup Q_2(M,D,\varepsilon) \cup Q_3(M,D,\varepsilon)$,

$$\begin{aligned}
Q_1(M) &= \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 \le M^2, x_3 = M \text{ or } x_3 = -M\}\\
&\quad \cup \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = M^2, x_3 \in [-M,M] \setminus \{0\}\},\\
Q_2(M,D,\varepsilon) &= \{x \in \mathbb{R}^3 : (x_1,x_2) \in D, \delta_D((x_1,x_2)) \le h_1, x_3 \in [-\eta,\eta]\},\\
Q_3(M,D,\varepsilon) &= \{x \in \mathbb{R}^3 : (x_1,x_2) \in D^c, x_1^2 + x_2^2 \le M^2, x_3 \in [-\eta,\eta] \setminus \{0\}\},\\
\Omega(M,D) &= \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < M^2, x_3 \in (-M,M)\} \setminus (D^c \times \{0\}),\\
Q_4(D) &= \{x \in \mathbb{R}^3 : (x_1,x_2) \in D, \delta_D((x_1,x_2)) \le h, x_3 = 0\}.
\end{aligned}$$

proof of Theorem 1.1.

Step 1.

In this step we will use the notation from Corollary 6.6. We will show that for any $\Lambda = \{C_1, R_1, \kappa_1, \kappa_2\}$, $D \in F(\Lambda)$ and $x \in \mathbb{R}^3 \setminus (D^c \times \{0\})$ we have $H(u^{(D)})(x) > 0$.

Fix $\Lambda = \{C_1, R_1, \kappa_1, \kappa_2\}$ where $C_1 > 0$, $R_1 > 0$, $\kappa_2 \ge \kappa_1 > 0$ and fix $D_0 \in F(\Lambda)$. Let $\{D(t)\}_{t\in[0,1]}$, $D(0) = D_0$, $D(1) = B(0,1)$ be the family of domains defined by (16). By Lemma 2.4 there exists $\Lambda' = \{C_1', R_1', \kappa_1', \kappa_2'\}$ where $C_1' > 0$, $R_1' > 0$, $\kappa_2' \ge \kappa_1' > 0$ such that $\forall t \in [0,1]$ $D(t) \in F(\Lambda')$. Put $v^{(\varepsilon,t)} = v^{(\varepsilon,D(t))}$.

We will use Corollary 6.6 applied to $\Lambda' = \{C_1', R_1', \kappa_1', \kappa_1'\}$. Fix $M \ge M_0 \ge 10$ and $\varepsilon \in (0, c_1 M^{-7}]$. Let

$$T = \{t \in [0,1] : H(v^{(\varepsilon,t)})(x) > 0 \quad \text{for all} \quad x \in \Omega(M, D(t))\}.$$

Let $\Omega_+(M) = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < M^2, x_3 \in (0,M)\}$ and $\Omega_-(M) = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < M^2, x_3 \in (-M,0)\}$. Let us make the following observation: $H(v^{(\varepsilon,t)})(x) > 0$ for all $x \in \Omega(M, D(t))$ if and only if $H(v^{(\varepsilon,t)})(x) > 0$ for all $x \in \Omega_+(M)$. Indeed, if the latest inequality holds then $H(v^{(\varepsilon,t)})(x) > 0$ for all $x \in \Omega_-(M)$ by Lemma 6.1 and $H(v^{(\varepsilon,t)})(x) > 0$ for all $x \in D(t) \times \{0\}$ by Lewy's theorem. It follows that

$$T = \{t \in [0,1] : H(v^{(\varepsilon,t)})(x) > 0 \quad \text{for all} \quad x \in \Omega_+(M)\}.$$

The reason to consider $\Omega_+(M)$ instead of $\Omega(M, D(t))$ is that $\Omega_+(M)$ does not depend on t. By Corollary 6.6 we have $1 \in T$ so T is nonempty. We will show that T is both open and closed (relatively in $[0,1]$), which implies that $T = [0,1]$.

