Estimates and structure of α-harmonic functions

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Abstract

We prove a uniform boundary Harnack inequality for nonnegative harmonic functions of the fractional Laplacian on arbitrary open sets D. This yields a unique representation of such functions as integrals against measures on $D^c \cup \{\infty\}$ satisfying an integrability condition. The corresponding Martin boundary of D is a subset of the Euclidean boundary determined by an integral test.

1 Main results and introduction

Let $d = 1, 2, \ldots$, and $0 < \alpha < 2$. The boundary Harnack principle (BHP) for nonnegative harmonic functions of the fractional Laplacian

$$\Delta^{\alpha/2}\varphi(x) = \lim_{\varepsilon\to 0^+} \int\limits_{|y|>\varepsilon} [\varphi(y) - \varphi(x)]\, \nu(x,y) dy\,, \tag{1}$$

was proved for Lipschitz domains in 1997 ([9]). Here

$$\nu(x,y) = \mathcal{A}_{d,-\alpha}|y-x|^{-d-\alpha}\,,$$

$\mathcal{A}_{d,\gamma} = \Gamma((d-\gamma)/2)/(2^\gamma \pi^{d/2}|\Gamma(\gamma/2)|)$ for $-2 < \gamma < 2$, and, say, $\varphi \in C_c^\infty(\mathbf{R}^d)$. BHP was extended to all open sets in 1999 ([38]), with the constant in the estimate depending on local geometry of their boundary (compare Corollary 1 below). The question whether the constant may be chosen independently of the domain, or uniformly, was since open.

In what follows D is a *domain* i.e. an open nonempty subset of $\mathbf{R}^d$. Let G_D be the Green function of D for $\Delta^{\alpha/2}$ ([32], [7], [36]). We define the Poisson kernel of D:

$$P_D(x,y) = \int_D G_D(x,v)\nu(v,y)\, dv\,, \quad x \in \mathbf{R}^d\,,\ y \in D^c\,. \tag{2}$$

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By a calculation of M. Riesz (see [8], [37]), for the ball $B_r = \{x \in \mathbf{R}^d : |x| < r\}$ we have

$$P_{B_r}(x,y) = C_{d,\alpha} \left(\frac{r^2 - |x|^2}{|y|^2 - r^2} \right)^{\alpha/2} \frac{1}{|x-y|^d}, \quad x \in B_r, \ y \in B_r^c. \tag{3}$$

Note that $P_{B_r}(x,y) \approx f(x)g(y)$ at ∂B_r provided x and y are not too close. Similar approximate factorization of general P_D underlies the following theorem which is equivalent to the uniform BHP (UBHP) for $\Delta^{\alpha/2}$.

Theorem 1 (UBHP) *There is a constant $C_{d,\alpha}$, depending only on d and α, such that*

$$P_D(x_1,y_1)P_D(x_2,y_2) \le C_{d,\alpha}\, P_D(x_1,y_2)P_D(x_2,y_1), \tag{4}$$

for every $D \subset B_1$ provided $x_1, x_2 \in D \cap B_{1/2}$ and $y_1, y_2 \in B_1^c$.

We will often use the following auxiliary function

$$s_D(x) = \int_D G_D(x,v)dv. \tag{5}$$

Our next result is a refinement of (4).

Theorem 2 *If $0 \in \partial D$, $D \subset B_1$, and $|y| \ge 1$ then*

$$\lim_{D \ni x \to 0} \frac{P_D(x,y)}{s_D(x)} \ \textit{exists}. \tag{6}$$

We say that D is *thin* at a point $y \in \mathbf{R}^d$ if

$$\int_D s_{D \cap B(y,1)}(v)\, \nu(v,y)dv < \infty, \tag{7}$$

and we say that D is *thick* at y if

$$\int_D s_{D \cap B(y,1)}(v)\, \nu(v,y)dv = \infty. \tag{8}$$

We say D is *thin at infinity* if $s_D(x) < \infty$ for all $x \in D$; otherwise D is *thick at infinity.*

We consider the set ∂D^* of *limit* points of D: we let $\partial D^* = \partial D$ if D is bounded and $\partial D^* = \partial D \cup \{\infty\}$ if D is unbounded. Here, for unbounded D, $D \ni v \to \infty$ means that $v \in D$ and $|v| \to \infty$. We also let $D^* = D \cup \partial D^*$.

Theorem 1 and Theorem 2 apply to the asymptotics of G_D at ∂D, and to the structure of nonnegative functions harmonic for $\Delta^{\alpha/2}$, or α-harmonic, on D (for definitions see below). If $D \subset \mathbf{R}^d$ is *Greenian* we fix an arbitrary reference point $x_0 \in D$ and by using UBHP we define the Martin kernel of D:

$$M_D(x,y) = \lim_{D \ni v \to y} \frac{G_D(x,v)}{G_D(x_0,v)}, \quad x \in \mathbf{R}^d, \ y \in \partial D^*. \tag{9}$$

Theorem 3 *$M_D(x,y)$ is α-harmonic in x on D if and only if D is thick at y. If D is thin at $y \in \partial D$ then $M_D(x,y) = P_D(x,y)/P(x_0,y)$. If D is thin at infinity then $M_D(x,\infty) = s_D(x)/s_D(x_0)$.*

We define $\partial D_M = \{y \in \partial D^* : D \text{ is thick at } y\}$ and $D^c_M = \{y \in D^c : D \text{ is thin at } y\}$.

Theorem 4 *Let D be Greenian. For every $f \geq 0$ which is α-harmonic in D there are unique nonnegative measures λ on D^c_M, and μ on $\partial_M D$, such that*

$$f(x) = \int_{D^c_M} P_D(x,y)\lambda(dy) + \int_{\partial_M D} M_D(x,y)\mu(dy)\,, \quad x \in D\,. \tag{10}$$

As a part of the above statement we have that $|\mu| < \infty$, and

$$\int_{D^c} P_D(x_0,y)\lambda(dy) < \infty\,. \tag{11}$$

We remark that for non-Greenian D nonnegative harmonic functions are constant, see Lemma 15 below.

The above theorems complete and extend in several directions part of the results of [9], [30], [38], [31], [10], [20], [34]. The role of BHP in explicit determination of the Martin boundary in the classical potential theory is well recognized, see recent [1] and [2]; see also [6], [5], and [3] for more references. The role of BHP in estimating the Green function and studying Schrödinger-type operators is also well understood. We refer the reader to [11], [28], [12], [19], and [13], [14], [21]; and [24], [24] for a general viewpoint.

Theorem 3 and Theorem 4 contrast sharply with the corresponding results in the classical potential theory ([3], [35]), where the Martin kernel, if not as explicitly defined as in (9), is always harmonic, and the Martin boundary, tantamount to the domain of integration in the second integral in (10), is generally finer than the Euclidean boundary (see also [5] for Lipschitz domains). The first integral in (10) reflects the fact that $\Delta^{\alpha/2}$ is a representative of *nonlocal* integro-differential operators. The paper is primarily addressed to the readers interested in the potential theory of such operators. The theory presently undergoes a rapid development, see [27] and the references given there. The outline and notions which we propose below may likely apply to such operators and their nonnegative functions quite generally. Technically, the development hinges on Lemma 7 and Lemma 8 below, and extensions of these should be sought for in the more general context.

The remainder of the paper is organized as follows. In Section 2 we give preliminary definitions and results. In Section 3 we prove Theorem 1. We also state our UBHP in a more traditional form, see Corollary 1. Theorem 2 is verified, in a much stronger form, in Section 4. In Section 5 we define harmonicity. In Section 6 we verify Theorem 3 and joint continuity of $M_D(x,y)$. In Section 7 we obtain the Martin representation (10) along with its converse. In Section 8 we prove absolute continuity of harmonic measure on D^c_M and give examples of thin and thick boundary points. For instance $D = \{(x,y) \in \mathbf{R}^2 : y > |x|^\gamma\}$ is thin at 0 if and only if $\gamma < 1$.

2 Preliminaries

For $x \in \mathbf{R}^d$ and $r > 0$ we let $|x| = \sqrt{\sum_{i=1}^d x_i^2}$, $B(x,r) = \{y \in \mathbf{R} : |y-x| < r\}$, $B_r = B(0,r)$, and $B = B_1$. All the sets, functions and measures considered in the sequel will be Borel. For $U \subset \mathbf{R}^d$ we write $U^c = \mathbf{R}^d \setminus U$. If $k > 0$ then $kU = \{kx : x \in U\}$. For a measure λ on $\mathbf{R}^d$, $|\lambda|$ denotes its total mass. For a function f we let $\lambda(f) = \int f d\lambda$ if the integral makes sense. The probabilistic measure concentrated at x will be denoted by ε_x. For nonnegative f and g and a positive number C we write $f \asymp C\, g$ for $C^{-1} f \le g \le Cf$. In what follows U will be an arbitrary domain. We will say that U is Greenian if $G_U(x,v)$ is finite almost everywhere on $U \times U$. U is always Greenian when $\alpha < d$. If $\alpha \ge d = 1$, then U is Greenian if and only if U^c is non-polar. In particular, if $\alpha > d = 1$, then U is Greenian unless $U = \mathbf{R}$. Here and below we refer the reader to [36], [32], and [7].

If U is Greenian then

$$\int_U G_U(x,v)\Delta^{\alpha/2}\varphi(v)dv = -\varphi(x)\,, \quad x \in \mathbf{R}^d\,,\, \varphi \in C_c^\infty(U)\,. \tag{12}$$

Furthermore, $G_U(x,v) = G_U(v,x)$ for $x, v \in \mathbf{R}^d$. The harmonic measure, ω, may be used to negotiate betweeen Green functions of Greenian domains:

$$G_D(x,v) = G_U(x,v) + \int G_D(w,v)\omega_U^x(dw)\,, \quad x,v \in \mathbf{R}^d\,,\, U \subset D\,. \tag{13}$$

By integrating (13) against the Lebesgue measure we obtain

$$s_D(x) = s_U(x) + \int s_D(y)\omega_U^x(dy)\,, \quad x \in \mathbf{R}^d\,,\; U \subset D\,. \tag{14}$$

Recall that $\operatorname{supp}\omega_U^x \subset U^c$, $x \in \mathbf{R}^d$. If $U \subset D$ then

$$\omega_D^x(A) = \omega_U^x(A) + \int_{D\setminus U} \omega_D^y(A)\omega_U^x(dy)\,, \quad A \subset D^c\,, \tag{15}$$

in particular $G_U(x,v) \le G_D(x,v)$ and $\omega_U^x(A) \le \omega_D^x(A)$ provided $A \subset D^c$, $x, v \in U$. Moreover, if $D_1 \subset D_2 \subset \ldots$ and $D = \bigcup D_n$, then $G_{D_n}(x,v) \uparrow G_D(x,v)$ and $\omega_{D_n}^x(\varphi) \to \omega_D^x(\varphi)$ whenever $x, v \in D$ and $\varphi \in C_0(\mathbf{R}^d)$ ([7]).

