Optimization Theory

Applied Mathematics

Problem set 3

1. Write equivalent linear programming problem using its classical form (i.e. the form of linear programs presented during the lecture) and its standard form (i.e. a form where all constraints except nonnegativity constraints are written in form of equalities):

$$\begin{array}{ll} \text{Minimize} & 5x_1 - 3x_2 + x_3 \\ \text{subject to} & x_1 + x_2 \geq 4, \\ & 2x_1 + 3x_3 \leq 15, \\ & x_2 - x_3 = 9 \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

2. Consider the following linear programming problem with 3 variables x_1, x_2, x_3 unconstrained with respect to their sign:

$$\begin{array}{ll} \text{Maximize} & x_1 + 2x_2 - 3x_3 \\ \text{subject to} & 4x_1 + 6x_2 - x_3 \leq 10, \\ & x_1 - x_2 \leq 4, \\ & 3x_1 - 4x_2 - 5x_3 \leq 9. \end{array}$$

   Rewrite it equivalently using 4 nonnegative variables.

3. Consider the linear program:

$$\begin{array}{ll} \text{Maximize} & \alpha x_1 + 2x_2 + x_3 - 4x_4 \\ \text{subject to} & x_1 + x_2 - x_4 = 4 + 2\Delta, \\ & 2x_1 - x_2 + 3x_3 - 2x_4 = 5 + 7\Delta, \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, \end{array}$$

   where $\alpha, \Delta \in \mathbb{R}$ are viewed as parameters.

   (a) Using the fact that the two first constraints are equalities, rewrite the problem using only 2 variables.

   (b) Using graphical interpretation of the problem, identify when the problem is unbounded (has no optimal solution) and when it has an optimal solution. What is it then (there will be several cases depending on the value of Δ)?

4. Consider the linear program:

$$\begin{array}{ll} \text{Maximize} & 4x_1 + 2x_2 \\ \text{subject to} & 2x_1 + x_2 \leq 6, \\ & -x_1 + 3x_2 \leq 6, \\ & x_1 - x_2 \leq 1, \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$

   (a) Solve this problem graphically. Check that there multiple optimal solutions.

   (b) Solve it using the simplex method. How the fact that there are multiple optimal solutions can be determined using the final simplex tableau?

5. Solve the linear programming problem

$$\begin{array}{ll} \text{maximize} & -2x_1 + x_2 \\ \text{subject to} & x_1 \geq 2, \\ & x_1 - x_2 \leq 1, \\ & x_1 + x_2 \leq 5, \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$

   using the simplex algorithm (you will need to use both phases). Draw the graphical interpretation of the problem. At each step of the simplex procedure, draw the movements that you have made on the picture.

6. Solve the linear program

$$\begin{array}{ll} \text{maximize} & 4x_1 + x_2 \\ \text{subject to} & -x_1 + x_2 \le -1, \\ & x_1 \le 4, \\ & x_2 \ge 1, \\ & 2x_1 + x_2 \le 10, \\ & x_1 \ge 0, x_2 \ge 0. \end{array}$$

using the simpex method (again, you will need both phases). Make a graph of the feasible set and indicate on it which vertices are visited by the simplex procedure in its successive steps (and in what order).

7. Formulate the following problem as a linear program: Murderville has specified the minimum requirements for the number of patrolmen on duty during each 4-hour period as seen in the table below:

Time of day	Number of patrolmen
Midnight–4A.M.	36
4A.M.–8A.M	18
8A.M.–Noon	12
Noon–4P.M.	10
4P.M.–8P.M	20
8P.M.–Midnight	32

Each policeman works for consecutive 8 hours during the day, and his assignment is repeated on each day (in particular, he may be assigned to a shift starting on one day and ending on the next one). Moreover, at least 30% of policemen working at any time must be officers with at least 4-year experience in the force. The total number of such officers available is 23. The number of less experienced officers can be considered unlimited. Find the assignment of experienced and unexperienced officers to each of the 8-hour shifts which minimizes the total number of policemen used and satisfies all the requirements imposed by the town.