By Lemma 2.5 and standard arguments, $v^{(\varepsilon,t)}(x) \to v^{(\varepsilon,s)}(x)$ for $x \in \Omega_+(M)$, when $[0,1] \ni t \to s$.

Let us assume that $\{t_n : n = 1,2,\ldots\} \subset T$ and $t_n \to t_0$ as $n \to \infty$. Then $H(v^{(\varepsilon,t_0)})(x) \ge 0$ for all $x \in \Omega_+(M)$. By Corollary 6.6 $H(v^{(\varepsilon,t_0)})(x)$ does not vanish identically in $\Omega_+(M)$. By Lewy's theorem $H(v^{(\varepsilon,t)})(x) > 0$ for all $x \in \Omega_+(M)$. Hence $t_0 \in T$, which implies that T is closed.

Now, on the contrary, assume that T is not open. Then there exists $t_0 \in T$ and a sequence $\{t_n\}$ such that $[0,1] \ni t_n \to t_0$ as $n \to \infty$ and $t_n \notin T$ for any $n = 1,2,\ldots$. Hence there exists a sequence of points $x_n \in \Omega_+(M)$ such that $H(v^{(\varepsilon,t_n)})(x_n) \le 0$. After taking a subsequence, if necessary, we may assume that $x_n \to x_0 \in \overline{\Omega_+(M)}$ as $n \to \infty$. If $x_0 \in$

$(D(t_0))^c \times \{0\}$ then for sufficiently large n we get $x_n \in Q_2(M, D(t_n), \varepsilon) \cup Q_3(M, D(t_n), \varepsilon)$ and we get a contradiction to Corollary 6.6. If $x_0 \in \Omega_+(M) \cup Q_1(M) \cup (D(t_0) \times \{0\})$ then by standard arguments $H(v^{(\varepsilon,t_n)})(x_n) \to H(v^{(\varepsilon,t_0)})(x_0) \leq 0$ as $n \to \infty$. If $x_0 \in \Omega_+(M) \cup (D(t_0) \times \{0\})$ then we get a contradiction with our assumption that $t_0 \in T$. If $x_0 \in \Omega_1(M)$ we get a contradiction to Corollary 6.6. So T is open.

It follows that for any fixed $M \geq M_0 \geq 10$ and $\varepsilon \in (0, c_1 M^{-7})$ we have $H(v^{(\varepsilon,D_0)})(x) > 0$ for all $x \in \Omega(M, D_0)$. By taking $\varepsilon \to 0$ we obtain that $H(u^{(D_0)})(x) \geq 0$ for all $x \in \Omega(M, D_0)$. By the estimates of $H(u^{(D_0)})$ on $Q_4(D_0)$ from Corollary 6.6 we obtain that $H(u^{(D_0)})(x)$ does not vanish near $\partial D_0 \times \{0\}$. Hence Lewy's theorem implies that $H(u^{(D_0)})(x) > 0$ for all $x \in \Omega(M, D_0)$. Since $M \geq M_0 \geq 10$ was arbitrary we get that $H(u^{(D_0)})(x) > 0$ for all $x \in \mathbb{R}^3 \setminus (D_0^c \times \{0\})$.

Step 2.

By sign(Hess$(u(y))$) we denote a signature of the Hessian matrix of $u(y)$. In this step we will show that for arbitrary $\Lambda = \{C_1, R_1, \kappa_1, \kappa_1\}$, $D \in F(\Lambda)$ and $y \in \mathbb{R}^3 \setminus (D^c \times \{0\})$ we have sign(Hess$(u(y))) = (1, 2)$ and φ is strictly concave on D.

