On the 'high spots' of fundamental sloshing modes in a trough

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We study an eigenvalue problem with a spectral parameter in a boundary condition. The problem describes sloshing of a heavy liquid in a container, which means that the unknowns are the frequencies and modes of the liquid's free oscillations. The question of 'high spots' (the points on the mean free surface, where its elevation attains the maximum and minimum values) is considered for fundamental sloshing modes in troughs of uniform cross-section. For troughs, whose cross-sections are such that the horizontal, top interval is the one-to-one orthogonal projection of the bottom, the following result is obtained: any fundamental eigenfunction attains its maximum and minimum values only on the boundary of the rectangular free surface of the trough.

Keywords: sloshing problem; eigenvalue; fundamental eigenfunction

1. Introduction

Linear water wave theory is a well-known approach to describing surface waves in the presence of rigid boundaries. In particular, this theory is widely used for determining sloshing frequencies and modes in a container. They are obtained from a mixed Steklov problem involving a spectral parameter in a boundary condition. This boundary value problem (usually referred to as the sloshing problem) has been the subject of a great number of studies over more than two centuries (see Fox & Kuttler (1983) for a historical review). Advanced mathematical techniques based on spectral theory of operators in a Hilbert space are developed for this problem; they can be found in the book by Kopachevsky & Krein (2001).

An inviscid, incompressible, heavy liquid occupies a three-dimensional container bounded from above by a free surface, which in its mean position is a simply connected two-dimensional domain of finite diameter. Let Cartesian coordinates (x, y, z) be chosen so that the mean free surface (it is assumed to coincide with the undisturbed one) lies in the (x, z)-plane and the y-axis is directed upwards.

† T. Kulczycki was supported in part by MNiSW grant # N N201 373136; N. Kuznetsov acknowledges the hospitality of the Institute of Mathematics and Computer Science, Technical University of Wroclaw, where the work on this paper was initiated and completed during the second visit.

The surface tension is neglected on the free surface, and we assume the liquid motion to be irrotational and of small-amplitude. The latter assumption allows us to linearise the boundary conditions on the free surface and this leads to the following boundary value problem for $\phi(x, y, z)$ — the velocity potential of the flow with a time-harmonic factor removed:

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \quad \text{in } W, \tag{1.1}$$

$$\phi_y = \nu u \quad \text{on } F, \tag{1.2}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } B. \tag{1.3}$$

Here $W \subset \mathbb{R}^3_- = \{(x, z) \in \mathbb{R}^2, y < 0\}$ is the liquid domain which is bounded and has no cusps on its piecewise smooth boundary ∂W. The latter includes a two-dimensional domain $F \subset \partial\mathbb{R}^3_-$ referred to as the free surface; $B = \partial W \setminus \overline{F}$ denotes the rigid container's bottom. The zero eigenvalue of the spectral parameter ν obviously exists for problem (1.1)–(1.3), but we exclude it with the help of the following orthogonality condition:

$$\int_F \phi \, \mathrm{d}x\mathrm{d}z = 0. \tag{1.4}$$

In condition (1.2), $\nu = \omega^2/g$, where ω is the radian frequency of liquid's oscillations and g is the acceleration due to gravity.

It has been known since the 1950s that problem (1.1)–(1.4) has a discrete spectrum; that is, there exists a sequence of eigenvalues

$$0 < \nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_n \leqslant \ldots, \tag{1.5}$$

each having a finite multiplicity equal to the number of repetitions in (1.5). These eigenvalues are such that

$$\nu_n \to \infty \quad \text{as } n \to \infty,$$

and they can be found by means of a variational principle described in § 2(*b*). The corresponding eigenfunctions ϕ_n $(n = 1, 2, \ldots)$ belong to the Sobolev space $H^1(W)$ and form a complete system in an appropriate Hilbert space (see, for example, Kopachevsky & Krein (2001)).

A substantial body of work on problem (1.1)–(1.4) (a great part of which is numerical computation of eigenfrequencies) deals with either two-dimensional oscillations or with motions in containers having a vertical axis of symmetry (see Fox & Kuttler (1983) and references cited therein; references to papers published in Russian can be found in Kopachevsky & Krein (2001)). Here we consider this problem for containers having another geometry, namely for horizontal cylinders of uniform cross-section or troughs. To the authors' knowledge, the sloshing problem for such containers has received much less attention; it was investigated by McIver & McIver (1993) and (mainly numerically) by several Russian authors (see the book by Lukovsky *et al.* (1984) for the references).

