Optimization Theory

Applied Mathematics

Laboratory assignments

LP9. The Education Council in a town in the mid-west of the United States tries to plan the assignment of students from 3 neighbourhoods to the 2 public schools in town. The assignment is supposed to minimize the total cost of school busing subject to the constraints related to the number of students each school can accomodate as well as requirements of the racial balance policy. The data realated to the problem is given in the table below (costs are given in dollars per person):

Neighbourhood	No. of black students	No. of white students	Cost of bus to school 1	Cost of bus to school 2
A	1100	0	0	13
B	720	960	15	10
C	510	760	17	5

The capacity of school 1 is 2500 students, while the capacity of school 2 is 2200 students. The racial balance policy requires that there are no more than 70% students of each race in each school.

Formulate and solve the integer programming model that will find the optimal assignment of students to schools satisfying all the constraints.

LP10. (Based on [AEP07]) A computer company has estimated the number of service hours needed during the next five months, according to the following table:

Month	No. of service hours
January	6000
February	12500
March	7000
April	8500
May	10500

The service is performed by hired technicians; their number is 60 at the beginning of January. Each technician can work up to 160 hours per month. In order to cover the future demand of technicians new ones must be hired. Before a technician is hired he/she undergoes a period of training, which takes a month and requires 40 hours of supervision by a trained technician. A trained technician has a salary of $16000 per month (regardless of the number of working hours) and a trainee has a monthly salary of $6500. At the end of each month a certain number of the technicians quit to work for another company. From the historical data we know that we can estimate this number as coming from normal distribution with average of 12% and variance of $4\%^2$.

Formulate an integer programming problem whose optimal solution will give the numbers of trainees to be hired at the beginning of each month that will minimize the total salary costs during the given time period, given that the number of available service hours are enough to cover the demand in each month with probability 0.95.

LP11. (Based on [MK80]) The company providing cleaning services for the hospitals has found out that the demand for its services is quite seasonal. The number of hours of service needed in each month is given in the table below:

Month	Hours
January	1800
February	2500
March	1500
April	900
May	700
June	1000
July	700
August	600
September	1300
October	1600
November	1800
December	2300

Each person is hired at the beginning of a month for a 3- or 6-month contract. Each person needs to pass the training in the first month, which decreases the number of hours for which he/she is available to 80. In any other month he/she can work for 196 hours. The monthly payment of any worker is $1700. Write an integer program allowing you to find the optimal plan of hireing people for 3- and 6-month contracts. Your goal should be the minimization of cost subject to covering all the demand.

LP12. The head of Civil Defense in a town endangered by earthquakes wants to create the plan of assigning the casualties of a possible earthquake to four hospitals in town. The simulations suggests that almost all the casualties will come from three neighbourhoods: the historical neighbourhood A (up to 250 people), the inner city B (up to 450), and the city center (C; up to 90). The times of transporting people to each hospital (in minutes) and their estimated capacities (in number of beds) are given below:

Hospital	Beds available	Time from A	Time from B	Time from C
State Hospital	400	25	23	20
Saint Lucy	120	12	16	8
Saint Paul	250	20	26	16
Children's Hospital	210	30	35	21

Write a linear program allowing you to find the assignment of casulaties to the hospitals, assuming that your goal is to minimize the expected number of minutes necessary to transport all the casualties to the hospitals.

LP13. (Airline Scheduling Problem, based on [Tr08]) A small airline would like to find an optimal schedule of flights to maximize profit. The airline operates flights on 3 lines between 4 cities: Cincinatti–Roanoke, Atlanta–Roanoke and New York–Roanoke. The table below shows the distances (in nautical miles), daily demands and market-clearing fares (in dollars) per seat for each of these lines:

	distance	demand (to)	demand (fro)	fare per seat
New York–Roanoke	375	420	450	250
Atlanta–Roanoke	310	880	730	230
Cincinatti–Roanoke	220	400	460	180

The airline has decided to purchase two types of aircraft to satisfy its needs: Embraer 145 and Avro RJ-100. The characteristics of each machine are given in the table below:

Aircraft	EMB-145	Avro RJ-100
number of seats	50	100
block speed (knots)	350	475
operating cost per hour of usage	$1750	$3800
maximum utilization per day	15 hrs.	12 hrs.