A point $y \in D^c$ is called *regular* for D if $G_D(x,y) = 0$ for $x \in D$, and it is called *irregular* otherwise ([32], [36]).

Let $\varphi \in C_c^\infty(\mathbf{R}^d)$, $\operatorname{supp}\varphi \cap \overline{D} = \emptyset$, and let open Greenian D' contain both D and the support of φ. Using (12) for D and D', (13), and Fubini we obtain

$$\int_D G_D(x,v)\Delta^{\alpha/2}\varphi(v)dv = \int_{D^c} [\varphi(y) - \varphi(x)]\omega_D^x(dy)\,, \quad x \in D\,,\, \varphi \in C_c^\infty(\mathbf{R}^d). \tag{16}$$

By considering φ supported away from $\overline{D}$, and by (1) we conclude that on $(\overline{D})^c$, ω_D^x is absolutely continuous with respect to the Lebesgue measure, and has density $P_D(x,y)$ given by (2). This is the Ikeda-Watanabe formula [26]:

$$\omega_D^x(A) = \int_A P_D(x,y)dy\,, \quad \text{if } \operatorname{dist}(A,D) > 0\,. \tag{17}$$

If $D' \subset D$ is a Lipschitz domain (e.g. a ball) then $\omega_D^x(\partial D') \le \omega_{D'}^x(\partial D') = 0$ ([9]), hence

$$\omega_D^x(dy) = P_D(x,y)dy \text{ on } D'^c \quad \text{provided } x \in D \subset D' \text{ and } D' \text{ is Lipschitz}. \qquad (18)$$

The Green function of the ball is known explicitly:

$$G_{B_r}(x,v) = \mathcal{B}_{d,\alpha}\, |x-v|^{\alpha-d} \int_0^w \frac{s^{\alpha/2-1}}{(s+1)^{d/2}}\, ds\,, \quad x,v \in B_r, \qquad (19)$$

where

$$w = (r^2 - |x|^2)(r^2 - |v|^2)/|x-v|^2,$$

and $\mathcal{B}_{d,\alpha} = \Gamma(d/2)/(2^\alpha \pi^{d/2}[\Gamma(\alpha/2)]^2)$, see [8], [37]. It is also known ([14], [18]) that

$$s_{B_r}(x) = \frac{C_{d,\alpha}}{\mathcal{A}_{d,-\alpha}}(r^2 - |x|^2)^{\alpha/2}\,, \quad |x| \le r\,. \qquad (20)$$

For a nonnegative measure λ we define its *Poisson integral* on D:

$$P_D[\lambda](x) = \int_{D^c} P_D(x,y)\lambda(dy)\,, \quad x \in D\,,$$

compare (10). Furthermore we define

$$H_D[\lambda] = P_D[\lambda] + \lambda\,, \qquad (21)$$

as the function $P_D[\lambda]$ on D, and the measure λ restricted to D^c on D^c. We will regard λ (on D^c) as the *external values* (or the "boundary condition") of $H_D[\lambda]$.

If $U \subset D$, and $v \in U^c$ is such that $G_U(x,v) = 0$ for $x \in \mathbf{R}^d$, then by (13) we have

$$G_D(x,v) = \int G_D(w,v)\omega_U^x(dw)\,, \quad x \in U\,. \qquad (22)$$

This, and (24) below may be considered a mean value property.

For nonempty open $U \subset D$ we denote

$$\int H_D[\lambda](dy)\, \omega_U^x(dy) = \int_{D\setminus U} P_D[\lambda](y)\omega_U^x(dy) + \int_{D^c} P_U(x,y)\lambda(dy)\,, \quad x \in U\,. \qquad (23)$$

Lemma 1 *If $U \subset D$ and $\lambda \ge 0$ then*

$$H_D[\lambda](x) = \int H_D[\lambda](dy)\, \omega_U^x(dy)\,, \quad x \in U\,. \qquad (24)$$

PROOF: Let $x \in U$, $y \in D^c$. By integrating (13) against $\nu(v,y)dv$ on $\mathbf{R}^d$ we get

$$H_D[\varepsilon_y](x) = P_D(x,y) = P_U(x,y) + \int P_D(z,y)\omega_U^x(dz) = \int H_D[\varepsilon_y](dy)\, \omega_U^x(dy)\,. \qquad (25)$$

The case of general $\lambda \ge 0$ follows from Fubini-Tonelli theorem. $\square$

The next two lemmas are versions of Harnack inequality.

Lemma 2 *If* $\lambda \geq 0$ *and* $x_1, x_2 \in B_r \subset B_s \subset D$ *then*

$$P_D[\lambda](x_1) \leq \left(\frac{1+r/s}{1-r/s}\right)^{d-\alpha/2} P_D[\lambda](x_2)\,. \tag{26}$$

PROOF: By (3) we have $P_{B_r}(x_1, z) \leq (1+r/s)^{d-\alpha/2}(1-r/s)^{\alpha/2-d} P_{B_r}(x_2, z)$ if $|z| \geq r$. Using (25), (18), and (3) we prove the result. $\square$

Lemma 3 *If* $x_1, x_2 \in D$ *then there is* c_{x_1,x_2} *such that for every* $\lambda \geq 0$

$$H_D[\lambda](x_1) \leq c_{x_1,x_2} H_D[\lambda](x_2)\,. \tag{27}$$

PROOF: If $x_1, x_2 \in B_r \subset B_{2r} \subset D$ for some $r > 0$ then we are done by Lemma 2 with $c = c_{x_1,x_2}$ depending only on d and α. Assume that $B(x_1, 2r) \subset D$, $B(x_2, 2r) \subset D$, $B(x_1, 2r) \cap B(x_2, 2r) = \emptyset$ for some $r > 0$, and consider (25) with $U = B(x_1, r)$. Let $y \in D^c$. By (18) we obtain $P_D(x_1, y) \geq \int_{B(x_2,r)} cP_D(x_1, y)P_{B_r}(0, x - x_1)dx$. $\square$
If $K \subset D$ is compact and $x_1, x_2 \in K$ then c_{x_1,x_2} in Harnack's inequality above may be so chosen to depend only on K, D, and α. This follows from the same proof. Note that D may be disconnected.

Remark 1 If $\lambda \geq 0$ and $P_D[\lambda](x)$ is finite (positive) for some $x \in D$ then it is finite (positive, resp.) for all $x \in D$. This follows from Lemma 3. Note that if (11) holds then $P_D[\lambda]$ is finite and continuous on D, a consequence of (26) (see also the proof of Lemma 4 below).

The proof of the following well-known result is given for reader's convenience.

Lemma 4 G_D *is positive and jointly continuous:* $D \times D \mapsto (0, \infty]$.

PROOF: If $\overline{B}(z, s) \subset D$, $\lambda \geq 0$, and

$$f = H_{B(z,s)}[\lambda] \tag{28}$$

is finite on $B(z, s)$ then by (26)

$$(1 - |u|^2/s^2)^{d-\alpha/2} \leq \frac{f(z+u)}{f(x)} \leq (1 - |u|^2/s^2)^{\alpha/2-d}\,, \quad |u| < s\,. \tag{29}$$

This may be applied to the second term on the right of (13), where we use $U = B(z, s)$ with $z = x$ and $z = v$, and symmetry. Note that for this U the first term on the right of (13) is explicitly given by (19) and also positive on $U \times U$. Thus $G_D(x, y)$ is jointly continuous $D \times D \mapsto [0, \infty]$ and, by Lemma 3, $G_D(x, y) > 0$ on $D \times D$. $\square$

For clarity we note that G_D it is finite and locally uniformly continuous on $D \times D \setminus \{(x, x) :\ x \in D\}$ (on $D \times D$ if $\alpha > d = 1$), see (19).

Scaling will be important in what follows. Let $k > 0$. We have

$$\int_{kU} \nu(0, y)dy = k^{-\alpha} \int_U \nu(0, y)dy\,.$$

Similarly, if $\varphi_k(x) = \varphi(x/k)$ and $\varphi \in C_c^\infty(\mathbf{R}^d)$ then

$$\Delta^{\alpha/2}\varphi_k(x) = k^{-\alpha}\Delta^{\alpha/2}\varphi(x/k)\,, \quad x \in \mathbf{R}^d\,.$$

By (12) and uniqueness of the Green function we see that

$$G_{kU}(kx, kv) = k^{\alpha-d}G_U(x, v)\,, \quad x, v \in \mathbf{R}^d\,, \tag{30}$$

hence

$$s_{kU}(kx) = k^\alpha s_U(x)\,, \quad x \in \mathbf{R}^d\,, \tag{31}$$

and

$$P_{kD}(kx, ky) = k^{-d}P_D(x, y)\,, \quad x, y \in \mathbf{R}^d\,. \tag{32}$$

By (16) we also have that

$$\omega_{kD}^{kx}(kA) = \omega_D^x(A)\,, \quad x \in \mathbf{R}^d,\ A \subset \mathbf{R}^d\,. \tag{33}$$

Translation invariance is equally important but easier to observe; for example we have $G_{U+y}(x + y, v + y) = G_U(x, v)$. Both properties enable us to reduce many of the considerations below to the context of the unit ball centered at the origin.

3 Factorization of Poisson kernel

We keep assuming that D is a domain. Note that the constants in the estimates below are independent of D. When $0 < r \le 1$ we denote $D_r = D \cap B_r$ and $D'_r = B^c \cup D \setminus B_r$. Our first estimate is an extension of an observation made in [38, the proof of Lemma 3.1].

