'HIGH SPOTS' THEOREMS FOR SLOSHING PROBLEMS

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Abstract. We investigate several 2D and 3D cases of the classical eigenvalue problem that arises in hydrodynamics and is referred to as the sloshing problem. In particular, for a domain $W \subset \mathbf{R}^2$ (canal's cross-section), where $\partial W = F \cup B$ and F (cross-section of the free surface of fluid) is an interval of the x-axis, whereas B (bottom's cross-section) is the graph of a negative function, the following result is proved. The fundamental eigenfunction u_1 of the sloshing problem (the corresponding eigenvalue is simple) has monotonic traces on F and B; moreover, u_1 attains its maximum and minimum values at the end-points of F.

It is established that for the 2D (3D) ice-fishing problem with a single (circular) hole the function u_1 (both fundamental eigenfunctions) attains its maximum value at an inner point of F.

A relationship between the high spots and hot spots theorems is considered.

1. Introduction

The present paper deals with boundary value problems for the Laplace equation when there is a spectral parameter in a boundary condition. In particular, we consider the problem that gives natural frequencies and the corresponding two-dimensional modes of the free wave motion in an infinitely long canal having a uniform cross-section. This and more general versions of the problem (it is usually referred to as the sloshing problem), has been the subject of a great number of studies over more than two centuries; see the paper [8] for a historical review and early results are described by Lamb in his *Hydrodynamics* [15]. Among recent works on this problem we mention the book [11] by Kopachevsky and Krein, and the papers [12], [13], and [14]. The latter two papers are, in particular, concerned with the question (related to the topic of our paper) when eigenvalues are simple.

In its simplest form the 2D sloshing problem is as follows. Let an inviscid, incompressible, heavy fluid (water) occupies a canal bounded from above by a free surface of finite width. The surface tension is neglected there and we assume the water motion to be irrotational and of small-amplitude. The latter assumption allows us to linearize boundary conditions on the free surface which leads to the following statement of the problem in the case of the two-dimensional motion in planes normal to the generators of the canal's bottom. Let rectangular Cartesian coordinates (x, y) be taken

[0]2000 *MS Classification*: 35Q35, 76B10, 35P99.

The first author was partially supported by KBN grant.

in the plane of the motion with the origin and the x-axis in the mean free surface, whereas the y-axis is directed upwards. With a time-harmonic factor removed, the velocity potential $u(x, y)$ for the flow must satisfy the boundary value problem:

$$u_{xx} + u_{yy} = 0 \quad \text{in } W, \tag{1.1}$$

$$u_y = \nu u \quad \text{on } F, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } B. \tag{1.3}$$

Here the cross-section W of the canal is a bounded simply connected domain whose piecewise smooth boundary ∂W has no cusps. One of the open arcs forming ∂W is an interval F of the x-axis and is called the free surface of water (without loss of generality $F = \{|x| < a, \ y = 0\}$. The bottom $B = \partial W \setminus \bar{F}$ is the union of open arcs, lying in the half-plane $y < 0$, complemented by corner points (if there are any) connecting these arcs. (More precisely, F is the cross-section of the free-surface of fluid, B is the bottom's cross-section.) We suppose that the orthogonality condition

$$\int_F u \, \mathrm{d}x = 0 \tag{1.4}$$

holds, thus excluding the zero eigenvalue of (1.1)–(1.3), in which case the spectral parameter ν is equal to ω^2/g, where ω is the radian frequency of the water oscillations and g is the acceleration due to gravity.

It has been well-known since the 1950s (these results can be found in many sources the most recent of which is the book [11]) that problem (1.1)–(1.4) has a discrete spectrum; that is, there exists a sequence of eigenvalues

$$0 < \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n \leq \dots, \tag{1.5}$$

each having a finite multiplicity equal to the number of repetitions in (1.5), and such that $\nu_n \to \infty$ as $n \to \infty$. These eigenvalues can be found by means of variational principle. The corresponding eigenfunctions u_n, $n = 1, 2, \dots$, belong to the Sobolev space $H^1(W)$ and form a complete system in an appropriate Hilbert space. It was demonstrated in [13] that the fundamental eigenvalue ν_1 is simple. The corresponding eigenfunction u_1 has only one nodal line connecting F and $\bar{B}$.