Our considerations of the sloshing problem in a trough are concerned with the question of the so-called 'high spots' for the eigenmodes corresponding to the fundamental eigenvalue. We recall that at every instant of time the free-surface elevation is proportional to the trace $\phi_1(x, 0, z)$ (see, for example, Lamb (1932)). Therefore, during one half-period of oscillation, the elevation has its high spot at the point that belongs to the closure $\bar{F}$ of the free surface and delivers the maximum value to the trace; during the other half-period, the high spot is at the point of $\bar{F}$, where $\phi_1(x, 0, z)$ attains its minimum.

One may ask whether our formulation of the problem must be modified when the wall-sides of the trough are not vertical so that the edges of the free surface can move horizontally as the fluid oscillates? The answer is no as long as the amplitude of the waves is small enough which is confirmed by the fact that ϕ satisfying problem (1.1)–(1.4) has no singularities at the edges. This is a consequence of the general results about elliptic boundary value problems in domains with piecewise smooth boundaries (see, for example, Nazarov & Plamenevsky (1994)). However, the problem with non-vertical wall-sides is much more difficult because the separation of variables is impossible in this case.

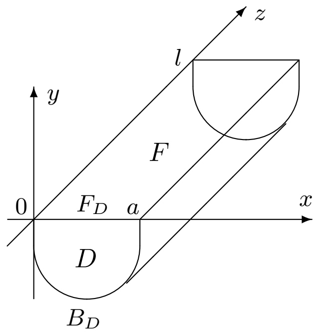

Figure 1.

First results about high spots for fundamental eigenfunctions of the sloshing problem were obtained by Kulczycki & Kuznetsov (2009), who proved some theorems about the location of these points in the two-dimensional case and for some vertical axisymmetric containers. Moreover, it was shown that for vertical-walled containers with horizontal bottom the question about high spots is equivalent to the 'hot spots' conjecture of J. Rauch (see Burdzy (2006) for a survey of results about the latter conjecture).

Here the question about high spots is solved for a class of troughs whose cross-sections are two-dimensional domains such that each point of the free surface has a single projection on the bottom (see theorem 3.1 for the exact formulation). Some required auxiliary results are proved in § 2, which begins with reformulation of the problem for the particular geometry of troughs.

2. The problem of sloshing in a trough; auxiliary results

(*a*) *Statement of the problem*

Let $D \subset \mathbb{R}^2_- = \{(x, y) : y < 0\}$ be a bounded, simply connected domain (see figure 1). We assume that the piecewise smooth boundary ∂D has no cusps and one of the open arcs forming ∂D is a horizontal interval

$$F_D = \{(x, y) : x \in (0, a),\, y = 0\}, \quad a > 0.$$

By $B_D = \partial D \setminus \overline{F_D}$ we denote the union of open arcs lying in $\mathbb{R}^2_-$ and complemented by corner points (if there are any) connecting these arcs.

A *trough* is the domain $W = D \times (0, l)$ (see figure 1); $l > 0$ and a are the length and width, respectively, of the trough's free surface $F = (0, a) \times \{0\} \times (0, l)$, which thus is a rectangle immersed in $\mathbb{R}^3$. For the sake of simplicity F will also denote the same rectangular domain as a subset of the (x, z)-plane. For the trough W, the rigid part of its boundary $B = \partial W \setminus \bar{F}$ consists of the bottom $B_D \times \{z \in [0, l]\}$ and the two side walls $D \times \{z = 0\}$ and $D \times \{z = l\}$.

