Optimization Theory

Applied Mathematics

Problem set 6

1. ([Be99]) Consider the problem

$$\text{minimize } x_1^2 + x_2^2 \text{ subject to } x_1 = 1.$$

   (a) Find its solution x^* and the value of the Lagrange multiplier λ^*.

   (b) For any fixed λ and c find the minimizer of L_c and check that for any bounded sequence (λ_n) and $c_n \nearrow \infty$ the sequence of minimizers of L_{c_n} converges to x^*. Show that the sequence which should converge to λ^* according to the theorem given on the lecture has the desired limit.

   (c) Compute the eigenvalues of $\nabla_{xx} L_c(x; \lambda)$. Suppose the auxiliary unconstrained problems solved in the method of multipliers are solved using the steepest descent method. Apply the theorem from the lecture to find out how its speed of convergence when applied to search for the minimizer of $L_c(\cdot; \lambda_n)$ will depend on the magnitude of c (in the method of multipliers we will need to find them also for very large c).

2. ([Be99]) Consider the problem

$$\text{minimize } \frac{1}{2}(-x_1^2 + x_2^2) \text{ subject to } x_1 = 1.$$

   (a) Find its solution x^* and the value of the Lagrange multiplier λ^*.

   (b) Show that for any $c < 1$ the augmented Lagrangian function L_c has no local minima for any $\lambda \neq \lambda^*$.

   (c) Calculate the successive iterations of the method of multipliers for $c_0 > 1$ and $\lambda_0 = 0$, assuming that the computations of all unconstrained minima are done using some method giving their exact value (provide only the recursive formulas for x^{k+1} and λ_{k+1} as the functions of x^k, λ_k and c_k).

   (d) Show that for any nondecreasing sequence $\{c_k\}$ such that $c_0 \geq 2$ and $c_k \to \infty$, $\lambda_k \to \lambda^*$ and $x^k \to x^*$.

   (e) Show (using the definition of superlinear convergence) that the rate of convergence of sequences $\{\lambda_k\}$ and $\{x^k\}$ is superlinear if $c_k \to \infty$.

3. ([Be99]) Consider the problem

$$\text{minimize } \frac{1}{2}(x_1^2 - x_2^2) - 3x_2 \text{ subject to } x_2 = 0.$$

   (a) Calculate the optimal solution and the Lagrange multiplier.

   (b) For $k = 0, 1, 2$ and $c_k = 10^{l+1}$ calculate and compare the iterates of the quadratic penalty method with $\lambda_k \equiv 0$ and the method of multipliers with $\lambda_0 = 0$. How far from the correct optimal solution are they? As before, assume the gradient method used to compute the unconstrained minima gives exact values of minima for quadratic functions.

4. Design a method to find a projection of a point $x \in \mathbb{R}^n$ on

   (a) a closed ball with center x_0 and radius r;

   (b) a hyperrectangle $\{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}$ for some given values a_i, b_i, $i = 1, \ldots, n$;

5. Show that whenever C is a linear subspace of $\mathbb{R}^n$, z is the projection of x on C (as a convex set) iff $(x - z)^T y = 0$ for every $y \in C$ (i.e. when z is the orthogonal projection of x on C).

6. Show using Karush-Kuhn-Tucker conditions, that finding a projection of a point $x \in \mathbb{R}^n$ on the set

$$C := \left\{ x \in \mathbb{R}^n : x^T Q x \leq 1 \right\}$$

   defined for a symmetric positive defnite matrix Q can be reduced to finding a zero of a function of one variable.

7. Consider the problem

$$\begin{array}{ll} \text{minimize} & f(x_1, x_2) = \frac{1}{2}(x_1 - 3)^2 + \frac{1}{2}(x_2 - 4)^2 \\ \text{subject to} & -x_1 + x_2 \leq 2 \\ & x_1 + 2x_2 \leq 7 \\ & 2x_1 - x_2 \leq 4 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

(a) Solve it using the Frank-Wolfe's method with starting point $x^0 = (0, 0)$ and α_k computed optimally.

(b) Solve it again using the gradient projection method with the same starting point, $s = 1$ and α_k computed optimally.

In both cases draw the subsequent iterations of the method on a picture.
Hint: In both cases, the minimum will be found after a finite (and small) number of steps.

References:

[Be99] D.P. Bertsekas, Nonlinear Programming, Athena Scientific, Belmont, MA: 1999.