Here we study properties of the traces of u_1 on $\bar{F}$ and $\bar{B}$ for domains such that $\bar{B}$ is the graph of a sufficiently smooth function given on $\bar{F}$. In particular, we show that the traces of u_1 are monotonic functions of the same kind (say, increasing) of x and of the arc length s measured from $(-a, 0)$, on $\bar{F}$ and on $\bar{B}$, respectively. Thus u_1 attains its maximum and minimum values at the end-points of $\bar{F}$. This property has the following hydrodynamic meaning. If water oscillates freely with the fundamental frequency, then at every moment the free-surface elevation is proportional to $u_1(x, 0)$ (see e.g. [15]). Therefore, the elevation has its maximum (minimum) value at the right (left, respectively) end-point of $\bar{F}$ during one half-period of oscillation, whereas during the other half-period maximum and minimum exchange places with one another. (This is the reason to call this property 'high spots' theorem.) Moreover, the vertical component of velocity has also maximum absolute value at the end-points of $\bar{F}$ except for the moments when the velocity changes its direction.

For the 3D sloshing problem in the case of an open bounded container that has vertical walls and constant depth the question of high spots reduces (see details in Section 3) to proving the so-called 'hot spots' theorem for the Neumann Laplacian in the 2D domain which is the horizontal cross-section of container. The hot spots conjecture formulated by J. Rauch in 1974 (see e.g. [3]), says that any eigenfunction, corresponding to the smallest non-zero eigenvalue the Neumann Laplacian in a domain $D \subset \mathbf{R}^d$, attains its maximum and minimum values on ∂D. During the past decade, the hot spots conjecture has been intensively studied (see [5] for a survey). It was proved for several classes of 2D domains (see [2], [3], [10], [20]), and some 3D domains [1], but it is still an open question for an arbitrary convex 2D domain. On the other hand, there exist multiply connected domains that serve as counterexamples to the hot spots conjecture (see [6], [7]).

Along with the 2D and 3D sloshing problems for bounded water domains, we consider the so-called 2D and 3D ice-fishing problems. For the latter problems the high spots theorems distinguish from those obtained for bounded containers. Let us formulate the 2D ice-fishing problem for which

$$W = \mathbf{R}^2_- = \{x \in \mathbf{R},\, y < 0\},$$

is the cross-section of the water domain, the cross-section F of the free surface consists of a finite number of bounded intervals of the x-axis, and $B = \partial \mathbf{R}^2_- \setminus \bar{F}$ is the cross-section of rigid ice covering water. Relations (1.1)–(1.4) must be complemented by the following one:

$$\int_W |\nabla u|^2 \, \mathrm{d}x\mathrm{d}y < \infty, \tag{1.6}$$

that means that the kinetic energy per unit span is finite. Note that this condition implies that the potential energy is also finite:

$$\int_F |u|^2 \, \mathrm{d}x < \infty$$

(see e.g. [17], ch. 3). For problem (1.1)–(1.4) and (1.6) with F consisting of one and two equal intervals all eigenvalues (1.5) are simple (see [14] for the proof). For both these kinds of F it occurs that the corresponding fundamental eigenfunctions attain their maximum and minimum values at points that are symmetric about the origin and are inner points of F. Similar result takes place for the 3D ice-fishing problem when the free surface is a disc.

2. 2D PROBLEMS

2.1. **The 2D sloshing problem in a canal.** In this case the following theorem is true.