It is clear that the ansatz

$$\phi(x, y, z) = u(x, y) \cos \frac{m\pi}{l} z, \quad m = 0, 1, 2, \ldots, \tag{2.1}$$

reduces the eigenvalue problem (1.1)–(1.4) in the trough W to the following sequence of boundary value problems:

$$u_{xx} + u_{yy} - \left(\frac{m\pi}{l}\right)^2 u = 0 \quad \text{in } D, \tag{2.2}$$

$$u_y = \nu u \quad \text{on } F_D, \tag{2.3}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } B_D, \tag{2.4}$$

$$\int_{F_D} u(x, 0)\, \mathrm{d}x \mathrm{d}z = 0. \tag{2.5}$$

Here the first three relations must hold for all $m = 0, 1, 2, \ldots$, whereas the last condition should be imposed only for $m = 0$. Indeed, condition (2.5) arises from (1.4) and the factor $\cos \frac{m\pi}{l} z$ in the right-hand side of (2.1) yields that (1.4) is satisfied automatically for $m \neq 0$.

(*b*) *Auxiliary results*

The variational principle for determining eigenvalues of problem (1.1)–(1.4) (see, for example, Moiseev (1964)), implies that the following Rayleigh quotient

$$\frac{\int_D \left[u_x^2 + u_y^2 + \left(\frac{m\pi}{l}\right)^2 u^2\right] \mathrm{d}x\mathrm{d}y}{\int_{F_D} u^2 \,\mathrm{d}x} \tag{2.6}$$

is to be minimised in the case of problem (2.2)–(2.5). In particular, for obtaining the smallest eigenvalue, this quotient must be minimised over the Sobolev space $H^1(D)$ when $m \neq 0$; if $m = 0$, then one has to minimise (2.6) over the subspace of $H^1(D)$ consisting of functions for which condition (2.5) holds.

The variational method guarantees that for every $m = 0, 1, 2, \ldots$, the spectral problem (2.2)–(2.5) has a sequence of eigenvalues $\{\nu_{m,k}\}_{k=1}^\infty$ such that

$$0 < \nu_{m,1} \leq \nu_{m,2} \leq \nu_{m,3} \leq \ldots, \quad m = 0, 1, 2, \ldots, \tag{2.7}$$

and by $\{u_{m,k}\}_{m=0,k=1}^\infty$ we denote the double sequence of corresponding eigenfunctions. Every eigenvalue in (2.7) has a finite multiplicity equal to the number of repetitions; moreover, for every $m \geqslant 0$ we have that

$$\nu_{m,k} \to \infty \quad \text{as } k \to \infty.$$

It is clear that the sequence of eigenvalues $\{\nu_n\}_{n=1}^{\infty}$ of problem (1.1)–(1.4) for a trough W coincides with the double sequence $\{\nu_{m,k}\}_{m=0,k=1}^{\infty}$. Thus the sequence of problems (2.2)–(2.5) is equivalent to the original sloshing problem.

If W is a trough, then one of the following three options:

$$\text{(i) } \nu_1 = \nu_{0,1}, \quad \text{(ii) } \nu_1 = \nu_{1,1}, \quad \text{(iii) } \nu_1 = \nu_{0,1} = \nu_{1,1}$$

delivers the fundamental eigenvalue ν_1 of problem (1.1)–(1.4). Indeed, we minimise the Rayleigh quotient (2.6) over $H^1(D)$ for $m \geqslant 1$, thus getting the least value when $m = 1$. However, it might occur that minimising (2.6) with $m = 0$ over a proper subspace of $H^1(D)$ we get the minimum value even less than or equal to that in the former case. Let us consider two examples of troughs having simple geometries, from which one sees that each of these options is realised under certain conditions.

Example 1. Let W be a rectangular box, that is, $D = (0, a) \times (0, -h)$, where $h > 0$. It is well-known that in this case the eigenvalues in the double-sequence (2.7) take the form $\nu_{m,k} = \alpha_{m,k} \tanh \alpha_{m,k} h$, where

$$\alpha_{0,k} = \frac{k\pi}{a} \quad \text{and} \quad \alpha_{m,k} = \sqrt{\left[\frac{m\pi}{l}\right]^2 + \left[\frac{(k-1)\pi}{a}\right]^2} \quad \text{for } m \geqslant 1.$$

The eigenfunction corresponding to $\nu_{m,k}$ is given by formula (2.1), in which $u = u_{m,k}(x, y)$ is given by the following formulae:

$$u_{0,k}(x, y) = c_{0,k} \cos \frac{k\pi x}{a} \cosh \alpha_{0,k}(y + h),$$

and

$$u_{m,k}(x, y) = c_{m,k} \cos \frac{(k-1)\pi x}{a} \cosh \alpha_{m,k}(y + h) \quad \text{for } m \geqslant 1,$$

where $c_{m,k}$ is a non-zero real number.

