Optimization Theory

Applied Mathematics

Problem set 5

1. Find the distance between the plane $x_1 + 2x_2 + 3x_3 = 2$ and the paraboloid $x_3 = x_1^2 + x_2^2 + 1$ using the method of Lagrange multipliers.

2. Show that the minimum of $f(x_1, x_2, x_3) = x_1 + x_2$ subject to $(x_1-1)^2 + x_2^2 + x_3^2 = 1$ and $(x_1-2)^2 + x_2^2 + x_3^2 = 4$ cannot be found using Lagrange multipliers even though it exists. What assumption of the theorem we are using is not satisfied for this problem?

3. ([Tr13]) Find the minimum of the function $f(x) = \sum_{i=1}^n x_i^2$ subject to $\sum_{i=1}^n a_i x_i = c$ and $\sum_{i=1}^n b_i x_i = d$, where a and b are n-dimensional vectors such that $\|a\| = \|b\| = 1$ and a and b are orthogonal. Find the solution using the method of Lagrange multipliers.

4. Using the method introduced during the lecture find the minimum and the maximum of $f(x_1, x_2, x_3) = x_1^2 + 4x_2^2 + x_3^2 - 2x_1x_2 - 8x_1x_3 + 2x_2x_3$ subject to $x_1^2 + x_2^2 + x_3^2 = 1$.

5. Find the minimum and the maximum of $f(x, y, z) = 9x^2 + 4y^2 + 2z^2 - 12xy + 12xz + 8yz$ subject to $9x^2 + 4y^2 + z^2 = 1$.
   Hint: Change the variables in such a way that the problem can be solved using the technique from the previous problem.

6. Using Karush-Kuhn-Tucker conditions find the minimum of $x_1^2 + 4x_2$ subject to $\frac{x_1^2}{4} + x_2^2 \leq 1$ and $x_2 \geq x_1$.

7. Solve the problem
   $$\begin{array}{ll} \text{minimize} & x_1^2 + 2x_2^2 - x_3 \\ \text{subject to} & 2x_1 + x_2 - x_3 = 0 \\ & x_2 \geq x_1^2 - 2x_1 + 2 \end{array}$$
   using Karush-Kuhn-Tucker conditions.

8. Using Karush-Kuhn-Tucker conditions find the minimum of $f(x_1, x_2) = 2x_2 - x_1^3$ subject to $x_2 \leq (x_1-1)^2$, $x_1^2 + x_2^2 \leq 1$.

9. Using Karush-Kuhn-Tucker conditions find the minimum of $2x_2 - x_1$ subject to $4 - 2x_2^2 \geq x_1 \geq -x_2^2$.

10. Suppose Q is a symmetric positive definite matrix and c is a nonzero vector.
    (a) ([Be99]) Using Karush-Kuhn-Tucker conditions find (the matrix form of) the maximum value v and the optimal solution x of the problem
    $$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & x^T Q x \leq 1, \end{array}$$
    (b) Using the result from part (a) find the maximum value and the optimal solution of the more general problem
    $$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & x^T Q x + d^T x \leq \alpha, \end{array}$$
    with d being an arbitrary vector and α, a positive number.
    Hint: Rewrite the problem above using new vector of variables $y := x + \frac{1}{2}Q^{-1}d$.

11. Consider the problem
    $$\begin{array}{ll} \text{maximize} & (x_1 + 4)x_2 \\ \text{subject to} & x_1 + x_2 \leq 10, \\ & \frac{4}{3}x_1 + \frac{2}{3}x_2 \leq 12, \\ & \frac{1}{2}x_1 + \frac{3}{2}x_2 \leq 12, \\ & x_1 + x_2 + x_3 \geq 0. \end{array}$$
    Find its solution using Karush-Kuhn-Tucker conditions. Suppose that at cost of \$3 we can increase the RHS of exactly one of the constraints by 1. Does it make sense to increase any of them, if the value function is also denominated in dollars (give an answer based on the interpretation of Lagrange multipliers given during the Lecture)? If the answer is positive, which one should we increase?

References:

[Be99] D.P. Bertsekas, Nonlinear Programming, Athena Scientific, Belmont, MA: 1999.
[Tr13] W.F. Trench, The Method of Lagrange Multipliers, 2013, available at http://works.bepress.com/william_trench/130/