Theorem 2.1. *Let B be the graph of a C^2-function given on F, and let B form non-zero angles with F at their common endpoints.*

If u_1 is the fundamental eigenfunction of problem (1.1)–(1.4)*, then the trace $u_1(x,0)$ is a monotonic function on F. Moreover the trace of u_1 on B is also a monotonic function. More precisely if $u_1(x,0)$ is increasing, then the mapping $s \mapsto u_1(x(s), y(s))$ is also increasing on B; here s is the arc length on B measured from $(-a,0)$ (the left end-point of F).*

The geometric assumptions of Theorem 2.1 and the fact that $u_1 \in H^1(W)$ allow us to apply the general theory of elliptic boundary value problems in piecewise smooth domains (see e.g. [19]). It follows from this theory that u_1 is continuous throughout $\bar{W}$ and near each corner point of ∂W the following estimate holds:

$$|\nabla u_1(x,y)| = O(1) \quad \text{as } \rho \to 0. \tag{2.1}$$

Here ρ is the distance of a point $(x,y) \in W$ from the nearest corner point.

Since u_1 is harmonic in W and continuous in $\bar{W}$, it attains its minimum and maximum values on the boundary ∂W. Then the following assertion about 'high spots' of u_1 is an obvious consequence of Theorem 2.1.

Corollary 2.2. *Let B have the same properties as in Theorem 2.1 and let u_1 be the fundamental eigenfunction of problem* (1.1)–(1.4)*. If u_1 increases on F, then u_1 attains its maximum (minimum) value at the right (left, respectively) end-point of $\bar{F}$.*

There is another boundary value problem that is equivalent to (1.1)–(1.4) and has the form:

$$v_{xx} + v_{yy} = 0 \quad \text{in } W, \tag{2.2}$$

$$-v_{xx} = \nu v_y \quad \text{on } F, \tag{2.3}$$

$$v = 0 \quad \text{on } B. \tag{2.4}$$

Here v is the conjugate to u harmonic function (stream function), chosen as follows. By the Cauchy–Riemann equations we obtain from (1.3) that $v = \text{const}$ on B; the appropriate choice of the additive constant in v gives condition (2.4). Condition (2.3) follows from (1.2) by differentiation and application of the Cauchy–Riemann equations. It was demonstrated in [13] that the fundamental eigenfunction v_1 may be chosen to be positive on $W \cup F$. The second corollary of Theorem 2.1 concerns the trace of v_1 on F.

Corollary 2.3. *Let B have the same properties as in Theorem 2.1. If v_1 is the positive fundamental eigenfunction of problem* (2.2)–(2.4)*, then $v_1(x,0)$ is a concave function on F attaining its maximum value at the end-point of the only nodal line of u_1.*

2.2. **Auxiliary assertions.** Our proof of Theorem 2.1 is based on two assertions that involve another mixed Steklov problem (see e.g [9]), which distinguish from the sloshing problem only by the boundary condition on B, namely:

$$w_{xx} + w_{yy} = 0 \quad \text{in } W, \tag{2.5}$$

$$w_y = \lambda w \quad \text{on } F, \tag{2.6}$$

$$w = 0 \quad \text{on } B. \tag{2.7}$$

This problem also has discrete spectrum $0 < \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n \leq \dots$ such that $\lambda_n \to \infty$ as $n \to \infty$, and the first eigenvalue λ_1 is simple. The corresponding eigenfunctions w_n, $n = 1, 2, \dots$, belong to the subspace $H^1_B(W) \subset H^1(W)$ ($H^1_B(W)$ consists of functions in $H^1(W)$ that vanish on B), and the traces $w_n(x, 0)$ form an orthogonal basis in $L^2(F)$. Moreover, we choose w_n, $n = 1, 2, \dots$, so that the latter basis is orthonormal and w_1 is positive on $W \cup F$.

First we prove the following property of domain monotonicity for λ_1. This property is similar to that of the eigenvalues for the Dirichlet Laplacian (see e.g. [17], ch. 4).