Then we get for $m = 0, 1$ and $k = 1$

$$\nu_{0,1} = \frac{\pi}{a} \tanh \frac{\pi h}{a} \quad \text{and} \quad \nu_{1,1} = \frac{\pi}{l} \tanh \frac{\pi h}{l}.$$

Hence if $l < a$, then $\nu_{0,1} < \nu_{1,1}$, and so $\nu_1 = \nu_{0,1}$. On the other hand, $\nu_{1,1} < \nu_{0,1}$ when $l > a$, and so $\nu_1 = \nu_{1,1}$. Finally, if $l = a$, then $\nu_1 = \nu_{0,1} = \nu_{1,1}$, and so the multiplicity of the fundamental eigenvalue is two.

Example 2. Let W be a triangular trough with the bottom sides inclined at $\frac{\pi}{4}$ to the vertical, that is,

$$D = \left\{(x, y) : 0 < x < a, \left|x - \frac{a}{2}\right| - \frac{a}{2} < y < 0\right\}.$$

According to Kirchhoff's solution for this geometry (see Lamb (1932), § 258), we have that $\nu_{0,1} = \frac{2}{a}$ and the corresponding eigenfunction is as follows:

$$u_{0,1}(x, y) = c\left(x - \frac{a}{2}\right)\left(y + \frac{a}{2}\right), \tag{2.8}$$

where c is a non-zero constant.

On the other hand, Kelland's solution of problem (2.2)–(2.4) with $m = 1$ (see Lamb (1932), § 261) gives that the fundamental eigenvalue is

$$\nu_{1,1} = \frac{\pi}{l\sqrt{2}} \tanh \frac{\pi a}{l2\sqrt{2}}$$

and the corresponding eigenfunction is as follows:

$$u_{1,1}(x, y) = c \cosh \frac{\pi}{l\sqrt{2}} \left(x - \frac{a}{2} \right) \cosh \frac{\pi}{l\sqrt{2}} \left(y + \frac{a}{2} \right), \tag{2.9}$$

where c is again a non-zero constant. For obtaining a fundamental eigenfunction in W, one has to multiply the latter expression by $\cos \frac{\pi z}{l}$.

Let λ_+ be the only positive root of $\lambda \tanh \lambda = 1$, and so $\frac{\pi}{\lambda_+ 2\sqrt{2}} \approx 0.926$. Similar to example 1, we have that if $l < \frac{\pi a}{\lambda_+ 2\sqrt{2}}$, then $\nu_{0,1} < \nu_{1,1}$, in which case $\nu_1 = \nu_{0,1}$. Besides, if $l > \frac{\pi a}{\lambda_+ 2\sqrt{2}}$, then $\nu_{1,1} < \nu_{0,1}$, and so $\nu_1 = \nu_{1,1}$. Finally, if $l = \frac{\pi a}{\lambda_+ 2\sqrt{2}}$, then $\nu_1 = \nu_{0,1} = \nu_{1,1}$, and so the multiplicity of the fundamental eigenvalue is two.

For investigating the behaviour of high spots for eigenfunctions, corresponding to the fundamental eigenvalue of problem (1.1)–(1.4) in the case when W is a trough, we need several auxiliary results (some of them are known and are given here for the sake of completeness).

Lemma 2.1. *Let ϕ be an eigenfunction of problem* (1.1)–(1.4)*, then ϕ attains its maximum and minimum values on $\bar{F}$.*

Proof. The geometric assumptions imposed on W and the fact that $\phi \in H^1(W)$ allow us to apply the general theory of elliptic boundary value problems in piecewise smooth domains (see, for example, the book by Nazarov & Plamenevsky (1994)). It follows from this theory that ϕ is continuous throughout $\bar{W}$. Then the maximum principle for harmonic functions is applicable and it yields that the maximum and minimum values of ϕ are attained on ∂W. However, the points, where these values are attained cannot belong to B because otherwise Hopf's lemma† and condition (1.3) contradict each other. □

The next assertion will play an essential role in the proof of our main theorem.