Lemma 5 *For every $p \in (0, 1)$ there is a constant $C_{d,\alpha,p}$ such that if $D \subset B$ then*

$$\omega_D^x(B^c) \le C_{d,\alpha,p}\, s_D(x)\,, \quad x \in D_p\,.$$

Proof: Let $0 < p < 1$. We choose a function $\varphi \in C_c^\infty(\mathbf{R}^d)$ such that $0 \le \varphi \le 1$, $\varphi(x) = 1$ if $|x| \le p$, and $\varphi(x) = 0$ if $|x| \ge 1$. Let $x \in D_p$. By (16) we have

$$\begin{aligned}\omega_D^x(B^c) &= \int_{B^c}(\varphi(x) - \varphi(y))\omega_D^x(dy) \le \int_{D^c}(\varphi(x) - \varphi(y))\omega_D^x(dy)\\ &= -\int_D G_D(x, y)\Delta^{\alpha/2}\varphi(y)dy\,.\end{aligned}$$

It remains to observe that $\Delta^{\alpha/2}\varphi$ is bounded and the lemma follows. □

For $x \in \mathbf{R}^d$, $r > 0$, and a nonnegative measure f on $\mathbf{R}^d$, we let

$$\Lambda_x(f) = \int_{\mathbf{R}^d}\nu(x, y)f(dy)\,, \quad \text{and} \quad \Lambda_{x,r}(f) = \int_{B(x,r)^c}\nu(x, y)f(dy)\,.$$

Note that if $k > 0$ and f_k is the measure defined by

$$\int\varphi(y)f_k(dy) = \int\varphi(ky)f(dy) \tag{34}$$

then

$$\Lambda_{0,kr}(f_k) = k^{-\alpha}\Lambda_{0,r}(f)\,. \tag{35}$$

Lemma 6 *Let $0 < p < 1$. There is $C_{d,\alpha,p}$ such that if $D \subset B$ and $\lambda \geq 0$ then*

$$H_D[\lambda](x) \leq C_{d,\alpha,p}\, \Lambda_{0,p}(H_D[\lambda])\,, \quad x \in D_p\,. \tag{36}$$

PROOF: Let $0 < p < q < r \leq 1$ and $x \in D_p$. By (24),

$$H_D[\lambda](x) = \int_{D'_r} H_D[\lambda](dy)\omega^x_{D_r}(dy) \leq \int_{D'_r} H_D[\lambda](dy)\omega^x_{B_r}(dy)\,,$$

and so Fubini-Tonelli theorem yields

$$H_D[\lambda](x) \leq \frac{1}{1-q}\int_q^1 \int_{D'_r} H_D[\lambda](dy)\omega^x_{B_r}(dy)dr = \int_{D'_q} K(x,y)H_D[\lambda](dy)\,,$$

where, according to (3),

$$K(x,y) = \frac{1}{1-q}\int_q^{1\wedge|y|} P_{B_r}(x,y)dr = \frac{C_{d,\alpha}}{1-q}\int_q^{1\wedge|y|}\left(\frac{r^2-|x|^2}{|y|^2-r^2}\right)^{\alpha/2}\frac{1}{|x-y|^d}\,dr\,.$$

Here and below $|y| \geq q$ and $r \leq 1\wedge|y|$, which implies that

$$\frac{|x-y|}{|y|} \geq \frac{q-p}{q}\,, \quad \frac{|y|+r}{|y|} \geq 1\,, \text{ and } \quad r^2 - |x|^2 \leq 1\,.$$

Thus

$$K(x,y) \leq \frac{C_{d,\alpha,q/p}}{|y|^{d+\alpha/2}}\int_q^{1\wedge|y|}\frac{dr}{(|y|-r)^{\alpha/2}} \leq \frac{C_{d,\alpha,q/p}}{|y|^{d+\alpha}}\,.$$

We conclude the proof by choosing, e.g., $q = (1+p)/2$. □

The above *regularization* of $P_{B_r}(x,y)$ (first applied in [9]) is a close analogue of volume averages in classical potential theory.

Lemma 7 *Let $0 < p < 1$. There is $C_{d,\alpha,p}$ such that for $f = H_D[\lambda]$, $\lambda \geq 0$, and $D \subset B$,*

$$C^{-1}_{d,\alpha,p}\Lambda_{0,p}(f)s_D(x) \leq f(x) \leq C_{d,\alpha,p}\Lambda_{0,p}(f)s_D(x)\,, \quad x \in D_p\,. \tag{37}$$

Furthermore, the lower bound for f is valid for all $x \in D$.

PROOF: Let $0 < p < q < r < 1$ and $x \in D_p$. By (24) we have that

$$f(x) = \int_{D'_r} f(y)\omega^x_{D_q}(dy) + \int_{D_r\setminus D_q} f(y)\omega^x_{D_q}(dy)\,. \tag{38}$$

If $v \in D_q$ and $y \in D'_r$, then $(r-q)/q \leq |y-v|/|y| \leq (r+q)/q$. Hence (18) yields

$$\begin{aligned}\int_{D'_r} f(y)\omega^x_{D_q}(dy) &= \int_{D'_r}\int_{D_q} G_{D_q}(x,v)\nu(v,y)f(y)dvdy \\ &\asymp C_{d,\alpha,r,q}\, s_{D_q}(x)\int_{D'_r}\nu(0,y)f(y)dy\,. \end{aligned} \tag{39}$$

The second integral of (38) is estimated by using Lemma 5, 6, and scaling (33, 31, 35):

$$\begin{aligned}\int_{D_r\setminus D_q} f(y)\omega^x_{D_q}(dy) &\le \omega^x_{D_q}(B^c_q)\sup_{D_r\setminus D_q} f \\ &\le C_{d,\alpha,p/q,r/q}\, s_{D_q}(x)\int_{D'_r}\nu(0,y)f(dy)\,. \end{aligned} \tag{40}$$

Since f is nonnegative, (38), (39) and (40) yield:

$$f(x) \asymp C_{d,\alpha,p,q,r}\, s_{D_q}(x)\int_{D'_r}\nu(0,y)f(dz)\,.$$

Clearly, $s_{D_q}(x) \le s_D(x)$. In view of (14) and Lemma 5 we also have that

$$\begin{aligned} s_D(x) &= s_{D_q}(x) + \int_{D\setminus D_q} s_D(z)\omega^x_{D_q}(dz) \le s_{D_q}(x) + \omega^x_{D_q}(B^c_q)\sup_D s_D \\ &\le s_{D_q}(x)(1 + C_{d,\alpha,p,q}\sup_B s_B) = C_{d,\alpha,p,q}\, s_{D_q}(x)\,. \end{aligned}$$

Of course, $\int_{D'_r}\nu(0,y)f(y)dy \le \int_{B^c_p}\nu(0,y)f(y)dy$. Lemma 6 yields that also

$$\int_{B^c_p}\nu(0,y)f(y)dy \le \int_{D'_r}\nu(0,y)f(y)dy + \frac{C_{d,\alpha}|D_r|}{p^{d+\alpha}}\sup_{D_r} f \le C_{d,\alpha,p,r}\int_{D'_r}\nu(0,y)f(y)dy\,.$$

This proves (37). Moreover, for any $x \in D$ we have

$$\begin{aligned} f(x) &= \int_{B^c}\int_D G_D(x,z)\nu(z,y)f(y)dzdy \ge C_{d,\alpha}\, s_D(x)\int_{B^c}\nu(0,y)f(y)dy \\ &\ge C_{d,\alpha,p}\, s_D(x)\int_{B^c_p}\nu(0,y)f(y)dy\,. \quad \square \end{aligned}$$

Remark 2 *Scaling* leaves (37) invariant. Indeed, let $f = H_D[\lambda]$, $k > 0$, and let f_k be defined by (34). By (33, 32) $f_k = H_{kD}[\lambda_k]$, in particular $f_k(x) = f(x/k)$ on kD. By (35) and (31) we have that $\Lambda_{0,kp}(f_k)s_{kD}(x) = \Lambda_{0,p}(f)s_D(x/k)$, which proves our claim.

Remark 3 The constant $C_{d,\alpha,p}$ in (37) may be considered nondecreasing in p. Indeed, if $0 < p_1 < p_2 < 1$ and $f \le C_{d,\alpha,p_2}\Lambda_{0,p_2}(f)s_D$ on D_{p_2} then $f \le C_{d,\alpha,p_2}\Lambda_{0,p_1}(f)s_D$ on D_{p_1}. Similarly, if $C^{-1}_{d,\alpha,p_1}\Lambda_{0,p_1}(f)s_D \le f$ on D then $C^{-1}_{d,\alpha,p_1}\Lambda_{0,p_2}(f)s_D \le f$ on D.

Proof of Theorem 1: Lemma 7 with $p = 1/2$ and $\lambda = \varepsilon_{y_i}$, $i = 1, 2$, yields

$$\begin{aligned} P_D(x_1,y_1)P_D(x_2,y_2) &\le C^2_{d,\alpha,1/2}\Lambda_{0,1/2}(H_D[\varepsilon_{y_1}])s_D(x_1)\Lambda_{0,1/2}(H_D[\varepsilon_{y_2}])s_D(x_2) \\ &\le C^4_{d,\alpha,1/2}P_D(x_1,y_2)P_D(x_2,y_1)\,. \quad \square \end{aligned}$$

We end this section with a simple corollary of Lemma 7, which generalizes Theorem 1 and states our uniform BHP in a more traditional form. Note that the constant in the estimate does not depend on D.

Corollary 1 *Let $G \subset \mathbf{R}^d$ be open and let $K \subset G$ be compact. There is a constant $C = C_{d,\alpha,G,K}$ with the following property. If $D \subset \mathbf{R}^d$ is open and f, g are nonnegative Poisson integrals on $D \cap G$ equal to 0 in $G \setminus D$, then*

$$C^{-1}\frac{f(y)}{g(y)} \le \frac{f(x)}{g(x)} \le C\frac{f(y)}{g(y)}, \quad x, y \in D \cap K. \tag{41}$$

PROOF: Let $p = 1/2$. For each $x \in K$ we take any ball $B(x, r_x) \subset G$. The family $\{B(x, p\,r_x) : x \in K\}$ is an open covering of K. We choose a finite sub-covering $\{B(x_1, p\,r_{x_1}), \ldots, B(x_n, p\,r_{x_n})\}$. We let $r_j = r_{x_j}$, $B_j = B(x_j, r_j)$, $\widetilde{B}_j = B(x_j, p\,r_j)$, $R = \operatorname{diam} K$ and $r = \min\{r_1, \ldots, r_n\}$. Let $x \in D \cap \widetilde{B}_i$, $y \in D \cap \widetilde{B}_j$ and let f be a nonnegative Poisson integral on $G \cap D$ and equal to 0 in $G \setminus D$. Applying Lemma 7 once in the first inequality below and twice in the third one, and using the inequality $|z - x_j| \le R + |z - x_i| \le \frac{R+r}{r}|z - x_i|$ in the second one, we obtain:

$$\begin{aligned}
\frac{f(x)}{s_{D\cap B_i}(x)} &\le C_{d,\alpha,p}\left(\int_{\widetilde{B}_i^c \cap \widetilde{B}_j^c} \nu(x_i, z) f(z)dz + \int_{\widetilde{B}_i^c \cap \widetilde{B}_j} \nu(x_i, z) f(z)dz\right)\\
&\le C_{d,\alpha,p}\left(C_{d,\alpha,r,R}\int_{\widetilde{B}_j^c} \nu(x_j, z) f(z)dy + C_{d,\alpha,r}\int_{D\cap\widetilde{B}_j} f(z)dy\right)\\
&\le C_{d,\alpha,p,r,R}\left(\frac{f(y)}{s_{D\cap B_j}(y)} + \Lambda_{x_j,p\,r_j}(f)\int_{D\cap\widetilde{B}_j} s_{D\cap B_j}(z)dz\right).
\end{aligned}$$

But $s_{D\cap B_j}(z) \le s_{B_j}(z)$, which does not exceed a constant dependent only on d, α, R, and, again by Lemma 7, $\Lambda_{x_j,p\,r_j}(f) \le C_{d,\alpha,p}\, f(y)/s_{D\cap B_j}(y)$. Therefore

$$\frac{f(x)}{s_{D\cap B_i}(x)} \le C_{d,\alpha,p,r,R}\frac{f(y)}{s_{D\cap B_j}(y)}.$$

Corollary 1 follows. In fact, C in (41) depends only on d, α, $\operatorname{diam} G$ and $\operatorname{dist}(K, G^c)$. □

Remark 4 Let $D \subset \mathbf{R}^d$ be open, $U \subset D$ bounded, $f = H_D[\lambda]$, $\operatorname{supp}\lambda = A \neq \emptyset$ and $\operatorname{dist}(A, D) > 0$. Then if f is finite at one point $x_0 \in U$ then f is bounded on U. This follows from Corollary 1 applied to $K = \overline{U}$, G an open set such that $K \subset G \subset A^c$ and $g = P_{D\cap G}(\mathbf{1}_A(x)dx)$.