Proposition 2.4. *Let R be a simply connected subdomain of W such that $R \neq W$ and $F \cap \partial R$ is an open nonempty set. Then $\lambda_1^R > \lambda_1^W$, where the superscript indicates the domain in which problem* (2.5)–(2.7) *is considered.*

Proof. Let w_1^R (here the superscript has the same meaning as for λ_1^R, and we will use superscripts in the same way below) be the normalised fundamental eigenfunction corresponding to λ_1^R. In order to argue in the same way as in the proof of Theorem 3.8, [4], we extend w_1^R by zero to the whole domain W (we will keep the same notation for the extended function). Then the following Green's formula holds:

$$\int_W \nabla w_1^R \nabla w_n^W \,\mathrm{d}x\,\mathrm{d}y = \lambda_n^W \int_F w_1^R w_n^W \,\mathrm{d}x, \quad n = 1, 2, \dots, \tag{2.8}$$

because $w_1^R \in H^1_B(W)$ and in view of relations (2.5) and (2.6).

It is clear that

$$\lambda_1^R = \int_R \left|\nabla w_1^R\right|^2 \mathrm{d}x\,\mathrm{d}y = Q\left(w_1^R, w_1^R\right), \quad \text{where } Q(u, v) = \int_W \nabla u \nabla v \,\mathrm{d}x\,\mathrm{d}y$$

is defined on $(H_1(W))^2$. For $(x, y) \in W$ we put

$$s_k(x, y) = \sum_{m=1}^{k} c_m w_m^W(x, y), \text{ where } c_m = \int_F w_1^R w_m^W \,\mathrm{d}x, \text{ and } k = 1, 2, \dots,$$

and write

$$\lambda_1^R = Q(s_k, s_k) + Q\left(w_1^R - s_k, w_1^R - s_k\right) + 2Q\left(w_1^R - s_k, s_k\right).$$

Here we have

$$Q(s_k, s_k) = \int_F \frac{\partial s_k}{\partial y} s_k \, \mathrm{d}x = \sum_{m=1}^{k} \lambda_m^W c_m^2 \, ,$$

and it follows from equality (2.8) that

$$Q\left(w_1^R, s_k\right) = \sum_{m=1}^{k} c_m Q\left(w_1^R, w_m^W\right) = \sum_{m=1}^{k} c_m \lambda_m^W \int_F w_1^R w_m^W \, \mathrm{d}x = \sum_{m=1}^{k} \lambda_m^W c_m^2 .$$

Therefore, $Q\left(w_1^R - s_k, s_k\right) = 0$, and so

$$\lambda_1^R \geq Q(s_k, s_k) = \sum_{m=1}^{k} \lambda_m^W c_m^2 \, .$$

Letting $k \to \infty$, we get that

$$\lambda_1^R \geq \sum_{m=1}^{\infty} \lambda_m^W c_m^2 .$$

Since λ_1^W is simple and $\sum_{m=1}^{\infty} c_m^2 = 1$ (this equality is a consequence of two facts: w_1^R is normalised on $F \cap \partial R$ and w_1^R is extended by zero to $W \setminus R$), we have to consider the following two cases:

(i) $\lambda_1^R = \lambda_1^W$ and $c_1 = 1$, whereas $c_2 = \cdots = c_m = \cdots = 0$;

(ii) $\lambda_1^R > \lambda_1^W$.

Let us show that case (i) is impossible, which proves the proposition.