Proposition 2.2. *The eigenvalue $\nu_{1,1}$ is simple and the trace $u_{1,1}(x, 0)$ of a corresponding eigenfunction does not vanish on F_D.*

Prior to proving this proposition we recall some notions and prove some lemmas. All of them will be used only for $m = 1$, but we formulate and prove them for all $m \geqslant 1$ in order to demonstrate their generality.

Let $N(u) = \{(x, y) \in \bar{D} : u(x, y) = 0\}$ be the set of nodal lines and/or points of a sloshing eigenfunction u for $m \geqslant 1$. A connected component of $D \setminus N$ will be called a nodal domain. On account of conditions (2.2) and (2.4), one concludes that each nodal domain has a piecewise smooth boundary without cusps.

Lemma 2.3. *Let u be an eigenfunction of problem* (2.2)–(2.4) *for $m \geqslant 0$* (*the problem must be complemented by condition* (2.5) *when $m = 0$*)*. If R is a nodal domain of u, then $\bar{R} \cap F_D$ contains an interval of the x-axis.*

† See Protter & Weinberger (1984), ch. 2, theorem 7, and comments on p. 156; it has come into wide use to cite this assertion as Hopf's lemma.

Proof. Let, contrary to the assertion, $\bar{R} \cap F_D$ be empty or consist of a finite number of points. Applying Green's identity to u in R, we get from the boundary conditions on ∂R that

$$\int_R \left[u_x^2 + u_y^2 + \left(\frac{m\pi}{l} \right)^2 u^2 \right] \mathrm{d}x\mathrm{d}y = 0.$$

Hence u vanishes in R, and so u is identically equal to zero in D by the analyticity of solutions to elliptic equations with constant coefficients (see, for example, John (1955), ch. VII, where even more general theorem can be found). The contradiction obtained proves the lemma's assertion. □

Lemma 2.4. *For every* $m = 1, 2, \ldots$ *the number of nodal domains corresponding to* $u_{m,k}$ *is less than or equal to* $k = 1, 2, \ldots$.

Proof. Let us assume the contrary, that is, the number of nodal domains is $K > k$; we denote them $R_1, \ldots, R_K$. Then we get by lemma 2.3 that for every $j = 1, \ldots, K$ the boundary ∂R_j contains a subinterval of F_D; the closure of the union of these intervals is $\overline{F_D}$. Now we define

$$w_j = \begin{cases} u_{m,k} & \text{in } R_j \\ 0 & \text{outside } R_j. \end{cases}$$

Thus the functions $w_1, \ldots, w_K$ belong to $H^1(D)$ and their traces on F_D are linearly independent; moreover, there exists a linear combination

$$w = c_1 w_{j_1} + \cdots + c_k w_{j_k},$$

which satisfies the orthogonality conditions

$$\int_{F_D} w\, u_{m,i}\, \mathrm{d}x = 0, \quad i = 1, \ldots, k-1.$$

It is clear that condition (2.3) holds for w with $\nu = \nu_{m,k}$, which means that w minimises the Rayleigh quotient (2.6). Therefore, the variational principle guarantees that w is an eigenfunction corresponding to $\nu_{m,k}$. Hence w solves equation (2.2) in D, which implies that w is a real-analytic function. Hence it vanishes identically in D being equal to zero on a subdomain of D. The contradiction obtained proves the lemma's assertion. □

The method for introducing the functions $w_1, \ldots, w_K$ that correspond to domains $R_1, \ldots, R_K$ originates from Courant's proof of the theorem about nodal lines of eigenfunctions for a fixed membrane (see Courant & Hilbert (1953)). In the proof of proposition 2.2 to which we turn now, lemma 2.4 is required only for $k = 1$.

Proof. According to lemma 2.4 the eigenfunction $u_{1,1}$ corresponding to $\nu_{1,1}$ does not change sign. Since the trace of $u_{1,2}$ must be orthogonal to that of $u_{1,1}$ on F_D, the eigenfunction $u_{1,2}$ must change sign, and so cannot be an eigenfunction corresponding to $\nu_{1,1}$. Indeed, assuming the contrary, we consider $u_{1,2}^{(+)}$ and $u_{1,2}^{(-)}$ which are the positive and negative parts of $u_{1,2}$, respectively. These functions vanish in non-empty complementary subdomains of D, but do not vanish identically.