(a) Solve the integer programming problem of maximizing the profit using the following sets of decision variables: numbers of aircrafts of each type in the fleet and numbers of flights between each pair of destinations (separately in each direction) using aircraft of each type. Take into account the aircraft availability and demand fufillment constraints. In particular, take into account that the number of flights between each pair of destinations should be the same in both directions.

(b) Do the same as in point (a) subject to an additional constraint that the minimum number of flights (in each direction) between Atlanta and Roanoke should be 12.

LP14. (Crew Scheduling Problem, based on [Tr08]) A small airline would like to find an optimal allocation of flight crews to flights. The airline operates flights on 4 lines between 4 cities: Denver–Dallas (2 flights per day: Denver–Dallas in the morning and Dallas–Denver in the afternoon), New York–Dallas (4 flights per day: New York–Dallas starting in the morning and finishing in the afternoon, starting in the afternoon and finishing at night, Dallas–New York from morning 'til afternoon, starting at night and finishing in the morning of the next day), Denver–New York (6 flights per day: Denver–New York from the morning 'til the afternoon, from the afternoon 'til the night and from night 'til the morning of the next day, New York–Denver from the morning 'til the afternoon, from the afternoon 'til the night and from the night 'til the morning of the next day) and Mexico–Dallas (4 flights: Dallas–Mexico in the morning and in the afternoon, and Mexico–Dallas in the afternoon and at night). A crew schedule consists of 1- or 2-flight rotations. Each rotation costs $2500 if it terminates in the originating city on the day of the departure (e.g. Dallas–Mexico + Mexico–Dallas on the same day), $4000 if it requires staying overnight (e.g. Dallas–Denver in the afternoon + Denver–Dallas in the morning of next day), $3000 if it terminates in a different city than it originates, but doesn't require staying overnight (any 1-flight rotation which doesn't finish at night or a 2-flight rotation like Denver–Dallas + Dallas–Mexico with both flights on the same day and the latter flight not finishing at night) or $4500 if it terminates in another city than it originates and needs staying overnight. No rotation can consist of two flights that are not on the same day or two consecutive days. The schedule is repeated in the same form every day. Write and solve the integer programming problem of minimizing the crew schedulling cost using binary variables R_i denoting whether a given (i-th) rotation is used. Solve it using an integer programming model.

LP15. The increase of the cost of coal has forced the Knoxville Municipal Power Company to seek some savings in their energy production plan. The company has to produce $8000MWh$ of energy every day subject to the state norms of emissions of particulate and SO_2. The cost of producing $1MWh$ of energy in \$ and the levels of emissions of particulate and SO_2 in μg per $1MWh$ of energy are as follows:

Coal	Production cost	Emision of particulate	Emission of SO_2
Colombian	42	3.5	80
Indonesian	18	12	270

The daily norm for the emission of particulate is $60000\mu g$, while that for the amission of SO_2 is $300000\mu g$.

There's also a possibility of installation of filters on the chimneys of the plant. In that case the cost of producing $1MWh$ of energy increases by \$20 (for each type of coal), but the emission of particulate decreases by 90%, while that of SO_2, by 80%. Some amount of energy can also be bought at the market price of \$57 per $1MWh$. Write an integer program allowing you to find out, how much Colombian and how much Indonesian coal should be used for production, as well as whether the filters should be installed or not and whether some fraction of energy should be bought at the market. Your goal is the minimization of the total cost.

LP16. The head of the building department is planning the inspections for the upcoming week. He has 3 inspectors at his disposal: plumbing inspector (28h per week for inspections), electrical inspector (30h for inspections) and building inspector (34h for inspections). The times necessary for inspecting different types of buildings are given in the table below:

Inspector	Gas station	Restaurant	Garage	Residential building
plumbing insp.	4	2	1	2
electrical insp.	2	5	4	2
building insp.	3	3	2	3

For some buildings all three inspections can be done. In that case the probability of finding a defect is 12%, but the inspection takes as much time as the work of the longest-woking inspector needs. If only one inspector is sent, the probability of finding a defect decreases to 2.5%. Write and solve the linear program allowing to find the optimal inspection plan for the upcoming week (i.e. how many buildings of what type should be inspected by which inspector and how many of them should be inspected by all inspectors). Your goal is to maximize the expected number of defects found by the inspectors.