The fact that $c_1 = 1$ and the other coefficients are equal to zero means that $w_1^R \equiv w_1^W$ on F. If $F \neq \partial R \cap F$, then w_1^W does vanish on the interval (or intervals) $F \setminus \partial R$, and $\frac{\partial w_1^W}{\partial y}$ also vanishes there by the boundary condition (2.6). Then $w_1^W \equiv 0$ in W by the uniqueness theorem for the Cauchy problem for the Laplace equation, which contradicts to the fact that w_1^W is eigenfunction. If $F = \partial R \cap F$, then the latter theorem implies that $w_1^R \equiv w_1^W$ in R. Hence w_1^W satisfies the homogeneous Dirichlet problem in $W \setminus \bar{R}$, which again contradicts to the fact that w_1^W is eigenfunction. The proof is complete. □

Remark 2.5. *The monotonicity property for the fundamental eigenvalue of the sloshing problem is well-known (see e.g. the classical survey paper* [18] *by Moiseev). It says that if R is a subdomain of W such that $R \neq W$ and $F \cap \partial R = F$, then $\nu_1^R < \nu_1^W$.*

Let us turn to our second auxiliary assertion that concerns the inequality which is analogous to that between the second eigenvalue of the Neumann Laplacian and the first eigenvalue of the Dirichlet Laplacian (see the note [21] by Pólya).

Theorem 2.6. *Let W be confined to the semi-strip $\{|x| < a, y < 0\}$. Then $\nu_1 \leq \lambda_1$ and the equality holds only for the infinitely deep rectangular domain $\{|x| < a, -\infty < y < 0\}$.*

Proof. Note that for the infinitely deep rectangular domain we have:

$$\nu_1 = \lambda_1 = \frac{\pi}{2a}, \quad \text{whereas } u_1 = \sin\frac{\pi x}{2a} \exp\frac{\pi y}{2a} \text{ and } w_1 = \cos\frac{\pi x}{2a} \exp\frac{\pi y}{2a},$$

which proves the second assertion.

Now the first assertion of theorem immediately follows from Proposition 2.4 and Remark 2.5. □

2.3. **Proofs of Theorem 2.1 and Corollary 2.3.** For the sake of brevity we will write u and v instead of u_1 and v_1, respectively.

Since v is positive on $W \cup F$ and B is the graph of a function given on F, we have that $v_y \geq 0$ on B. In view of the boundary condition (1.3) this implies

$$u_x = \frac{\partial u}{\partial t} t_x \geq 0 \quad \text{on } B,$$

where $\frac{\partial}{\partial t}$ denotes the tangential derivative and t_x is the projection of the unit tangent on the x-axis (the direction of tangent coincide with the increasing of the arc length s on B). Note that t_x is non-negative because B is the graph, and so the same is true for $\frac{\partial u}{\partial t}$. This proves the second assertion of Theorem 2.1 that $s \mapsto u(x(s), y(s))$ is increasing on B.

Let us show that $u_x \geq 0$ on F which proves the first assertion of Theorem 2.1 and also the concavity of $v(x, 0)$ (see Corollary 2.3). Indeed, the latter fact is true because $u_x = v_y = -\frac{v_{xx}}{\nu_1}$.

If we assume the contrary, i.e., that u_x changes sign at some inner point of F then there exists an open nonempty subset of W such that $u_x < 0$ on this set. Let R be a connected component of this subset. Note that $u_x = 0$ on $\partial R \cap B$ because $u_x \geq 0$ on B and $u_x < 0$ on R. It follows that u_x satisfies the boundary value problem (2.5)–(2.7) on the domain R. Moreover, $\lambda_1^R = \nu_1^W$ in condition (2.6) for u_x, which one obtains differentiating condition (1.2) with respect to x. By Proposition 2.4 we get $\lambda_1^W < \lambda_1^R = \nu_1^W$ which contradicts Theorem 2.6 since W is not the infinitely deep rectangular domain. Hence u_x does not change sign on F.

It remains to note that v, being harmonic in W and having concave trace on F, has only one maximum point on F and $v_x = u_y = \nu_1 u$ vanishes at that point. Therefore, the only nodal line of u emanates from this point. Proofs are complete.

2.4. **The 2D ice-fishing problem.** We consider the ice-fishing problem for infinitely deep water and the following two kinds of the free surface:

(i) $F = \{|x| < 1, y = 0\}$;

(ii) $F = \{b < |x| < b + 1, y = 0\}$.