Hence they are not real-analytic in D, and so are not eigenfunctions corresponding to $\nu_{1,1}$. Therefore, we have

$$\int_D \left[\left(\frac{u_{1,2}^{(\pm)}}{\partial x} \right)^2 + \left(\frac{u_{1,2}^{(\pm)}}{\partial y} \right)^2 + \left(\frac{\pi}{l} \right)^2 \left(u_{1,2}^{(\pm)} \right)^2 \right] \mathrm{d}x\mathrm{d}y > \nu_{1,1} \int_{F_D} \left(u_{1,2}^{(\pm)} \right)^2 \mathrm{d}x.$$

On the other hand the equality

$$\int_D \left[\left(\frac{u_{1,2}}{\partial x} \right)^2 + \left(\frac{u_{1,2}}{\partial y} \right)^2 + \left(\frac{\pi}{l} \right)^2 u_{1,2}^2 \right] \mathrm{d}x\mathrm{d}y = \nu_{1,1} \int_{F_D} u_{1,2}^2 \, \mathrm{d}x$$

follows from the assumption made. Summing up the two inequalities and subtracting the last inequality from the sum, we arrive at a contradiction. Thus the eigenvalue $\nu_{1,1}$ is simple.

Without loss of generality, we suppose $u_{1,1}$ to be non-positive on $\bar{D}$. If this function vanishes at $(x_0, 0) \in F_D$, then this point delivers the maximum value to the function, and so $\frac{\partial u_{1,1}}{\partial y}(x_0, 0) > 0$ by the Hopf lemma. On the other hand, this value of the derivative vanishes in view of condition (2.3). The contradiction obtained proves the proposition's assertion. □

3. High spots theorem for troughs

Having proposition 2.2 proved, we are in a position to turn to our main assertion.

Theorem 3.1. *Let W be a trough such that its side-walls have the following properties: B_D is the graph of a C^2-function given on F_D, and B_D forms non-zero angles with F_D at their common end-points. If ϕ_1 is an eigenfunction corresponding to the fundamental eigenvalue ν_1 of problem* (1.1)–(1.4) *in W, then the trace $\phi_1(x, 0, z)$ attains its maximum and minimum values on ∂F only* (*see figure* 2).

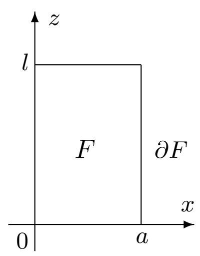

Figure 2.

Proof. Let us consider each of the three options (i)–(iii) listed prior to example 1.

First, if $\nu_1 = \nu_{0,1}$, then this eigenvalue is simple by theorem 3.1 proved in Kozlov *et al.* (2004). According to formula (2.1), the corresponding eigenfunction is $\phi_1(x, y, z) = u_{0,1}(x, y)$, and it solves problem (2.2)–(2.5) for $m = 0$ and $\nu = \nu_{0,1}$. Theorem 2.1 proved by Kulczycki & Kuznetsov (2009) says that under the geometric assumptions imposed on D the endpoints of $\overline{F_D}$ are the only points, where the trace $u_{0,1}(x, 0)$ attains its extremum values, say, $(0, 0)$ and $(a, 0)$ are the minimum and maximum points, respectively. Then $\phi_1(x, 0, z)$ is equal to $u_{0,1}(0, 0)$ and $u_{0,1}(a, 0)$ on the whole edges

$$\{(x, y, z) : x = y = 0, z \in [0, l]\} \quad \text{and} \quad \{(x, y, z) : x = a, y = 0, z \in [0, l]\},$$

respectively, and only there. Thus the theorem's assertion is proved under the assumption of case (i).