Fix $\Lambda = \{C_1, R_1, \kappa_1, \kappa_1\}$ where $C_1 > 0$, $R_1 > 0$, $\kappa_2 \geq \kappa_1 > 0$ and fix $D \in F(\Lambda)$. Let φ be the solution of (1-2) for D, u the harmonic extension of φ. Let $(x_1, x_2) \in D$, put $x = (x_1, x_2, 0)$. Denote $f(x) = u_{11}(x)u_{22}(x) - u_{12}^2(x)$. By Lemma 4.7 $u_{13}(x) = u_{23}(x) = 0$, $u_{33}(x) > 0$. By Step 1 $H(u)(x) > 0$. Hence $f(x) > 0$. We have $u_{11}(x) + u_{22}(x) + u_{33}(x) = 0$ so $u_{11}(x) + u_{22}(x) < 0$. This and $f(x) > 0$ implies that $u_{11}(x) < 0$, $u_{22}(x) < 0$. Hence sign(Hess$(u(x))) = (1, 2)$. Since $H(u)(y) > 0$ for any $y \in \mathbb{R}^3 \setminus (D^c \times \{0\})$ we get sign(Hess$(u(y))) = (1, 2)$.

Inequalities $f(x) > 0$, $u_{11}(x) < 0$, $u_{22}(x) < 0$ give that $\varphi(x_1, x_2) = u(x_1, x_2, 0)$ is strictly concave on D.

Step 3.

In this step we will show that for any open bounded convex set $D \subset \mathbb{R}^2$ φ is concave on D.

Fix an open bounded convex set $D \subset B(0, 1) \subset \mathbb{R}^2$. It is well known (see e.g. [9, page 451]) that there exists a sequence of sets D_n such that $D_n \in F(\Lambda_n)$ for some $\Lambda_n = \{C_{1,n}, R_{1,n}, \kappa_{1,n}, \kappa_{2,n}\}$ and $\bigcup_{n=1}^\infty D_n = D$, $D_n \subset D_{n+1}$, $n \in \mathbb{N}$, $d(D_n, D) \to 0$ as $n \to \infty$ (where $C_{1,n} > 0$, $R_{1,n} > 0$, $\kappa_{2,n} \geq \kappa_{1,n} > 0$). Let $\varphi^{(n)}$, φ denote solutions of (1-2) for D_n and D. By Step 2 $\varphi^{(n)}$ are concave on D_n. By Lemma 2.5 we have $\lim_{n\to\infty} \varphi^{(n)}(x) = \varphi(x)$ for $x \in D$. So φ is concave on D.

By scaling we may relax the assumption $D \subset B(0, 1)$. □

7. Extensions and conjectures

proof of Theorem 1.5. a) It is well known that if $\psi_r(x) = \psi(rx)$, for some $r > 0$ and all $x \in \mathbb{R}^d$ then $(-\Delta)^{\alpha/2}\psi_r(x) = r^\alpha(-\Delta)^{\alpha/2}\psi(rx)$ (see e.g. [4, page 9]). Fix $x_0 \in \partial D$ and $\lambda \in (0, 1)$. Put $f(x) = \varphi(\lambda x + (1 - \lambda)x_0) - \lambda^\alpha\varphi(x)$. We have $(-\Delta)^{\alpha/2} f(x) = 0$ for $x \in D$ and $f(x) \geq 0$ for $x \in D^c$. Hence $f(x) \geq 0$ for $x \in D$.

b) Fix $x, y \in D$ and $\lambda \in (0, 1)$. Put $z = \lambda x + (1 - \lambda)y$. Let l be the line which contains x and y. Let $x_0 \in \partial D$ be the point on l which is closer to x than to y and $y_0 \in \partial D$ be the point on l which is closer to y than to x. We have

$$z = y\frac{|z - x_0|}{|y - x_0|} + x_0\left(1 - \frac{|z - x_0|}{|y - x_0|}\right).$$

By a) we get

$$\varphi(z) \geq \left(\frac{|z - x_0|}{|y - x_0|}\right)^\alpha \varphi(y) \geq \left(\frac{|z - x|}{|y - x|}\right)^\alpha \varphi(y) = (1 - \lambda)^\alpha\varphi(y).$$

We also have

$$z = x\frac{|z-y_0|}{|x-y_0|} + y_0\left(1 - \frac{|z-y_0|}{|x-y_0|}\right).$$

Again by a) we get

$$\varphi(z) \geq \left(\frac{|z-y_0|}{|x-y_0|}\right)^{\alpha}\varphi(x) \geq \left(\frac{|z-y|}{|x-y|}\right)^{\alpha}\varphi(x) = \lambda^{\alpha}\varphi(x).$$

□

Now we present some conjectures concerning solutions of (3-4).