Here b is a non-negative parameter; for $b = 0$ the ice-fishing problem with F of the second kind coincides with the problem for F of the first kind. The fact that the total length of the free surface is normalised to be equal to two is not a restriction in view of domain similarity.

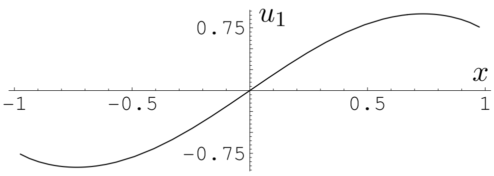

FIGURE 1. The trace $u_1(x,0)$ for the ice-fishing problem with F of the kind (i); $x \in [-0.99, 0.99]$.

It is established (see [14]) that all eigenvalues are simple for all non-negative values of b. Moreover, all eigenvalues with odd numbers coincide with the characteristic values of the following integral equation:

$$z(x) = \frac{\nu}{\pi} \int_0^1 [\log(x+\xi+2b) - \log|x-\xi|]\, z(\xi)\, \mathrm{d}\xi, \quad x \in [0,1]. \tag{2.9}$$

Here $z(x) = u(x+b,0)$ and the latter function is the trace on right half of F of the corresponding odd eigenfunction. Equation (2.9) is the basis for proving the following high spots assertion.

Proposition 2.7. *Let F be either of the kind* (i) *or of the kind* (ii) *with sufficiently large b. Then the points, where the fundamental eigenfunction of the corresponding ice-fishing problem attains its maximum and minimum values, are inner points of F and symmetric with respect to* 0.

Proof. Let us consider case (i) first (Figure 1 illustrates the assertion in this case). Then u_1 satisfies the equality $u_1(-x,y) = -u_1(x,y)$ and we have (changing the sign if necessary), that $u_1(x,y) > 0$ for $x > 0$. In order to analyse the behaviour of $u_1(x,0)$ near $x = 1$, we put $\tilde{u}_1(\rho,\theta) := u_1(x,y)$, where (ρ,θ) are the polar coordinates such that $(x,y) = (1+\rho\cos\theta, \rho\sin\theta)$. By formula (2.1) in [14], there is a neighborhood of the point $(1,0)$, where the following representation

$$u_1(x,y) = \tilde{u}_1(\rho,\theta) = c(\pi/\nu_1 + \rho(\cos\theta\log\rho - \theta\sin\theta)) + d\rho\cos\theta + \psi(\rho,\theta)$$

is valid. Here $\theta \in [-\pi, 0]$, c, d are some constants, the function ψ and their derivatives have estimates $\psi = O(\rho^{1+\delta})$, $|\nabla\psi| = O(\rho^\delta)$, as $\rho \to 0$, $\delta > 0$.

For the point $(x,0)$, where $0 < x < 1$, we have that $\rho = 1-x$ and $\theta = -\pi$, and so

$$u_1(x,0) = \tilde{u}_1(1-x,-\pi) = c\pi/\nu_1 - c(1-x)\log(1-x) - d(1-x) + \psi(1-x,-\pi).$$

This implies that

$$\frac{\partial u_1}{\partial x}(x,0) = c\log(1-x) + c + d + O\left((1-x)^\delta\right),$$

as $x \to 1$ and $0 < x < 1$.

Note that $c \geq 0$ because otherwise $u_1(x,0) < 0$ for the values of x sufficiently close to 1. If $c > 0$ then $(\partial u_1/\partial x)(x,0) \to -\infty$ as $x \to 1$ and if $c = 0$ then $u_1(x,0) \to 0$ as $x \to 1$. Since $u_1(-x,0) = -u_1(x,0)$ and $u_1(x,0) > 0$ for $x > 0$, in both cases ($c > 0$ or $c = 0$) the maximum of $x \mapsto u_1(x,0)$ is attained at an inner point of F. Moreover, the same property holds for the minimum of u_1 because u_1 is antisymmetric.