8. The factory produces 3 products A, B and C from ingredients X, Y and Z. A unit of product A is then sold for \$2, a unit of B for \$3, while a unit of C for \$5. Each product can be sold in fractional amounts. A unit of material X costs \$1, a unit of Y, \$2, while a unit of Z, \$3. To produce a unit of product C a mix of $\frac{1}{6}$ of material X, $\frac{1}{3}$ of Y and $\frac{1}{3}$ of Z is necessary. To produce a unit of B a mix of $\frac{1}{2}$ of X with $\frac{1}{4}$ of Z is needed, while to produce A – $\frac{1}{2}$ of X must be mixed with $\frac{1}{3}$ of Y. The processing of X, Y and Z to obtain A, B and C does not cost anything. The demand for the product B is limited to 1500 units, while that for product C to 700 units. Suppose also that the owner of the factory can spend at most \$10000 for the ingredients. Write (without solving) a linear program maximizing the profit of the factory (the value of the products sold minus the value of the ingredients bought) subject to all the constraints. Write also its matrix form (with constraints in form of inequalities).

9. Consider the following (non-linear) problem:

$$\begin{array}{ll} \text{Minimize} & \max\{|3x_1 - x_2|, |2x_1 - 5x_2|, x_1 + x_2\} \\ \text{subject to} & x_1 + 5x_2 \le 7, \\ & x_1 + 3x_2 \ge 1, \\ & 2x_1 - x_2 \le 3 \\ & x_1 \ge 0, x_2 \ge 0. \end{array}$$

Transform it into an equivalent linear program by introducing 1 additional variable (plus changing the objective function and adding some constraints).

10. Consider the (non-linear) problem

$$\begin{array}{ll} \text{Maximize} & \frac{2x_1 - x_2 + 3}{x_1 + 2x_2 + 2} \\ \text{subject to} & x_1 \le 1, \\ & x_2 \ge 1, \\ & 2x_1 + x_2 \le 2, \\ & x_1 \ge 0, x_2 \ge 0 \end{array}$$

Introduce 3 new variables which will enable you to transform it into a linear program. Solve it using the simplex method. Compute the values of the original variables corresponding to the solution obtained.

Hint: Before solving the LP reduce the number of variables using the fact that one of the constraints is an equality.

11. Write the dual problem associated with the problem from exercise 6.

    (a) Read the solution to the problem from the final simplex tableau. Check that indeed it satisfies all the constraints of the dual problem and that it gives the value equal to the optimal value of the primary problem from exercise 6.

    (b) Suppose that the RHS of any constraint in the primal problem from exercise 6. (when written in the standard form) can be increased by 1 for \$0.5. Should any of them be increased? If yes, than which one?

12. Suppose a glass producer has a molding machine with dwo different forms that fit into machine: one that can be used to produce juice glasses, and the other one to produce cocktail glasses. Suppose he works 60 hours per week in which he can produce 2400 cases of juice glasses or 1800 cases of cocktail glasses (or he can split the production between the two proportionally to these values). Moreover, he has space to store the maximum of 2000 cocktail glasses or half of that of juice glasses. His seller takes the entire weekly production of glasses only once a week, so he needs storage space for the entire weekly production. Finally, he should take into account that the demand for the juice glasses is limited by 800 cases per week. A case of juice glasses gives the profit of \$4, while that of cocktail glasses gives the profit of \$5. Formulate the linear program that allows to find the optimal weekly production of both types of glasses. Solve it using the simplex algorithm.

    Formulate the dual problem associated with the problem. Read the optimal solution of the dual from the final simplex tableau. Given the interpretation of the optimal dual variables as shadow prices try to answer the question: Which of the following three possibilities is more profitable for the glass factory owner:

    (a) Invest in extra storage space, given that expanding it by each 1% costs approximately \$2 a week.

    (b) Invest in advertisment of juice glasses, given that an investment of \$12 per week will increase the demand by 1%.

    (c) Pay for extra hours of the employees, given that paying extra \$80 per week will increase their working time by an hour.

    What profit would it give per each dollar invested?