Second, let either $\nu_1 = \nu_{1,1}$ or $\nu_1 = \nu_{0,1} = \nu_{1,1}$. Then any corresponding eigenfunction is of the form

$$\phi_1(x, y, z) = c_{0,1} u_{0,1}(x, y) + c_{1,1} u_{1,1}(x, y) \cos \frac{\pi}{l} z. \tag{3.1}$$

Here $c_{0,1}$ and $c_{1,1}$ are constants such that $c_{0,1} = 0$ and $c_{1,1} \neq 0$ when $\nu_1 = \nu_{1,1}$, whilst if $\nu_1 = \nu_{0,1} = \nu_{1,1}$, then these constants do not vanish simultaneously. The functions $u_{0,1}$ and $u_{1,1}$ solve problem (2.2)–(2.5) with ν equal to ν_1 as indicated above, and $m = 0$ and $m = 1$, respectively.

Let us assume that the trace $\phi_1(x, 0, z)$ attains its maximum value at an inner point of the free surface, say, $(\hat{x}, \hat{z}) \in F$. Then we have

$$\frac{\partial \phi_1}{\partial z}(\hat{x}, 0, \hat{z}) = -c_{1,1} \frac{\pi}{l} u_{1,1}(\hat{x}, 0) \sin \frac{\pi}{l} \hat{z} = 0.$$

However proposition 2.2 shows that this is impossible unless $c_{1,1} = 0$, which leads to a contradiction when $\nu_1 = \nu_{1,1}$, and gives that $c_{0,1} \neq 0$ in the case when $\nu_1 = \nu_{0,1} = \nu_{1,1}$. According to formula (3.1), we have $\phi_1(x, y, z) = c_{0,1} u_{0,1}(x, y)$ in the latter case, and so the considerations at the beginning of the proof lead to the result.

Since the minimum of $\phi_1(x, 0, z)$ is the maximum of $-\phi_1(x, 0, z)$, what was said above proves the theorem for the minimum case as well. □

4. Discussion

For the sloshing problem in a trough of uniform cross-section the following main result is obtained (see theorem 3.1). Any eigenfunction corresponding to the fundamental eigenvalue attains its maximum and minimum values only on the boundary of the rectangular free surface of the trough (see figure 2), provided its side-wall domain has the following property. Vertical lines give a one-to-one mapping of the upper horizontal segment onto the side-wall's curvilinear boundary. Some — but by no means complete — information about the location of points, where the extremum values are attained, follows from the proof of theorem 3.1. Let us discuss what can be said about the location in each of the three cases introduced in § 2(*b*).

(i) If $\nu_1 = \nu_{0,1} < \nu_{1,1}$ (see (2.7) for the definition of these eigenvalues), then there exists only one (up to a non-zero factor) fundamental eigenfunction which attains a constant value along each of the two edges between the free surface and the bottom — maximum on one edge and minimum on the other.

(ii) It is shown that the eigenvalue $\nu_{1,1}$ is simple (like $\nu_{0,1}$) for an arbitrary domain D. Moreover, if D is fixed, then the inequality $\nu_{1,1} < \nu_{0,1}$ holds for all troughs with the side-walls $D \times \{z = 0\}$ and $D \times \{z = l\}$ when l sufficiently large.

Indeed, using a non-zero constant as a trial function in the Rayleigh quotient (2.6), we get

$$\nu_{1,1} < \left(\frac{\pi}{l}\right)^2 \frac{|D|}{a},$$

where $|D|$ is the Lebesgue measure of D. The right-hand side of the latter inequality tends to zero as $l \to +\infty$, and so $\nu_{1,1}$ can be made as small as one pleases for sufficiently long troughs. However $\nu_{0,1}$ is fixed being independent of l.

If the fundamental eigenvalue is $\nu_1 = \nu_{1,1} < \nu_{0,1}$, then the proof of theorem 3.1 gives no information about the location of extremum values on ∂F for the corresponding eigenfunction ϕ_1 (unlike the case when $\nu_1 = \nu_{0,1}$). Besides, examples 1 and 2 show that if $\nu_{1,1} < \nu_{0,1}$, then ϕ_1 attains these values only at the corners of the rectangular free surface. Thus another question arises: whether this property of ϕ_1 is general or there exists a domain D such that the extremum values of ϕ_1 are attained at inner points on the edges $F_D \times \{z = 0\}$ and $F_D \times \{z = l\}$ when $\nu_{1,1} < \nu_{0,1}$.