Conjecture 7.1. Let $\alpha = 1$, $d \geq 3$. If $D \subset \mathbb{R}^d$ is an arbitrary bounded convex set then the solution of (3-4) is concave on D.

It seems that using the generalization of H. Lewy's result obtained by S. Gleason and T. Wolff [20, Theorem 1] one can show this conjecture. Let $\alpha = 1$, $d \geq 3$ and $D \subset \mathbb{R}^d$ be a sufficiently smooth bounded convex set such that ∂D has a strictly positive curvature, φ the solution of (3-4) and u its harmonic extension in $\mathbb{R}^{d+1}$. It seems that using the method of continuity, in the similar way as in this paper, one can show that the Hessian matrix of u has a constant signature $(1, d-1)$. This implies concavity of φ on D. Anyway, Conjecture 7.1 remains an open challenging problem.

Conjecture 7.2. Let $d \geq 2$, $D \subset \mathbb{R}^d$ be an arbitrary bounded convex set and φ be the solution of (3-4).

a) If $\alpha \in (1,2)$ then φ is $1/\alpha$-concave on D.

b) If $\alpha \in (0,1)$ then φ is concave on D.

Remark 7.3. For any $\alpha \in (1,2)$, $\eta \in (0, 1-1/\alpha)$ and $d \geq 2$ there exists a bounded convex set $D \subset \mathbb{R}^d$ (a sufficiently narrow bounded cone) such that the solution of (3-4) is not $1/\alpha + \eta$ concave on D.

Justification of Remarks 1.4 and 7.3. It is clear that it is sufficient to show Remark 7.3. For any $\theta \in (0, \pi/2)$, $d \geq 2$ let

$$D(\theta) = \{(x_1, \ldots, x_d) : \sqrt{x_2^2 + \ldots + x_d^2} < x_1 \tan\theta, |x| < 1\}.$$

Let $\alpha \in (0,2)$ and φ be the solution of (3-4) for $D(\theta)$.

By [29, Theorem 3.13, Lemma 3.7] for any $\varepsilon > 0$ there exists $\theta \in (0, \pi/2)$ and $c > 0$ such that

$$\varphi(x) \leq c|x|^{\alpha-\varepsilon}, \qquad x \in D(\theta). \tag{71}$$

Theorem 3.13 and Lemma 3.7 in [29] are formulated only for $d \geq 3$ but small modifications of proofs in [29] give these results also for $d = 2$. (71) for any $d \geq 2$ also follows from the recent paper [7].

Fix $d \geq 2$, $\alpha \in (1,2)$, $\eta \in (0, 1-1/\alpha)$ and $\varepsilon \in \left(0, \frac{\alpha^2\eta}{1+\eta\alpha}\right)$. There exists $\theta \in (0, \pi/2)$ and $c > 0$ such that the solution φ of (3-4) for $D(\theta)$ satisfies $\varphi(x) \leq c|x|^{\alpha-\varepsilon}$. Fix $x_0 = (a, 0, \ldots, 0) \in D(\theta)$. If φ is $1/\alpha + \eta$ concave on $D(\theta)$ then for any $\lambda \in (0,1)$ we have

$$\varphi(\lambda x_0) \geq \lambda^{\frac{\alpha}{1+\eta\alpha}}\varphi(x_0) = \lambda^{\alpha - \frac{\alpha^2\eta}{1+\eta\alpha}}\varphi(x_0).$$

On the other hand $\varphi(\lambda x_0) \leq c\lambda^{\alpha-\varepsilon}|x_0|^{\alpha-\varepsilon}$, so

$$c\lambda^{\alpha-\varepsilon}|x_0|^{\alpha-\varepsilon} \geq \lambda^{\alpha-\frac{\alpha^2\eta}{1+\eta\alpha}}\varphi(x_0),$$

which gives

$$\lambda^{\frac{\alpha^2\eta}{1+\eta\alpha}-\varepsilon} \geq \varphi(x_0)c^{-1}|x_0|^{\varepsilon-\alpha}$$

for any $\lambda \in (0,1)$, contradiction. □

We finish this section with an open problem concerning p-concavity of the first eigenfunction for the fractional Laplacian with Dirichlet boundary condition.