Remark 5 Let $D \subset \mathbf{R}^d$ be an open Greenian set, $x_0 \in D$ a fixed point and $f(x) = G_D(x, x_0)$. It is well known that the set $\{x \in \partial D : f(x) > 0\}$ is polar so it is of Lebesgue measure zero. Let G be an open bounded Lipschitz domain, $D \cap G \neq \emptyset$, and assume that $x_0 \notin G$. Then $\omega_G^x(\partial G) = 0$ for $x \in G$. It follows from the above and (22) that

$$f(x) = \int_{D\setminus\overline{G}} f(w) P_{D\cap G}(x, w)\, dw, \quad x \in D \cap G,$$

so f is a Poisson integral on $D \cap G$.

Therefore one may apply Corollary 1 to D, G and f as above. In particular, one may also use Remark 4 to function f. It follows that for arbitrary $r > 0$ the function f is bounded on any bounded subset of $B^c(x_0, r) \cap D$.

4 Existence of limits

For a positive function q on a nonempty set U we define its relative oscillation:

$$\mathrm{RO}_U\, q = \mathrm{RO}_{x\in U}\, q(x) = \frac{\sup_{x\in U} q(x)}{\inf_{x\in U} q(x)}\,.$$

For notational convenience, we put $\mathrm{RO}_U\, q = 1$ if $U = \emptyset$.

The main result of this section addresses the asymptotics of Poisson integrals at $x = 0$. (26) gives a motivation for (42), but here $x = 0$ may be, e.g., a boundary point of D.

Lemma 8 *For every $\eta > 0$ there exists $r > 0$ such that*

$$\mathrm{RO}_{D\cap B_r} \frac{H_D[\lambda_1]}{H_D[\lambda_2]} \le 1 + \eta \tag{42}$$

for all open $D \subset B$ and nonzero nonnegative measures λ_1, λ_2 on B^c satisfying (11).

Proof: Let c denote the constant $C_{d,\alpha,1/2}$ of Lemma 7. Recall from the proof of Theorem 1 that (4) holds with $C_{d,\alpha} = c^4$. Thus, (42) holds with $1+\varepsilon$ replaced by c^4. We will show that the left hand side of (42) is self-improving when $r \to 0^+$. This will be done under each of the two complementary assumptions: (44) and (48) below. First, however, we need some preparation. For $0 < p < q < 1/2$ and a measure f we let $D_{p,q} = D_q \setminus D_p$,

$$\Lambda_{x,p,q}(f) = \int_{D_{p,q}} \nu(x,y) f(dy)\,, \quad f^{p,q} = H_{D_p}[\mathbf{1}_{D_{p,q}} f]\,, \quad \text{and} \quad \widetilde{f}^{p,q} = H_{D_p}[\mathbf{1}_{D'_q} f]\,.$$

What follows will be valid with $i = 1$ *and* with $i = 2$. Let $f_i = H_D[\lambda_i]$. By (24) we have $f_i = f_i^{p,q} + \widetilde{f}_i^{p,q}$. For $r \in (0, 1/2]$ we denote $m_r = \inf_{D_r}(f_1/f_2)$ and $M_r = \sup_{D_r}(f_1/f_2)$. As we noted above, $M_r \le c^4 m_r$. Let $\varepsilon > 0$.

Let $q \in (0, 1/2]$ and let $p = p(q) \in (0, q/2)$ (depending on p and ε) be given by

$$(q + 2p)/(q - 2p) = 1 + \varepsilon\,, \tag{43}$$

so that if $z \in D_{2p}$ and $y \in D'_q$ then $(1+\varepsilon)^{-d-\alpha}\nu(0,y) \le \nu(z,y) \le (1+\varepsilon)^{d+\alpha}\nu(0,y)$. Thus, for $x \in D_{2p}$ we have

$$\widetilde{f}_i^{2p,q}(x) = \int_{D'_q}\int_{D_{2p}} G_{D_{2p}}(x,z)\nu(z,y) f_i(y) dz dy \le (1+\varepsilon)^{d+\alpha} \Lambda_{0,q}(f_i) s_{D_{2p}}(x)\,,$$

and

$$\widetilde{f}_i^{2p,q}(x) \ge (1+\varepsilon)^{-d-\alpha} \Lambda_{0,q}(f_i) s_{D_{2p}}(x)\,.$$

We will now examine consequences of the following assumption:

$$\Lambda_{0,p,q}(f_i) \le \varepsilon\, \Lambda_{0,q}(f_i)\,, \quad i = 1, 2\,. \tag{44}$$

If (44) holds then using the full statement of Lemma 7, and Remark 2 we obtain

$$f_i^{2p,q}(x) \le c\, s_{D_{2p}}(x) \Lambda_{0,p}(f_i^{2p,q}) \le c\, s_{D_{2p}}(x)\Lambda_{0,p,q}(f_i) \le c\,\varepsilon\, s_{D_{2p}}(x) \Lambda_{0,q}(f_i)\,, \quad x \in D_p\,.$$

Recall that $f_i = f_i^{2p,q} + \widetilde{f}_i^{2p,q}$. Thus, if (44) holds then we have

$$\frac{(1+\varepsilon)^{-d-\alpha}\Lambda_{0,q}(f_1)}{(c\,\varepsilon+(1+\varepsilon)^{d+\alpha})\Lambda_{0,q}(f_2)} \le \frac{f_1(x)}{f_2(x)} \le \frac{(c\,\varepsilon+(1+\varepsilon)^{d+\alpha})\Lambda_{0,q}(f_1)}{(1+\varepsilon)^{-d-\alpha}\Lambda_{0,q}(f_2)}\,, \quad x\in D_p\,, \tag{45}$$

and, finally,

$$\mathrm{RO}_{D_p}\frac{f_1}{f_2} \le (c\,\varepsilon+(1+\varepsilon)^{d+\alpha})^2(1+\varepsilon)^{2d+2\alpha}\,. \tag{46}$$

We are satisfied with (46) for the moment. We consider $0 < p' < q'/4 < q' < 1/2$, $g = f_1^{2p',q'} - m_{q'} f_2^{2p',q'}$, and $h = M_{q'} f_2^{2p',q'} - f_1^{2p',q'}$. Note that on $D_{2p'}$ both g and h are Poisson integrals of nonnegative measures. By Theorem 1,

$$\sup_{D_{p'}}\frac{f_1^{2p',q'}}{f_2^{2p',q'}} - m_{q'} = \sup_{D_{p'}}\frac{g}{f_2^{2p',q'}} \le c^4\inf_{D_{p'}}\frac{g}{f_2^{2p',q'}} = c^4\left(\inf_{D_{p'}}\frac{f_1^{2p',q'}}{f_2^{2p',q'}} - m_{q'}\right)\,,$$

and

$$M_{q'} - \inf_{D_{p'}}\frac{f_1^{2p',q'}}{f_2^{2p',q'}} = \sup_{D_{p'}}\frac{h}{f_2^{2p',q'}} \le c^4\inf_{D_{p'}}\frac{h}{f_2^{2p',q'}} = c^4\left(M_{q'} - \sup_{D_{p'}}\frac{f_1^{2p',q'}}{f_2^{2p',q'}}\right)\,,$$

hence

$$(c^4+1)\left(\sup_{D_{p'}}\frac{f_1^{2p',q'}}{f_2^{2p',q'}} - \inf_{D_{p'}}\frac{f_1^{2p',q'}}{f_2^{2p',q'}}\right) \le (c^4-1)(M_{q'}-m_{q'})\,. \tag{47}$$

We will now examine consequences of the following assumption:

$$\Lambda_{0,q'}(f_i) \le \varepsilon\Lambda_{0,2p',q'}(f_i)\,. \tag{48}$$

By Lemma 7 and Remark 2 applied to $D_{2p'} \subset B_{q'}$ for $x \in D_{p'}$ we have

$$\widetilde{f}_i^{2p',q'}(x) \le c\,s_{D_{2p'}}(x)\Lambda_{0,2p'}(\widetilde{f}_i^{2p',q'}) = c\,s_{D_{2p'}}(x)\Lambda_{0,q'/2}(\widetilde{f}_i^{2p',q'}) \le c\,s_{D_{2p'}}(x)\Lambda_{0,q'}(f_i)\,.$$

Hence, by our assumption (48), Lemma 7 and Remark 2 applied to $D_{p'} \subset B_{2p'}$

$$\widetilde{f}_i^{2p',q'}(x) \le c\,\varepsilon\,s_{D_{2p'}}(x)\Lambda_{0,2p',q'}(f_i) \le c\,\varepsilon\,s_{D_{2p'}}(x)\Lambda_{0,p'}(f_i^{2p',q'}) \le c^2\varepsilon\,f_i^{2p',q'}(x)\,, \quad x\in D_{p'}\,.$$

Since $f_i = f_i^{2p',q'} + \widetilde{f}_i^{2p',q'}$, this and (47) yield

$$(c^4+1)\left(M_{p'}/(1+c^2\varepsilon) - m_{p'}(1+c^2\varepsilon)\right) \le (c^4-1)(M_{q'}-m_{q'})\,.$$

Note that $m_{p'} \ge m_{q'}$. Dividing by $m_{q'}$ finally gives

$$\mathrm{RO}_{D_{p'}}\frac{f_1}{f_2} \le (1+c^2\varepsilon)^2 + (1+c^2\varepsilon)\frac{c^4-1}{c^4+1}\left(\mathrm{RO}_{D_{q'}}\frac{f_1}{f_2} - 1\right)\,. \tag{49}$$