(iii) If $\nu_1 = \nu_{0,1} = \nu_{1,1}$, then there are two linearly independent eigenfunctions, one of which does not depend on z, and its behaviour is the same as in case (i) — maximum along one edge between the free surface and the bottom and minimum along the other. Besides, the theorem's proof provides no information about the location on ∂F of points, where the second linearly independent eigenfunction (cf. case (ii)), and any combination of two linearly independent eigenfunctions with non-zero coefficients attain their extremum values.

Assuming that $\nu_1 = \nu_{0,1} = \nu_{1,1}$ in example 2, which is equivalent to the equality

$$\frac{l}{a} = \frac{\pi}{\lambda_+ 2\sqrt{2}} \approx 0.926,$$

where λ_+ is the only positive root of $\lambda \tanh \lambda = 1$, let us investigate those fundamental eigenfunctions that are combinations of two linearly independent eigenfunctions with non-zero coefficients. It is obvious that no extremum values of these eigenfunctions can occur at inner points of the edges $\left\{x = \frac{a}{2} \pm \frac{a}{2}\right\} \times \{y = 0\} \times \{0 < z < l\}$.

Our aim is to show that any such eigenfunction cannot have its extremum values at inner points on the edges $F_D \times \{z = 0\}$ and $F_D \times \{z = l\}$ as well (like in the case when $\nu_1 = \nu_{1,1} < \nu_{0,1}$). For this purpose we normalise the eigenfunctions $u_{0,1}$ and $u_{1,1}$ by putting c equal to

$$\left(\frac{2}{a}\right)^2 \quad \text{and} \quad \cosh^{-2} \frac{\lambda_+}{\sqrt{2}}$$

in formulae (2.8) and (2.9), respectively, thus obtaining $u_{0,1}(a, 0) = u_{1,1}(a, 0) = 1$. Substituting these normalised eigenfunctions into formula (3.1), we get that the trace on $F_D \times \{z = 0\}$ is of the form:

$$\phi_1(x, 0, 0) = \frac{2\, c_{0,1}}{a} \left(x - \frac{a}{2}\right) + \frac{c_{1,1}}{\cosh \frac{\lambda_+}{\sqrt{2}}} \cosh \frac{\lambda_+ \sqrt{2}}{a} \left(x - \frac{a}{2}\right),$$

where $c_{0,1}$ and $c_{1,1}$ are non-zero constants. Let us choose them so that

$$c_{0,1} = -c_{1,1} \beta(\mu), \quad \text{where } \mu \in (0, 1) \cup (1, 2) \text{ and } \beta(\mu) = \frac{\lambda_+}{\sqrt{2}} \frac{\sinh \frac{\lambda_+}{\sqrt{2}}(\mu - 1)}{\cosh \frac{\lambda_+}{\sqrt{2}}}.$$

Since $c_{0,1} = 0$ for $\mu = 1$, the result is obvious in this case. Moreover, $\beta(\mu)$ is chosen so that $\phi_1'\left(\mu\frac{a}{2}, 0, 0\right) = 0$, whereas the sign of $\phi_1''\left(\mu\frac{a}{2}, 0, 0\right)$ coincides with that of $c_{1,1}$. Finally, we have

$$\phi_1\left(\frac{a}{2} \pm \frac{a}{2}, 0, 0\right) = c_{1,1}\left[1 \mp \beta(\mu)\right],$$

and

$$\phi_1\left(\mu\frac{a}{2}, 0, 0\right) = c_{1,1}\left[\frac{\cosh\frac{\lambda_+}{\sqrt{2}}(\mu - 1)}{\cosh\frac{\lambda_+}{\sqrt{2}}} - (\mu - 1)\beta(\mu)\right].$$

These facts immediately yield that any linear combination with non-zero coefficients of independent eigenfunctions corresponding to $\nu_1 = \nu_{0,1} = \nu_{1,1}$ cannot attain extremum values at inner points of $F_D \times \{z = 0\}$. Similar considerations give the same result for $F_D \times \{z = l\}$.

Now we are in a position to formulate our last question. Is it true that if $\nu_1 = \nu_{0,1} = \nu_{1,1}$, then any linear combination with non-zero coefficients of independent eigenfunctions attains its extremal values only at the corners of the free surface?

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