Let $\alpha \in (0,2)$, $d \geq 1$, $D \subset \mathbb{R}^d$ be a bounded open set and let us consider the following Dirichlet eigenvalue problem for $(-\Delta)^{\alpha/2}$

$$(-\Delta)^{\alpha/2}\varphi_n(x) = \lambda_n \varphi_n(x), \qquad x \in D, \tag{72}$$

$$\varphi_n(x) = 0, \qquad x \in D^c. \tag{73}$$

It is well known (see e.g. [13], [27]) that there exists a sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$, $\lambda_n \to \infty$ and corresponding eigenfunctions $\varphi_n \in L^2(D)$. $\{\varphi_n\}_{n=1}^\infty$ form an orthonormal basis in $L^2(D)$, all φ_n are continuous and bounded on D, one may assume that $\varphi_1 > 0$ on D.

Open problem. For any $\alpha \in (0,2)$, $d \geq 2$ find $p = p(d,\alpha) \in [-\infty, 1]$ such that for arbitrary open bounded convex set $D \subset \mathbb{R}^d$ the first eigenfunction of (72-73) is p-concave on D. It is not clear whether such $p = p(d,\alpha) \in [-\infty, 1]$ exists.

Any results, even numerical, concerning this problem would be very interesting.

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References

[1] R. Bañuelos, R. D. DeBlassie, *On the First Eigenfunction of the Symmetric Stable Process in a Bounded Lipschitz Domain*, Potential Anal. 42 (2015), no. 2, 573-583.

[2] R. Bañuelos, T. Kulczycki, *The Cauchy process and the Steklov problem*, J. Funct. Anal. 211 (2004), 355-423.

[3] R. Bañuelos, T. Kulczycki, P. J. Méndez-Hernández, *On the shape of the ground state eigenfunction for stable processes*, Potential Anal. 24 (2006), 205-221.

[4] K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R. Song, Z. Vondraček, *Potential analysis of stable processes and its extensions*, Lecture Notes in Mathematics 1980, Springer-Verlag, Berlin, (2009).

[5] K. Bogdan, T. Kulczycki, M. Kwaśnicki, *Estimates and structure of α-harmonic functions*, Probab. Theory Related Fields 140 (2008), 345-381.

[6] K. Bogdan, T. Kulczycki, A. Nowak, *Gradient estimates for harmonic and q-harmonic functions of symmetric stable processes*, Illinois J. Math. 46 (2002), 541-556.

[7] K. Bogdan, B. Siudeja, A. Stos, *Martin kernel for fractional Laplacian in narrow cones*, Potential Anal. 42 (2015), no. 4, 839-859.

[8] Ch. Borell, *Greenian potentials and concavity*, Math. Anal. 272, (1985), 155-160.

[9] L. A. Caffarelli, A. Friedman, *Convexity of solutions of semilinear elliptic equations*, Duke Math. J. 52 (1985), 431-456.

[10] L. A. Caffarelli, L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations 32 (2007), 1245-1260.

[11] Z.-Q. Chen, P. Kim, R. Song, *Heat kernel estimates for the Dirichlet fractional Laplacian*, J. Eur. Math. Soc. (JEMS) 12 (2010), 1307-1329.

[12] Z.-Q. Chen, R. Song, *Estimates on Green functions and Poisson kernels for symmetric stable processes*, Math. Ann. 312 (1998), 465-501.

[13] Z.-Q. Chen, R. Song, *Intrinsic ultracontractivity and conditional gauge for symmetric stable processes*, J. Funct. Anal. 150 (1997), 204-239.

[14] R. D. DeBlassie, *The first exit time of a two-dimensional symmetric stable process from a wedge*, Ann. Probab. 18 (1990), 1034-1070.