We now come to the conclusion of our considerations. Let $\eta > 0$. If ε is small enough then the right side of (46) is smaller that $1+\eta$ and right side of (49) does not exceed $\varphi(\mathrm{RO}_{D_{q'}}(f_1/f_2))$, where

$$\varphi(t) = 1 + \frac{\eta}{2} + \frac{c^4}{c^4+1}(t-1)\,.$$

Let $\varphi^1 = \varphi$, $\varphi^{l+1} = \varphi(\varphi^l)$, $l = 1, 2, \ldots$. Observe that $\varphi(t) = t$ for $t = 1 + \eta(c^4+1)/2$, and $\varphi(t) < t$ for $t > 1 + \eta(c^4+1)/2$. Thus the l-fold compositions $\varphi^l(c^4)$ converge to $1 + \eta(c^4+1)/2$ as $l \to \infty$. In what follows let l be such that

$$\varphi^l(c^4) < 1 + \eta(c^4+1)\,.$$

Let k be the least integer such that $k - 1 > c^2/\varepsilon^2$. We denote $n = lk$. Let $q_0 = 1/2$, $q_{j+1} = p(q_j)$ for $j = 0, \ldots, n-1$ (see (43)), and $r = q_n$. If for any $j < n$, (44) holds with $q = q_j$ and $p = p(q) = q_{j+1}$, then

$$\mathrm{RO}_{D_r} \frac{f_1}{f_2} \le \mathrm{RO}_{D_{q_{j+1}}} \frac{f_1}{f_2} \le 1 + \eta\,,$$

and we are done. Otherwise for $j = 0, \ldots, n-1$, we have $\Lambda_{0,q_{j+1},q_j}(f_i) > \varepsilon \Lambda_{0,q_j}(f_i)$ for $i = 1$ or $i = 2$. Note that by Lemma 7

$$c^{-1} \frac{f_i(x)}{\Lambda_{0,q_j}(f_i)} \le s_{D_{2q_j}}(x) \le c \frac{f_{3-i}(x)}{\Lambda_{0,q_j}(f_{3-i})}\,, \quad x \in D_{q_{j+1},q_j}\,.$$

Hence $\Lambda_{0,q_{j+1},q_j}(f_i) > c^{-2}\varepsilon\, \Lambda_{0,q_j}(f_i)$ for both $i = 1, 2$. Let $p' = q_{(j+1)k}$ and $q' = q_{jk}$ for some j, $0 \le j < l$. Then:

$$\Lambda_{0,2p',q'}(f_i) \ge \Lambda_{0,q_{(j+1)k-1},q_{jk}}(f_i) \ge (k-1)c^{-2}\varepsilon\, \Lambda_{0,q'}(f_i) \ge \varepsilon^{-1} \Lambda_{0,q'}(f_i)\,,$$

that is (48) is satisfied. We conclude that (49) holds. Recall that $\mathrm{RO}_{D_{1/2}}(f_1/f_2) \le c^4$. By the definition of l and monotonicity of φ

$$\mathrm{RO}_{D_{q_{lk}}} \frac{f_1}{f_2} \le \varphi\left(\mathrm{RO}_{D_{q_{(l-1)k}}} \frac{f_1}{f_2}\right) \le \ldots \le \varphi^l\left(\mathrm{RO}_{D_{q_0}} \frac{f_1}{f_2}\right) \le 1 + \eta(c^4+1)\,,$$

i.e. $\mathrm{RO}_{D_r}(f_1/f_2) \le 1 + \eta(c^4+1)$. Since $\eta > 0$ was arbitrary, the proof is complete. □

PROOF OF THEOREM 2: For bounded D, by (2) and (5) we have that $s_D(x) = \lim_{|z|\to\infty} P_D(x,z)/\nu(0,z)$. We apply Lemma 8 to $\lambda_1 = \varepsilon_y$ and $\lambda_2 = \varepsilon_z$. It follows that $\mathrm{RO}_{D_r} f/s_D \to 1$ as $r \to 0^+$, which is equivalent to the convergence of f/s_D to a finite, positive limit. □

As an addition to Theorem 2 we note that if 0 is thin for D, then we have:

$$\lim_{D\ni x\to 0} \frac{f_1(x)}{f_2(x)} = \frac{\int \nu(0,y) f_1(y)dy}{\int \nu(0,y) f_2(y)dy} \quad \text{and} \quad \lim_{D\ni x\to 0} \frac{f_1(x)}{s_D(x)} = \frac{\int \nu(0,y) f_1(y)dy}{\int \nu(0,y) s_D(y)dy + 1} \tag{50}$$

for every nonnegative Poisson integrals f_1, f_2 on D. Indeed, by Theorem 1 the integrals $\int \nu(0,y) f_i(y)dy$ are finite. Hence, for every $\varepsilon > 0$ we can find $q > 0$ such that (44) is satisfied with $p = q/2$. It follows that (45) holds. Since ε was arbitrary, the first equality is proved. The second one follows easily by using $s_D(x) = \lim_{|z|\to\infty} P_D(x,z)/\nu(0,z)$.

5 Harmonicity

Let f be a nonnegative continuous function on D and a nonnegative measure on D^c.

Definition 1 *We say that f is α-harmonic in D if for every open U precompact in D*

$$f(x) = \int f(y)\omega_U^x(dy)\,, \quad x \in U\,, \tag{51}$$

where

$$\int f(y)\omega_U^x(dy) = \int_{D\setminus U} f(y)\omega_U^x(dy) + \int_{D^c} P_U(x,y)f(dy)\,.$$

The definition slightly extends the usual definition of a harmonic function (see, e.g., [9]) by allowing measure values on D^c. This is natural in view of the definition of $H_D[\lambda]$, see (21), (23), and the next paragraph. The letter H in (21) suggests "hybrid harmonic function". The case of genuine functions f is of course included by letting $f(dy) = f(y)dy$.

Formula (12) yields that the function $x \mapsto G_D(x,y)$ is α-harmonic on $D \setminus \{y\}$. Also, $x \mapsto \omega_D^x(A)$ is α-harmonic on D for every set A by (15). Lastly, it follows from (2) and (13) that $f = P_D(\cdot, y) + \varepsilon_y$ is α-harmonic in D. If a measure λ is nonnegative and the hybrid $H_D[\lambda]$ is finite on D then by Fubini-Tonelli and Remark 1 it is α-harmonic in D.

By the same token, $x \mapsto P_D(x,y)$ is *not* α-harmonic in D. To be absolutely clear on this, we consider $D = B$. We note that the function $x \mapsto P_B(x,y)$ vanishes on B^c, and has a maximum inside B. Thus the mean-value property (51) cannot hold.

We note that for (nonnegative) f which is α-harmonic on (nonempty open) D by (3) and (18) we necessarily have that

$$\int_{\mathbf{R}^d} (1+|y|)^{-d-\alpha} f(dy) < \infty\,. \tag{52}$$

If f is a (genuine) function on $\mathbf{R}^d$ continuous on D then (51) is equivalent to

$$\Delta^{\alpha/2} f(x) = 0\,, \quad x \in D\,. \tag{53}$$

The result is given in [13], and its proof can be extended without difficulty to the present more general setting. However, we will not use (53) in the sequel, and we leave the verification to the interested reader. We also refer the reader to [16] to see the limitations of (53) in defining harmonic functions for other operators as opposed to (51).

To further deal with U touching ∂D or f charging ∂D in (51) we need auxiliary considerations. Let $D^{(r)}$ be the set of *regular* boundary points for D (see Section 2). It is known that $\omega_U^x(\partial D \setminus D^{(r)}) = 0$ and $|\partial D \setminus D^{(r)}| = 0$ for every open U and $x \in U$ ([32]). For open $U \subset D$ suppose that f has a (necessarily unique) continuous extension to $U \cup U^{(r)}$ and does not charge $\partial U \setminus U^{(r)}$. For a set A we then define

$$\int_A f(y)\omega_U^x(dy) = \int_{A\cap U^{(r)}} f(y)\omega_U^x(dy) + \int_{A\setminus \overline{U}} P_U(x,y)f(dy)\,.$$

Here in the first integral on the right we employ the continuous extension of f. We note that if f is a genuine function then the second integral equals $\int_{A\setminus\overline{U}} f(y)\omega_U^x(dy)$ by (17).

Lemma 9 *Let f be α-harmonic in bounded D. Suppose that f has a density function on ∂D which continuously extends f to a bounded function on $D \cup D^{(r)}$. Then*

$$f(x) = \int_{D^c} f(y)\omega_D^x(dy)\,, \quad x \in D\,. \tag{54}$$

PROOF: Let D_n be an increasing sequence of open sets precompact in D such that $\bigcup D_n = D$. Recall that $P_{D_n}(x,y) \nearrow P_D(x,y)$ and $\omega^x_{D_n} \to \omega^x_D$ weakly as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \int\limits_{\overline{D}^c} f(y)\omega^x_{D_n}(dy) = \lim_{n\to\infty} \int\limits_{\overline{D}^c} P_{D_n}(x,y)f(dy) = \int\limits_{\overline{D}^c} P_D(x,y)f(dy) = \int\limits_{\overline{D}^c} f(y)\omega^x_D(dy)\,,$$

and, since f is bounded and continuous on $\overline{D}$ except on a set of zero harmonic measure,

$$\lim_{n\to\infty} \int\limits_{\overline{D}} f(y)\omega^x_{D_n}(dy) = \lim_{n\to\infty} \int\limits_{D\cup D^{(r)}} f(y)\omega^x_{D_n}(dy) = \int\limits_{D\cup D^{(r)}} f(y)\omega^x_D(dy) = \int\limits_{\overline{D}} f(y)\omega^x_D(dy)\,.$$

This proves (54). □

We note in passing that (54) implies α-harmonicity through (15). Generally, the reverse implication is not true as we will see is the case for the Martin kernel with the pole at a thick boundary point. (54) was coined *regular α-harmonicity* in [9] and is a useful concept in boundary potential theory, see also [17], [10].