Let us turn to the case when F is of the kind (ii). Since the fundamental eigenfunction is odd, we restrict ourselves to considering the point of maximum for the fundamental eigenfunction z_1 of equation (2.9) choosing that it is normalised as follows:

$$\int_0^1 z_1(x)\,\mathrm{d}x = 1.$$

According to Theorem 3.1, [14], the following asymptotic formulae hold as $b \to \infty$:

$$1 - z_1(x) = \frac{\nu_1}{\pi}\left[\frac{1}{2} + x\log x + (1-x)\log(1-x)\right] + O\left(\frac{1}{\log^2 b}\right), \tag{2.10}$$

where

$$\nu_1 = \pi\left[\log(2b) + \frac{3}{2} + O\left(\frac{1}{\log b}\right)\right]^{-1}.$$

Therefore, if b is sufficiently large, then the point $x_0(b)$, where z_1 attains its maximum, is close to the point of minimum of $x\log x + (1-x)\log(1-x)$, that is, $x = \frac{1}{2}$. Hence $x_0(b)$ is an inner point of $(0,1)$. □

3. 3D problems

Here we consider two cases of the sloshing problem for 3D water domains, namely:

• the 'glass' problem when a container has constant depth and vertical walls, i.e.,

$$W = \{x \in D,\ y \in (-d,0)\},\ F = D \times \{y=0\},\ B = \partial W \setminus \bar{F}. \tag{3.1}$$

where $x = (x_1, x_2) \in \mathbf{R}^2$, D is a bounded 2D domain, and $d \in (0,\infty]$;

• the ice-fishing problem for infinitely deep water when there is a single circular hole in the ice sheet, i.e.,

$$W = \mathbf{R}^3_- = \{x \in \mathbf{R}^2,\ y \in (-\infty,0)\},\ F = \{|x| < 1,\ y = 0\},\ B = \partial\mathbf{R}^3_- \setminus \bar{F}.$$

The fact that the radius of hole is equal to one is not a restriction in view of similarity.

Conditions (1.2), (1.3), and (1.4) must hold for both these problems, whereas the Laplace equation is now as follows:

$$\nabla_x^2 u + u_{yy} = 0 \quad \text{in } W, \quad \nabla_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right).$$

Moreover, condition (1.6) must be included in the ice-fishing problem.

Since one can separate the y-dependence in the glass problem, there is the following relationship between the questions of high spots for this problem and of hot spots for the 2D Neumann Laplacian.

Proposition 3.1. *A point $x \in \partial D$ is a hot spot for the Neumann Laplacian in D if and only if $(x,0) \in \partial F$ is a high spot for the glass problem.*

Proof. It is sufficient to note that

$$\phi(x) \cosh k(y+d) \text{ and } \nu = k \tanh kd \quad \Big(\phi(x) \mathrm{e}^{ky} \text{ and } \nu = k \text{ when } d = \infty\Big)$$

are the fundamental eigenfunction and eigenvalue, respectively, for the glass problem, if and only if we have

$$\nabla_x^2 \phi + k^2 \phi = 0 \ \text{ in } D, \quad \frac{\partial \phi}{\partial n_x} = 0 \ \text{ on } \partial D, \quad \int_D \phi \, \mathrm{d}x = 0,$$

which means that ϕ is the fundamental eigenfunction for the 2D Neumann Laplacian. □

Unfortunately, not too much is known about 2D domains for which the hot spots conjecture is true. In particular, this conjecture is proved for convex domains with two axes of symmetry [3] (see also [10]), and for the so-called lip-domains [2]. Such a domain is enclosed between the graphs of two functions that satisfy the Lipschitz condition with constant equal to one.