[15] Yu. V. Egorov, M. A. Shubin, *Linear partial differential equations. Foundations of the classical theory. Partial differential equations I*, Encyclopaedia Math. Sci. 30, Springer, Berlin (1992).

[16] A. El Hajj, H. Ibrahim, R. Monneau, *Dislocation dynamics: from microscopic models to macroscopic crystal plasticity*, Contin. Mech. Thermodyn. 21 (2009), 109-123.

[17] J. Elliot, *Absorbing barrier processes connected with the symmetric stable densities*, Illinois J. Math. vol. 3 (1959), 200-216.

[18] R. K. Getoor, *First passage times for symmetric stable processes in space*, Trans. Amer. Math. Soc. 101 (1961), 75-90.

[19] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften, Vol. 224. Springer-Verlag, Berlin-New York, (1977).

[20] S. Gleason, T. H. Wolff, *Lewy's harmonic gradient maps in higher dimensions*, Comm. Partial Differential Equations 16 (1991), 1925-1968.

[21] M. Kac, H. Pollard, *Partial sums of independent random variables*, Canad. J. Mat. vol. 11 (1950), 375-384.

[22] M. Kaßmann, L. Silvestre, *On the superharmonicity of the first eigenfunction of the fractional Laplacian for certain exponents*, preprint (2014), http://math.uchicago.edu/∼luis/preprints/cfe.pdf

[23] A. U. Kennington, *Power concavity and boundary value problems*, Indiana Univ. Math. J. 34 (1985), 687-704.

[24] A. U. Kennington, *Power concavity of solutions of Dirichlet problems*, Miniconference on nonlinear analysis (Canberra, 1983), 133-136, Proc. Centre Math. Anal. Austral. Nat. Univ., 8, Austral. Nat. Univ., Canberra, (1984).

[25] N. Korevaar, *Capillary surface convexity above convex domains*, Indiana Univ. Math. J. 32 (1983), 73-81.

[26] N. Korevaar, J. L. Lewis, *Convex solutions of certain elliptic equations have constant rank Hessians*, Arch. Rational Mech. Anal. 97 (1987), 19-32.

[27] T. Kulczycki, *Intrinsic ultracontractivity for symmetric stable processes*, Bull. Polish Acad. Sci. Math. 46 (1998), 325-334.

[28] T. Kulczycki *Properties of Green function of symmetric stable processes*, Probab. Math. Statist. (1997), 339-364.

[29] T. Kulczycki, *Exit time and Green function of cone for symmetric stable processes*, Probab. Math. Statist. 19 (1999), 337-374.

[30] T. Kulczycki, M. Ryznar, *Gradient estimates of harmonic functions and transition densities for Levy processes*, Trans. Amer. Math. Soc. (published electronically) DOI: http://dx.doi.org/10.1090/tran/6333

[31] H. Lewy, *On the non-vanishing of the jacobian of a homeomorphism by harmonic gradients*, Ann. of Math. (2) 88, (1968), 518-529.

[32] L. G. Makar-Limanov, *Solution of Dirichlet's problem for the equation $\Delta u = -1$ in a convex region*, Math. Notes Acad. Sci. USSR 9 (1971), 52-53.

[33] P. J. Méndez-Hernández, *Exit times from cones in $\mathbb{R}^n$ of symmetric stable processes*, Illinois J. Math. 46 (2002), 155-163.

[34] S. A. Molchanov, E. Ostrovskii, *Symmetric stable processes as traces of degenerate diffusion processes*, Theory Probab. Appl. 14 (1969), 128-131.

[35] F. Spitzer, *Some theorems concerning 2-dimensional Brownian motion*, Trans. Amer. Math. Soc. 87 (1958), 187-197.

[36] G. Żurek, *Concavity of α-harmonic functions* (in Polish), Master Thesis, Institute of Mathematics and Computer Science, Wrocław University of Technology (2014), http://prac.im.pwr.wroc.pl/∼zurek/papers/Funkcje_a_harmoniczne.pdf

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