Lemma 10 *Suppose that $0 \le g \le f$ on $\mathbf{R}^d$, $f = 0$ on D^c, and f, g are α-harmonic on D. If $U \subset D$ and $f(x) = \int_{U^c} f(y)\omega^x_U(dy)$, $x \in U$, then $g(x) = \int_{U^c} g(y)\omega^x_U(dy)$, $x \in U$.*

PROOF: Here f and g are genuine functions vanishing on D^c. Let D_n be an increasing sequence of open sets precompact in D such that $D = \bigcup D_n$. Let $U_n = U \cap D_n$. Then

$$0 \le \int_{U\setminus U_n} g(y)\omega^x_{U_n}(dy) \le \int_{U\setminus U_n} f(y)\omega^x_{U_n}(dy) = f(x) - \int_{D\setminus U} f(y)\omega^x_{U_n}(dy)\,, \quad x \in U_n\,.$$

By weak convergence of harmonic measures and disjointness of $D \setminus U$ and $\bigcap \overline{U \setminus U_n}$, we conclude that $\omega^x_{U_n}$ increase to ω^x_U on $D \setminus U$. Hence, by monotone convergence, the right side tends to 0 when $n \to \infty$. Thus

$$g(x) = \lim_{n\to\infty} \int_{D\setminus U} g(y)\omega^x_{U_n}(dy) = \int_{D\setminus U} g(y)\omega^x_U(dy)\,. \quad \square$$

Lemma 11 *Suppose that D_1 and D_2 are bounded open sets such that $\partial(D_1 \setminus D_2)$ and $\partial(D_2 \setminus D_1)$ are disjoint. Let f be a nonnegative measure with bounded density function on closure of $D = D_1 \cup D_2$ which satisfies $f(x) = \int_{D_i^c} f(y)\omega^x_{D_i}(dy)$ for $x \in D_i$, $i = 1, 2$. Then $f(x) = \int_{D^c} f(y)\omega^x_D(dy)$ for $x \in D$.*

PROOF: Let $\mu^x_0 = \varepsilon_x$ and $\mu^x_{n+1} = \int \mu^y_n \omega^x_{D_i}(dy)$ with $i = 1$ for n even and $i = 2$ for odd n. By our assumption $\text{dist}(D_1 \setminus D_2, D_2 \setminus D_1) > 0$ and so there is $c < 1$ such that

$$\omega^x_{D_i}(D) < c\,, \quad x \in D_i \setminus D_j\,, i \ne j\,.$$

Hence $\mu_n(D)$ tends to 0. On the other hand, μ_n is increasing on D^c and the limit measure satisfies (13), hence it equals ω^x_D. The result follows. □

Lemma 11 applies, e.g., if D_1, D_2 are overlapping finite open intervals on the line.

6 Martin kernel

We note that for open Greenian D,

$$\int_{\mathbf{R}^d} G_D(x,v)(1+|v|)^{-d-\alpha}dv < \infty\,, \quad x \in \mathbf{R}^d\,. \tag{55}$$

This follows from (52). Thus, by BHP, (7) is equivalent to

$$\int_{\mathbf{R}^d} G(x,v)\nu(0,v)dv < \infty\,, \qquad \text{for some (all) } \; x \in D\,. \tag{56}$$

Proof of Theorem 3: Let D be open and Greenian, and let $y \in \partial D$. By translation invariance, to study $M_D(\cdot, y)$ we may assume with no loss of generality, that $y = 0$.

Let $x \in D$, $\rho = 1 \wedge (|x| \wedge |x_0|)/2$, and $D_\rho = D \cap B(0,\rho)$. By Harnack inequality in the first variable, $G_D(x,v)/G_D(x_0,v)$ is bounded from above and below for $v \in D_\rho$. Also, $G_D(x,v) = P_{D_\rho}[G(x_0,u)du](v)$, $G_D(x_0,v) = P_{D_\rho}[G(x,u)du](v)$ for $v \in D_\rho$. Lemma 8, applied to D_ρ, yields that $M_D(x,0)$ is well-defined by (9). Clearly, $0 < M_D(x,0) < \infty$.

Denote $M(x) = M_D(x,0)$. If D is thin at 0, then by (50) we have

$$M(x) = \int_D \nu(0,y)G_D(x,y)dy \Big/ \int_D \nu(0,y)G_D(x_0,y)dy = P_D(x,0)/P_D(x_0,0)\,,$$

in particular $M(x)$ is *not* α-harmonic on D. However, if D is thick at 0 then

$$M(x) = \int_{D\setminus U} M(y)\omega_U^x(dy)\,, \quad x \in U\,, \tag{57}$$

for every $U = D \setminus \overline{B}_R$ with $R > 0$. Indeed, (57) is equivalent to uniform integrability of $G_D(y,z)/G_D(x_0,z)$ with respect to $\omega_U^x(dy)$ on the (bounded) set $D\setminus U$ as $D \ni z \to 0$. Let $0 < r < \min(R/4, 1, |x_0|/4)$ and $z_0 \in D_r$ be a fixed point. For $y \in D_R \setminus D_{3r}$ and $z \in D_r$, Remark 5 yields that

$$\frac{G_D(y,z)}{G_D(x_0,z)} \le C_{d,\alpha,r}\frac{G_D(y,z_0)}{G_D(x_0,z_0)}.$$

Again by Remark 5 we obtain that $\sup_{y\in D_R\setminus D_{3r}} G_D(y,z_0) < \infty$. Thus we only need to estimate $\int_{D_{3r}} G_D(z,y)\omega_U^x(dy)/G_D(x_0,z)$ for $z \in D_r$.

Since the density function (Poisson kernel) of ω_U^x is bounded on $D_{3R/4}$, we have

$$\int_{D_{3r}} G_D(y,z)\omega_U^x(dy) \le C_{d,\alpha,D,R} \int_{D_{3r}} G_D(y,z)dy\,. \tag{58}$$

On the other hand, $\omega^{x_0}_{D\setminus\overline{D}_{3r}}$ is absolutely continuous on $\overline{D}_{3r}$ with respect to the Lebesgue measure, and has $P_{D\setminus\overline{D}_{3r}}(x_0,\cdot)$ as density function. Thus

$$\begin{aligned} G_D(x_0,z) &= \int_{D_{3r}} G_D(y,z)P_{D\setminus\overline{D}_{3r}}(x_0,y)dy \\ &= \int_{D_{3r}}\int_{D\setminus\overline{D}_{3r}} G_D(y,z)G_{D\setminus\overline{D}_{3r}}(x_0,\zeta)\nu(\zeta,y)d\zeta dy \\ &\ge 2^{-d-\alpha}\left(\int_{D_{3r}} G_D(y,z)dy\right)\left(\int_{D\setminus\overline{D}_{3r}} G_{D\setminus\overline{D}_{3r}}(x_0,\zeta)\nu(\zeta,0)d\zeta\right)\,. \end{aligned} \tag{59}$$

The last integral becomes arbitrarily large when r is small enough. This is because $\int_D G_D(x_0,\zeta)\nu(\zeta,0)d\zeta=\infty$, D is thick at 0, and $G_D(x_0,\cdot)\approx s_D(\cdot)$ at 0 by Lemma 7.

Combining this, (59), and (58), we obtain the uniform integrability, and (57). In fact, (15) yields (57) for every open $U\subset D$ provided $0\notin\overline{U}$. In particular, M is a (genuine) function α-harmonic on D. Regarding a remark in Section 5 we note that $f=M$ violates (54) because M vanishes on D^c.

We now turn to the Martin kernel with the pole at infinity. Let $x\in D$. If $D=\mathbf{R}^d$ and $\mathbf{R}^d$ is Greenian, or $\alpha<d$, then $M_D(x,\infty)=\lim_{|v|\to\infty}|v-x|^{\alpha-d}/|v-x_0|^{\alpha-d}=1$, $s_D\equiv\infty$, and we are done. Without loosing generality we may suppose in what follows that D is a proper unbounded (Greenian) subset of $\mathbf{R}^d$, and $0\in D^c$. Consider the inversion with respect to the unit sphere:

$$Tx=\frac{1}{|x|^2}x\,,\quad x\neq 0\,.$$

Inversion is often used to reduce potential theoretic problems at infinity to those at 0 ([17]). In particular, the set $TD=\{Tx:\ x\in D\}$ is also Greenian and

$$G_D(x,v)=|x|^{\alpha-d}|v|^{\alpha-d}G_{TD}(Tx,Tv)\,,\quad x,v\neq 0\,. \tag{60}$$

We obtain

$$M_D(x,\infty)=\lim_{D\ni v\to\infty}\frac{|x|^{\alpha-d}|v|^{\alpha-d}G_{TD}(Tx,Tv)}{|x_0|^{\alpha-d}|v|^{\alpha-d}G_{TD}(Tx_0,Tv)}=\frac{|x|^{\alpha-d}}{|x_0|^{\alpha-d}}M_{TD}(Tx,0)\,. \tag{61}$$

The latter is a constant multiple of the *Kelvin transform* (see [17]) of the Martin kernel of TD with the pole at 0 and the reference point at Tx_0. The existence of $M_D(x,\infty)$ defined by (9) is proved. Also, $0<M_D(x,\infty)<\infty$. By [17], $M_D(x,\infty)$ is α-harmonic in D if and only if $M_{TD}(x,0)$ is α-harmonic in TD. But D is thick at infinity if and only if TD is thick at 0. Indeed, by (60) and a change of variable $v=Ty$ (with Jacobian $|y|^{-2d}$),

$$\int G_{TD}(Tx,y)\nu(0,y)dy=\mathcal{A}_{d,-\alpha}|x|^{d-\alpha}\int G_D(x,Ty)|y|^{-2d}dy=\mathcal{A}_{d,-\alpha}|x|^{d-\alpha}s_D(x)\,. \tag{62}$$

Thus α-harmonicity of $M_D(x,\infty)$ is equivalent to thickness of D at infinity, see (56). □

We let

$$M_D(x,y)=\frac{G_D(x,y)}{G_D(x_0,y)}\,,\quad x,y\in D\,, \tag{63}$$

so that $M_D(x,y)$ is now defined for all $x\in D$ and $y\in D^*$. Recall that $B_r=B(0,r)$.

Lemma 12 *For every $\rho>0$ and $\eta>0$ there is $r>0$ such that for every Greenian D*

$$\mathrm{RO}_{y\in\overline{D}\cap B_r}\,M_D(x,y)\leq 1+\eta\,,\quad \textit{if}\quad x,x_0\in D\setminus\overline{B}_\rho\,, \tag{64}$$

$$\mathrm{RO}_{y\in D^*\setminus\overline{B}_{1/r}}\,M_D(x,y)\leq 1+\eta\,,\quad \textit{if}\quad x,x_0\in D\cap\overline{B}_{1/\rho}\,. \tag{65}$$

The Martin kernel $M_D(x,y)$ is jointly continuous: $D\times D^*\setminus\{(x_0,x_0)\}\mapsto[0,\infty]$.

