Game Theory and Applications
Problem set 3

1. Find lower and upper values in pure strategies together with corresponding prudential strategies for matrix games defined by the following matrices. Next, find mixed Nash equilibria and value in mixed strategies for these games.

   - $A = \begin{bmatrix} 5 & -1 & -2 & 1 \\ 2 & 4 & -1 & 3 \\ 3 & 4 & 2 & 0 \end{bmatrix}$.
   - $A = \begin{bmatrix} 1 & 5 & 2 & 3 \\ 0 & 2 & 5 & 2 \\ 2 & 0 & 4 & 3 \\ -1 & 3 & 1 & -2 \end{bmatrix}$.

2. (a) Suppose in a matrix game with the payoff matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

   none of the players has a pure optimal strategy. What inequalities must the numbers a, b, c, d satisfy? Find the value in mixed strategies of this game.

   (b) Suppose the row player has a pure optimal strategy but $a \neq b$ and $c \neq d$. Show that in such a case the game has a unique equilibrium and that this equilibrium is pure.

   (c) Find a 2×2 bimatrix game having both pure and completely mixed (using both rows and both columns with a positive probability) Nash equilibria, even if each of the entries of payoff matrices is different.

3. Consider the following game: each of two players chooses a natural number. The player whose number was bigger receives one Euro from the other. Show that the lower value of this game is -1 and the upper value is 1, even if the players are allowed to use mixed strategies.

4. Find all pure-strategy Nash equilibria in a two-player game with strategy sets $X = Y = [0, 1]$ and utility functions $u_1(x, y) = -x^3 + 9xy^2 - 3x^2y$, $u_2(x, y) = 2\ln(y) - 9y^2$.

5. Find all pure-strategy Nash equilibria in a two-player game with strategy sets $X = Y = [0, 1]$ and utility functions $u_1(x, y) = \frac{4}{3}x^3 + xy^2 - 2x^2y - x^2 + xy$, $u_2(x, y) = 2y - 2xy - y^2$.

6. Consider a zero-sum game with strategy sets $X = Y = [0, 1]$ and utility function of player 1 $u(x, y) = -(x - y)^2$. Show that this game does not have a pure Nash equilibrium. Further, find its unique mixed-strategy equilibrium. Its form is rather intuitive. What properties of the utility functions will always imply the existence of a Nash equilibrium with similar structure?

7. Consider a two-player game with strategy sets $X = Y = [0, 1]$ and utility functions $u_1(x, y) = -(x-y)^2$, $u_2(x, y) = \begin{cases} y & \text{for } x < \frac{1}{2} \\ 1 - y & \text{for } x \geq \frac{1}{2} \end{cases}$. Show that this game has no (either pure or mixed) Nash equilibria.
   Hint: First show that the best response of player 1 against any mixed strategy of player 2 is pure.

8. Stackelberg duopoly game is a variant of Cournot's quantity competition where players have different priority. Player 1 chooses his strategy first, then Player 2, knowing the strategy chosen by his opponent, chooses his. This results in different procedure used to compute equilibria: we first compute the best response of Player 2 to any strategy of Player 1, we insert it into the utility of Player 1 and find strategies maximizing this new utility. Compute the equilibrium in Stackelberg duopoly game corresponding to Cournot game from the lecture. How do utilities of the players at the equilibrium change with respect to those at Cournot-Nash equilibrium? What about the price of the good? Is it more profitable for the consumer to have priority among the players?

9. Consider an asymmetric variant of the Bertrand duopoly game (price competition) from the lecture: Two producers of the same good set prices for it. When the price of producer 1 is smaller than that of producer 2, customers buy only from producer 1 and vice versa. If the prices are equal, half of the customers buy from each of the producers. The utility for a player is the revenue from all the good that he sells minus the cost of its production. The demand for the good, as in all other duopoly games that we have considered is $d(p) = (p_0 - p)^+$, where p is the market price of the good. Unit cost c_1 for player 1 is smaller than that for player 2 (c_2). Write down the strategy sets and the utility functions defining this game. Find Nash equilibria in this game. Do they always exist? What if we assume that the prices are denominated in come currency (and the strategy sets are discrete as a consequence)?

10. Consider the following game between the Police and a criminal. Police tries to find a place in criminal's home to plant a bug. The criminal suspects that his house is wired, so he tries to find a place where he can talk safely. The goal of the Police is to maximize the probability of obtaining some useful information, while the criminal wants to minimize this probability. As in other of our simplified models, we identify the house of the criminal with the interval $[0, 1]$. The probability of obtaining some useful information from the bug is $1 - e^{d-1}$, where d is the distance between the bug and criminals talking.

    (a) What are the optimal strategies for the Police and the criminal?

    (b) How will they change if the Police tries to find places for n bugs instead of one? (The probability of obtaining the information can then be computed as $1 - e^{\sum_{i=1}^{n}(d_i - 1)}$, where d_i is the distance between the criminal and the ith bug).

    (c) Assume that the Police tries to optimize the number of bugs. By that they mean maximizing the probability of obtaining the information minus the unit cost of a bug c multiplied by the number of bugs. What will be the optimal number of bugs then?

Programming Problem

L1. Write a program performing the following iterative procedure for a matrix game defined by an $m \times n$ matrix A given by the user (m and n must be arbitrary):

(1) Initiate two vectors $p = \mathbb{0}_{1\times m}$ and $q = \mathbb{0}_{1\times n}$. Start the procedure by choosing the first row in the matrix as the strategy a of player 1.

(2) For $i = 1$ to i_{max}

- Add 1 to p_a.
- Choose column b as a best response to mixed strategy $P = \frac{p}{\sum_{i=1}^{m} p_i}$ of player 1 in game defined by A.
- Add 1 to q_b.
- Choose row a as a best response to mixed strategy $Q = \frac{q}{\sum_{i=1}^{n} q_i}$ of player 2 in game defined by A.

For some chosen 6×6 matrix A with a known mixed Nash equilibrium draw a graph showing how P_i and Q_i change for $i = 1, 2, 3, \ldots$. The strategies should converge to a Nash equilibrium of the game. If you have problems with computing equilibria in your game (this should be dificult for a 6×6 game), try to solve it using Gambit software (it can be downloaded from http://gambit.sourceforge.net/).