Optimization Theory

Applied Mathematics

Problem set 1

1. Show that 2-dimensional function $f(x,y) = (x^2 - y)^2 + (x+y)^2$ has two global minima and one stationary point, which is neither a local maximum nor a local minimum.

2. ([Be99]) Find all local minima and all local maxima of the 2-dimensional function $f(x,y) = \sin x + \sin y + \sin(x+y)$ within the set
$$\{(x,y) \in \mathbb{R}^2 : x \in (0, 2\pi), y \in (0, 2\pi)\}.$$

3. ([Be99]) For each value of the scalar β, find the set of all stationary points of the following function of the two variables x and y
$$f(x,y) = x^2 + y^2 + \beta xy + x + 2y.$$
Which of those stationary points are local minima? Which are global minima and why? Does this function have a global maximum for some value of β?

4. ([Be99]) Show that the 2-dimensional function $f(x,y) = (y - x^2)^2 - x^2$ has only one stationary point, which is neither a local minimum nor a local maximum. Next, consider the minimization of the function f subject to the constraint $-1 \le y \le 1$. Show that now there exists at least one global minimum and find all global minima.
Hint: Rewrite the function f as a sum of a square of some expression and a function of the variable y, then find a lower bound on this sum and a pair of x and y giving the value of f equal to this lower bound.

5. Find all the local minima and all the local maxima of the 3-dimensional function
$$f(x,y,z) = x^3 - 4x^2 + y^2 + z^2 - 2xy + xz - yz + 5z.$$

6. Show that the n-dimensional function
$$f(x_1, \ldots, x_n) = \left(\|(x_1, x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1}, 2 - x_n)\|_2\right)^2$$
has exactly one stationary point which is a global minimum. Compute this minimum.

7. Show that the 2-dimensional function $f(x,y) = -(2x-y)^2 + x$ is concave. Find its global minimum subject to linear constraints
$$\begin{cases} x + 2y \le 8 \\ 4x - 5y \le 6 \\ -3x - y \le 5 \\ -3x + 2y \le 8 \end{cases}$$

8. Show that if $f : A \to \mathbb{R}$ is a convex function and A is a convex set, then:
   (a) Any local minimum of f over A is a global minimum of f.
   (b) If A is open and ∇f continuous, then $x^* \in A$ is a global minimum of f iff it is a stationary point.

   Is it possible that f has no global minimum if A is bounded? Is it possible that f has a global minimum which is not a stationary point of f if A is not open?

9. ([Be99]) Show that the following statements are true:
   (a) Any vector norm is a convex function.
   (b) The weighted sum of convex functions, with positive weights, is convex.
   (c) If I is an index set, $C \subset \mathbb{R}^n$ is a convex set, and $f_i : C \to \mathbb{R}$ is convex for each $i \in I$, then the function $h : C \to \mathbb{R} \cup \{+\infty\}$ defined by
   $$h(x) = \sup_{i \in I} f_i(x)$$
   is also convex.

10. Show that the function
$$f(x,y,z) = x^4 + 6x^2z^2 + 3z^4 + x^2 - 4xy + 4y^2 + z^2 + y$$
is strictly convex. Find its unique global minimum.

References:

[Be99] D.P. Bertsekas, Nonlinear Programming, Athena Scientific, Belmont, MA: 1999.