PROOF: Let $\rho > 0$ and $x, x_0 \in D \setminus \overline{B}_\rho$. We note that if $r > \rho$ then by (9)

$$\sup_{y \in \overline{D} \setminus \overline{B}_\rho} M_D(x,y) = \sup_{y \in D \setminus \overline{B}_\rho} \frac{G_D(x,y)}{G_D(x_0,y)}, \qquad \inf_{y \in \overline{D} \setminus \overline{B}_\rho} M_D(x,y) = \inf_{y \in D \setminus \overline{B}_\rho} \frac{G_D(x,y)}{G_D(x_0,y)},$$

hence $\mathrm{RO}_{y \in D^* \setminus \overline{B}_{1/r}} M_D(x,y) = \mathrm{RO}_{y \in D \setminus \overline{B}_{1/r}} M_D(x,y)$. As functions of y, $G_D(x,y)$ and $G_D(x_0,y)$ are nonnegative Poisson integrals on $D \cap B_\rho$ of measures on B_ρ^c. Thus (64) is an immediate consequence of Lemma 8 and scaling. To prove (65), as in the proof of Theorem 3 we may assume that $0 \in D^c$, and then (61) reduces (65) to (64) for TD.

By Lemma 4, M_D given by (63) is jointly continuous $D \times D \setminus \{(x_0,x_0)\} \mapsto [0,\infty]$. We consider the remaining case: $D \times D^* \ni (x', y') \to (x,y) \in D \times \partial D^*$. We have

$$\frac{M_D(x',y')}{M_D(x,y)} = \frac{M_D(x',y')}{M_D(x,y')} \cdot \frac{M_D(x,y')}{M_D(x,y)}.$$

Here the second factor on the right converges to 1 by (64) or (65). We will verify uniform continuity of the first factor at $x' = x$. If $\overline{B}(x,s) \subset D$, and $y' \in \overline{B}(x,s)^c$ then by (57) and (25) we see that $f = M_D(\cdot, y') + c\varepsilon_{y'}$ satisfies (28). Here $c = 1/P_D(x_0,y')$ if $y' \in D^c$ and D is thin at y', and $c = 0$ otherwise. The uniform continuity follows from (29) as in the proof of Lemma 4. Thus $M_D(x',y')/M_D(x,y) \to 1$. □

We remark that the section does not essentially depend on Section 5. Even the notion of harmonicity in the statement of Theorem 3 might be replaced by (57). Thus, the kernel functions G_D, P_D, and M_D, may be studied without this notion.

7 Structure of nonnegative harmonic functions

Lemma 13 *If $f \geq 0$ and f is α-harmonic measure on domain D, then there is a unique function f_s α-harmonic in D such that $f_s \geq 0$ on $\mathbf{R}^d$, $f_s = 0$ on D^c, and $f = H_D[f] + f_s$.*

PROOF: Let D_n be an increasing sequence of open precompact subsets of D such that $\bigcup_{n=1}^\infty D_n = D$. By (2), monotone convergence of G_{D_n} to G_D, α-harmonicity of f, and (17), we have

$$H_D[f](x) = \lim_{n\to\infty} \int_{D^c} \int_D G_{D_n}(x,z)\nu(z,y)dzf(dy) \leq f(x), \quad x \in D.$$

Let $f_s = f - H_D[f]$. Since $f_s = 0$ as measure on D^c, we may and will assume that it is a genuine function on $\mathbf{R}^d$: $f_s(x) = 0$, $x \in D^c$. The stated properties easily follow. □

By Fatou's lemma, the function $x \mapsto \Lambda_x(G_D(x_0,\cdot))$ is lower semicontinuous. Hence $\partial_M D = \{x \in \partial D : \Lambda_x(G_D(x_0,\cdot)) = \infty\}$ is Borel measurable, and in fact of type $\mathcal{G}_\delta$.

Lemma 14 *Let D be Greenian and let $\mu \geq 0$ be a finite measure on $\partial_M D$. Then*

$$f(x) = \int_{\partial_M D} M_D(x,y)\mu(dy), \quad x \in \mathbf{R}^d \tag{66}$$

is α-harmonic on D and vanishes on D^c. Conversely, if function $f \geq 0$ is α-harmonic on D and $f = 0$ on D^c then there is a unique finite measure $\mu \geq 0$ on $\partial_M D$ satisfying (66).

PROOF: It is a straightforward consequence of Theorem 3 that f given by (66) is α-harmonic in D and vanishes on D^c. We will write $f = M_D[\mu]$.

Let f be a nonnegative function α-harmonic in D such that $f = 0$ on D^c. Let D_n denote an increasing sequence of open sets precompact in D such that $\bigcup_{n=1}^\infty D_n = D$. For notational convenience we will also assume that $\omega^x_{D_n}(\partial D_n) = 0$ for $x \in D_n$ (this holds for example if D_n are Lipschitz domains). By (18) for $x \in D_n$ we then have

$$f(x) = \int_{D\setminus D_n} P_{D_n}(x,y)f(y)dy = \int_{D_n} M_{D_n}(x,v)\left(G_{D_n}(x_0,v)\int_{D\setminus D_n} \nu(v,y)f(y)dy\right)dv\,.$$

Let $\mu_n(dv) = \left(G_{D_n}(x_0,v)\int_{D\setminus D_n}\nu(v,y)f(y)dy\right)dv$. Since $\mu_n(D) = f(x_0) < \infty$, by considering a subsequence we may assume that μ_n weakly converge to a finite nonnegative measure μ on D^*. We claim that μ is supported in ∂D^*. Indeed, if $n > k$, $v \in D_k$, $y \in D\setminus D_n$, then $\nu(v,y) \le C_k$ and $G_{D_n}(x_0,v) \le G_D(x_0,v)$. Hence

$$\mu_n(D_k) \le C_k\left(\int_{D_k} G_D(x_0,v)dv\right)\left(\int_{D\setminus D_n} f(y)(1+|y|)^{-d-\alpha}dy\right) \to 0$$

as $n\to\infty$, see (52). This proves that $\mu(D_k) = 0$ and so μ is a measure on ∂D^*.

Let $\varepsilon > 0$ and $x \in D$. By Lemma 12 for every $y \in \partial D^*$ its neighborhood, V_y, exists such that

$$\mathrm{RO}_{V_y}\, M_U(x,\cdot) \le 1+\varepsilon\,,$$

with $U = D$ *and* $U = D_n$, $n = 1,\ldots$. From these, one selects a finite family $\{V_j,\ j = 1,\ldots,m\}$ such that $V = V_1\cup\ldots\cup V_m \supset \partial D^*$. For $j = 1,\ldots,m$, let $z_j \in V_j\cap D$. Let k be so large that for $n > k$ we have $z_j \in D_n$, and

$$(1+\varepsilon)^{-1} \le \frac{M_D(x,z_j)}{M_{D_n}(x,z_j)} \le 1+\varepsilon\,,\quad j = 1,\ldots,m\,.$$

If $v \in V_j\cap D_n$ for some j then

$$(1+\varepsilon)^{-3} \le \frac{M_D(v)}{M_D(z_j)}\cdot\frac{M_D(z_j)}{M_{D_n}(z_j)}\cdot\frac{M_{D_n}(z_j)}{M_{D_n}(v)} \le (1+\varepsilon)^3\,.$$

Therefore

$$(1+\varepsilon)^{-3} \le \frac{\int_{D\cap V} M_D(x,y)\mu_n(dy)}{\int_{D\cap V} M_{D_n}(x,y)\mu_n(dy)} \le (1+\varepsilon)^3\,,\quad n \ge k\,.$$

By letting $n\to\infty$ we obtain

$$(1+\varepsilon)^{-3} \le \frac{\int_{\partial D} M_D(x,y)\mu(dy)}{f(x)} \le (1+\varepsilon)^3\,,$$

which yields (66).

We will prove that μ is concentrated on $\partial_M D$. Let U be open and precompact in D and let $x \in U$. By Theorem 3 and (25), if $y \in \partial D^*$, then $M_D(x,y) \ge \int_{D\setminus B} M_D(z,y)\omega^x_U(dz)$ and equality holds if and only if $y \in \partial_M D$. By Fubini's theorem

$$0 = f(x) - \int_{D\setminus U} f(z)\omega^x_U(dz) = \int_{\partial D}\left(M_D(x,y) - \int_{D\setminus U} M_D(z,y)\omega^x_B(dz)\right)\mu(dy)\,,$$

hence $\mu(\partial D^* \setminus \partial_M D) = 0$.

We will prove the uniqueness of μ in the representation (66). We first consider $f = M_D[\varepsilon_{y_0}] = M_D(\cdot, y_0)$, where $y_0 \in \partial_M D$. To simplify notation, we assume as we may that $y_0 = 0$ (we use translation invariance if $0 \neq y_0 \in \mathbf{R}^d$ and inversion if $y_0 = \infty$).

Let $D_r = D \cap B_r$, $D'_r = D \setminus \overline{B}_r$. Suppose that f satisfies (66) for a nonnegative measure μ on $\partial_M D$. Let $r > 0$ and $g(x) = \int_{|y|>3r} M_D(x,y)\mu(dy)$. Considering $y \in \partial D_M$ such that $|y| > 3r$, by (57) we get

$$g(x) = \int_{D \setminus D_{2r}} g(z)\omega^x_{D_{2r}}(dz)\,, \quad x \in D_{2r}.$$

On the other hand, we may apply Lemma 10 to f, g, and D'_r to verify that

$$g(x) = \int_{D \setminus D'_r} g(z)\omega^x_{D'_r}(dz)\,, \quad x \in D'_r.$$

Lemma 11 yields $g(x) = \int_{D^c} g(z)\omega^x_D(dz) = 0$, that is, $\mu = 0$ on $\partial D_M \cap \{|y| > 3r\}$. In particular, the measures μ_n corresponding to $f = M_D(\cdot, y_0)$ weakly converge to ε_{y_0}. Fubini's theorem and dominated convergence yield that for general $f = M_D[\mu]$ the measures μ_n corresponding to f weakly converge to μ. Since μ_n are determined by f, so is μ. $\square$

We note that if f is α-harmonic in D and $0 \leq f \leq M_D(\cdot, y_0)$ then the proof of Lemma 14 yields $f = c\, M_D(\cdot, y_0)$ for some $c \in [0, 1]$. Thus, $M_D(\cdot, y_0)$ is *minimal harmonic* i.e. an extremal point of the class of nonnegative functions f (or hybrids) α-harmonic on D, such that $f(x_0) = 1$. The same is true of $(P_D(\cdot, y) + \varepsilon_y)/P_D(x_0, y)$, provided $y \in D^c$ is such that $P_D(x_0, y) < \infty$, because ε_y already determines the hybrid. We note, however, that our proof does not invoke Choquet's theorem. Instead it relies on (9) and Lemma 8.

Proof of Theorem 4: The theorem collects results of Lemma 13 and 14. $\square$

Noteworthy, if D is thin at infinity then $M(\cdot, \infty) = s_D$ is not α-harmonic in D, and the point at infinity is not charged by the measure μ in the representation (66).