Let us turn to the ice-fishing problem for a single circular hole. This problem was investigated in [11], and, in particular, it was proved that two eigenfunctions of the form:

$$\psi_1(r,y) \frac{x_i}{r}, \quad i = 1, 2, \quad \text{where } r^2 = x_1^2 + x_2^2, \tag{3.2}$$

correspond to the fundamental eigenvalue ν_1. Here

$$\psi_1(r,y) = \nu_1 \int_0^1 \psi_1(s,0) \, s \, \mathrm{d}s \int_0^\infty J_1(kr) \, J_1(ks) \, \mathrm{e}^{ky} \, \mathrm{d}k$$

and the trace of ψ_1 on F is the fundamental eigenfunction of the following integral equation:

$$\psi(r,0) = \nu \int_0^1 \psi(s,0) \, s \, I(r,s) \, \mathrm{d}s, \quad r \in (0,1), \tag{3.3}$$

which is similar to (2.9). The fundamental eigenvalue of (3.3) is also ν_1. Since the kernel

$$I(r,s) = \int_0^\infty J_1(kr) \, J_1(ks) \, \mathrm{d}k,$$

where J_1 is the usual Bessel function, is shown to be positive (there is also an expression of this kernel in terms of the complete elliptic integral $\mathbf{D}$), one can choose $\psi_1(r,0)$ to be positive for $r \in (0,1)$; it is clear that $\psi_1(0,0) = 0$ because $J_1(0) = 0$. Finally, choosing $\psi_1(1,0) = 1$ Proposition 4, [11], says that the following asymptotic formula holds:

$$\psi_1(r,0) = 1 + \frac{\nu_1}{\pi}(r-1) \log|r-1| + O\left(|r-1|\right) \quad \text{as } r \to 1,$$

and it can be differentiated. It is obvious that

$$\frac{\partial \psi_1}{\partial r}(r,0) \to -\infty \quad \text{as } r \to 1,$$

and taking into account formula (3.2), we arrive at the following

Theorem 3.2. *Both fundamental eigenfunctions of the ice-fishing problem for a single circular hole attain their maximum and minimum values at inner points of F.*

4. Discussion

There are many open questions regarding high spots for various versions of the sloshing problem. The first of them is whether Theorem 2.1 is true when B is not the graph of a function, but W is still confined to the semistrip $\{|x| < a, y < 0\}$ (the latter condition is usually referred to as the John condition). Of course, it looks plausible that Theorem 2.1 can be generalised to the 3D case, but one has to take into account that the fundamental eigenvalue is not more simple as the glass problem demonstrates (the multiplicity is two for the circular glass and it is reasonable to expect that the same is true for other containers with axial symmetry; such containers are the best candidates for proving the analogue of Theorem 2.1). It seems that the simplest case in generalizing Theorem 2.1 to 3D is when W is rotationally invariant (like a wine glass). One may expect the following assertion to hold.

Conjecture 4.1. ("a wine glass") Let the free surface of W be the unit disc

$$F = \{(x_1, x_2, y) : x_1^2 + x_2^2 \leq 1,\ y = 0\}$$

in the (x_1, x_2)-plane, and let B be the graph of a rotationally invariant, negative C^2-function given on F. If B forms a nonzero angle with F, then any eigenfunction u corresponding to the fundamental eigenvalue attains its minimum and maximum values on ∂F.

On the other hand, Proposition 2.7 shows the different behaviour of u_1 than in Theorem 2.1. On the basis of these two results, one may expect that the following assertion holds.

Conjecture 4.2. Let W be a bounded domain with smooth B such that at least one angle between B and F is greater than $\pi/2$. Then the fundamental eigenfunction u_1 attains its minimum or maximum value at an inner point of F.

It is also of interest to check the following

Conjecture 4.3. Let W satisfy the John condition, then inequality $\nu_n \leq \lambda_n$ holds for all positive integers n.

This conjecture is similar to the inequality that holds for the corresponding non-trivial eigenvalues of the Neumann and Dirichlet Laplacians (see [16]).

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