8 Miscelanea

Consider $f(x) = \omega^x_D(\partial_M D)$, $x \in \mathbf{R}^d$. By Lemma 13,

$$\int_{\partial_M D} P_D(x, y)dy \leq f(x) \leq 1\,, \quad x \in D\,.$$

Since $P_D(x, y) = \infty$ for $y \in \partial_M D$, we conclude that $|\partial_M D| = 0$.

We will now strengthen the result of Lemma 1 and (18).

Proposition 1 *For Greenian $D \subset \mathbf{R}^d$, and $x \in D$, the harmonic measure ω^x_D is absolutely continuous on $D^c \setminus \partial_M D$ with respect to the Lebesgue measure, with density $P_D(x, \cdot)$.*

Proof: Let $K \subset D^c \setminus \partial_M D$ be compact and let $f(x) = \omega^x_D(K) - P_D[\mathbf{1}_K](x) \geq 0$. We will verify that $f = 0$. By Theorem 4, $f(x) = \int_{\partial_M D} M_D(x, y)\mu(dy)$ for some nonnegative finite μ on ∂M_D. Let $L \subset \partial_M D$ be compact and let $g(x) = \int_L M_D(x,y)\mu(dx)$. It suffices to prove that $g = 0$. We let

$$U = \{x \in D \,:\, 2\,\mathrm{dist}(x,K) \le \mathrm{dist}(x,L)\}\,, \quad V = \{x \in D \,:\, 2\,\mathrm{dist}(x,L) \le \mathrm{dist}(x,K)\}\,.$$

Observe that by (57), $g(x) = \int_{D\setminus U} g(y)\omega_U^x(dy)$ for $x \in U$. On the other hand, Lemma 10 applied to $\omega_D^x(K)$, g, and $V \subset D$ yields $g(x) = \int_{D\setminus V} g(y)\omega_V^x(dy)$ for $x \in V$. Hence we may apply Lemma 11 to conclude that $g(x) = \int_{D^c} g(y)\omega_D^x(dy) = 0$. □

In particular, for any $f = H_D[\lambda]$ with nonnegative λ on D^c satisfying (11) and absolutely continuous with respect to the Lebesgue measure, we have

$$f(x) = \int_{D^c} f(y)\omega_D^x(dy)\,.$$

This, however, requires a convention that $f(y) = 0$ for $y \in \partial_M D$ on the right hand side, and should be used with caution.

We note that there are domains D for which the part of the harmonic measure, which is singular with respect to the Lebesgue measure (i.e. ω_D^x on $\partial_M D$) is positive. Indeed, such is the complement of every closed non-polar set of zero Lebesgue measure, for example, the complement of a point on the line if $1 < \alpha < 2$, see [36].

Lemma 15 *Every nonnegative f harmonic on non-Greenian D is constant on D.*

PROOF: If $\alpha < d$ then G_D is majorized by the Riesz kernel [32]. For $\alpha \ge d = 1$, by [36], if D is non-Greenian then D^c is polar. In this case, if $x, y \in D$ and $0 < r < \min(\mathrm{dist}(y, D^c), |y-x|)$, then by recurrence (see [36] for the definition) for every $\varepsilon > 0$ there is an open precompact $B \subset D$ such that $x \in B$ and $\omega^x_{B\setminus\overline{B}(y,r)}(B(y,r)) > 1-\varepsilon$. Using small r, and continuity of f at y we obtain $f(x) \ge f(y)$, hence f is constant on D. □

We will give examples of thin and thick boundary points. Let $d \ge 2$ and let $f : (0,1) \to (0,\infty)$ be any bounded increasing function. We define a *thorn* D_f by (cf. [18]):

$$D_f = \{(x_1,\ldots,x_d) \in \mathbf{R}^d \,:\, 0 < x_1 < 1,\, |(x_2,\ldots,x_d)| < f(x_1)\}\,.$$

Proposition 2 *The origin is thin for D_f if and only if $\int_0^1 t^{-d-\alpha} f(t)^{d+\alpha-1}dt < \infty$.*

PROOF: We denote the integral by I_f. Let $g(t) = \frac{1}{2}(f(t/2) \wedge t)$. Observe that for $x \in D_g$ we have $B(x, g(x_1)) \subset D_f$. Hence:

$$s_{D_f}(x) \ge s_{B(x,g(x_1))}(x) = C_{d,\alpha}(g(t))^\alpha.$$

If $I_f = \infty$, then $I_g = \infty$ too, and so $\Lambda_0(s_{D_f}) = \infty$.

Assume now that I_f is finite. We may assume that $f(t) \le |t|$. Let $D_{f,r} = D_f \cap B_r$. We have:

$$s_{D_f}(x) = s_{D_{f,4r}}(x) + \int_{D_f\setminus D_{f,4r}} s_{D_f}(y)\omega^x_{D_{f,4r}}(dy).$$

The latter component regarded as a function of x is a Poisson integral on $D_{f,4r}$ and so in view of Lemma 7:

$$s_{D_f}(x) \le s_{D_{f,4r}}(x)(1 + C_{d,\alpha}\Lambda_{0,3r}(s_{D_f}))\,, \quad x \in D_{f,3r}\,.$$

Let $M(t) = \sup_{x_1=t} s_{D_f}(x)/(f(4t))^\alpha$. Inscribing $D_{f,r}$ into a cylinder and using $G_D(x,y) \le C_{d,\alpha}|x-y|^{-d+\alpha}$ one can show that $s_{D_{f,t}}(x) \le C_{d,\alpha}(f(t))^\alpha$. We thus proved that:

$$M(r) \le c_1 + c_2 \int_{2r}^{1} M(t)(f(4t))^{d+\alpha-1}t^{-d-\alpha}dt\,,$$

where c_1 and c_2 are some constants depending on d and α. Let $R > 0$ satisfy

$$2c_2 \int_0^R (f(t))^{d+\alpha-1}t^{-d-\alpha}dt < 1\,.$$

Then:

$$M(r) \le c_1 + (1/2) \sup_{(2r,R)} M + c_2 I_f \sup_{(R,1)} M\,.$$

It follows that M is bounded by $2c_1 + 2c_2 I_f \sup_{(R,1)} M$ and hence $\Lambda_0(s_{D_f})$ is finite. □

The next result is an extension of [10, Lemma 7].

Proposition 3 *If $y \in \partial D_M \cap (\overline{D}^c)^*$ then*

$$M_D(x,y) = \lim_{\overline{D}^c \ni z \to y} \frac{P_D(x,z)}{P_D(x_0,z)}\,. \tag{67}$$

PROOF: Suppose that $y = 0$ is a limit point of D and of the interior of D^c, and D is thick at 0. We denote $D_r = D \cap B_r$, $D'_r = D \setminus D_r$. Assume that $t > 0$ and $4t < |x| \wedge |x_0|$, and let $z \in B_t \setminus \overline{D}$. By Lemma 7 and Remark 2

$$\int_{D_t} G_D(x,v)\nu(v,y)dv \ge C_{d,\alpha} \int_{D_{3t}} s_{D_{2t}}(v)\nu(v,y)dv \int_{D\setminus D_{2t}} G_D(x,v)\nu(v,y)dv\,.$$

This also holds for $x = x_0$. By Fatou's lemma we have $\lim_{\overline{D}^c \ni z \to 0} \int_{D_t} s_{D_{2t}}(v)\nu(v,y)dv = \infty$. Thus, (2) yields

$$\lim_{\overline{D}^c \ni z \to 0} \frac{P_D(x,z)}{P_D(x_0,z)} = \lim_{\overline{D}^c \ni z \to 0} \frac{\int_{D_t} G_D(x,v)\nu(v,y)dv}{\int_{D_t} G_D(x_0,v)\nu(v,y)dv}\,,$$

provided that limits exist. If $\delta > 0$ then for sufficiently small t by (9) we obtain

$$M_D(x,0) - \delta \le \frac{\int_{D_t} G_D(x,z)\nu(z,y)dz}{\int_{D_t} G_D(x_0,z)\nu(z,y)dz} \le M_D(x,0) + \delta\,,$$

which proves (67). For general $y \in \partial D$ we use translation invariance. If $y = \infty$ we use inversion. Namely, (60) and $|Tx - Tz| = |x-z|/(|x||z|)$ yield

$$P_D(x,z) = |x|^{\alpha-d}|z|^{-\alpha-d}P_{TD}(Tx,Tz)\,,$$

see [17]. This, and (61) yield (67). $\square$

If $D = B(0, r)$, $r > 0$, and $x_0 = 0$, then we have

$$M_D(x, Q) = r^{d-\alpha} \frac{(r^2 - |x|^2)^{\alpha/2}}{|x - Q|^d}\,, \quad |x| < r\,, \tag{68}$$

for every $Q \in \partial B(0, r)$. (68) follows from Proposition 3 and (3) or (9) and (19). The formula was given before in [25], [10], [20]. We note that B_r is thick at all its boundary points Q because $G_{B_r}(x, v) \approx (r - |v|)^{\alpha/2}$ as $B_r \ni v \to Q$, see (19). More generally, a Lipschitz (or even κ-fat) domain is thick at all its boundary points, as follows from [9] ([38]). For more information on the boundary potential theory in Lipschitz domains we refer to the papers [10], [4], [33], which may suggest further applications.

We note that by Fatou's lemma, if $y \in \partial D_M$ then $P_D(x, z) \to P_D(x, y) = \infty$ as $\overline{D}^c \ni z \to y$. If $y \in \partial D \setminus \partial D_M$ and $\overline{D}^c \ni z \to y$ non-tangentially (i.e. $|z - y| \leq c\,\mathrm{dist}(z, D)$ for some $c > 0$) then by dominated convergence we have $P_D(x, z) \to P_D(x, y) < \infty$.

Majority of our references below represent the probabilistic potential theory. For the interpretation of our results in probabilistic terms we refer, among others, to [22] and [7]. We wish to provide the following probabilistic connection. The (thickness) condition $\Lambda_x(s_D) = \infty$ has appeared implicitly in [18] and explicitly in [39]. Authors of these papers consider the following property of the symmetric α-stable process $\{X_t\}$ in $\mathbf{R}^d$ and a given domain D: *There exist a random time interval* $(\tau_0, \tau_0 + 1)$ *such that* $X(t) - X(\tau_0) \in D$ *for* $t \in (\tau_0, \tau_0 + 1)$. If D is a *thorn* then the property holds if and only if $\Lambda_0(s_D) = \infty$ ([18]). In [39] all open sets D are considered and the existence of such interval is established if $\Lambda_0(s_D)$ is infinite. We conjecture the the thickness of D at 0 is actually a characterization of this property.

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