LP17. The students from class 3A of the Desert City High School have been given an offer to take part in a summer camp organized by the school. The students in the class are given in the table below:

Name	Popul. Rank	Science Rank	Sports Rank	Gender
Andy Adams	13	2	8	M
Bernard Butler	12	4	18	M
Cindy Cox	6	18	3	F
Danny Davis	21	16	22	M
Ed Edwards	19	10	7	M
Fionna Felix	1	14	10	F
Garry Goldblum	20	3	25	M
Henry Houseman	2	24	2	M
Isabel Ingram	11	26	11	F
Jenny James	4	15	1	F
Kathy Killington	18	8	24	F
Larry Lewis	17	11	21	M
Mandy Moore	8	12	17	F
Nelson Noriega	9	7	4	M
Olivia Ohn	5	19	19	F
Patricia Peters	10	22	15	F
Quentin Quinn	25	13	26	M
Rhonda Rhodes	14	9	9	F
Samantha Smith	24	17	23	F
Tanya Thunderbaker	7	20	6	F
Uma Unger	26	21	14	F
Viola Vernon	15	23	16	F
Wei Wang	22	1	12	F
Xiang Xu	23	6	20	M
Yoko Yamamoto	3	5	13	F
Zack Zushinsky	16	25	5	M

The students may choose the people going to the camp, given that the following constraints (imposed by the school) are satisfied:

- Exactly 15 people can go to the camp.
- Gender parity rules are applied, i.e. at most 60% of students going to the camp are males and at most 60% are females.
- The average science ranking of kids going to the camp cannot be lower than 70% of the average science ranking in the entire class.
- The average sports ranking of kids going to the camp cannot be lower than 70% of the average sports ranking in the entire class.

Find the set of people going to the camp, assuming that the students have made their choice trying to minimize the average popularity ranking.

LP18. (Based on [NPTEL14]) A paper factory produces 2 types of standard sheets of paper whose widths are 20 inches and 25 inches respectively. Then it cuts these big sheets into smaller sheets according to the demand. Today there is a demand for 396 8-inch sheets, 612 7-inch sheets and 817 5-inch sheets. Find the cutting patterns and quantities of 20- and 25-inch sheets which should be used to produce the desired quantities of smaller sheets that minimize waste. In addition take into account that the factory can produce at most 300 20-inch and at most 250 25-inch sheets per day. Create the integer program allowing to solve the problem for any given demands for different types of sheets. Solve it for the data given above.
Hint: Find all the possible ways to cut a 25- and 20-inch sheets into 8-, 7- and 5-inch ones (including those which create some waste). The variables should be the numbers of 25- and 20-inch sheets cut using each way.

LP19. A sawmill produces standard boards which are 22 inches wide. Some clients order narrower boards (of the same length) though. The orders for today consist of 120 7-inch boards, 135 5-inch and 72 3-inch ones. Smaller boards are cut off from the standard ones: e.g. the firm may decide to cut a standard board into 2 7-inch boards, one 5-inch and one 3-inch one. It could also decide to cut it into 3 7-inch boards, but in that case a 1-inch strip is a waste. The firm wants to satisfy the demand in such a way that there is no waste produced. In this case it prefers to produce additional 7-, 5- or 3-inch boards and store them so that they can be used

on another day. The storage however induces storage cost proportional to the total sum of widths of boards stored. Create the integer program allowing to find the way to cut the boards using least storage space and solve it.
Hint: Find all the possible ways to cut a 22-inch board into 7-, 5- and 3-inch ones without waste. The variables should be the numbers of 22-inch boards cut using each way.

LP20. (Based on [Ec79]) National Transit operates buses between major cities and carries commercial packages on a space-available basis. A departing bus has room for up to 650 cubic feet of packages. Also, the packages that are included cannot exceed a total weight of 750 pounds. Finally, it has to be taken into account that any non-priority packages can be taken only if there are no priority packages left that can be taken. Packages awaiting shipment are described in the table:

Package	Priority	Volume (ft^2)	Weight (pounds)
1	Yes	40	60
2	Yes	35	120
3	No	130	210
4	Yes	200	250
5	No	70	100
6	No	100	150
7	No	25	60
8	Yes	300	170
9	Yes	90	150
10	No	75	40

Formulate and solve an integer programming model for selecting packages to be included in the shipment. The goal of National Transit is to maximize the total number of packages shipped provided all the constraints are satisfied.
Hint: The priority constraint can be taken into account by a proper construction of the objective function.

LP21. (Based on [Ny80]) A construction company from the north-east of Poland has a problem of shortage of (two types of) mobile cranes in some locations and their surplus in some other ones. The situation is presented in the table below:

Location	Shortage of T. 1	Shortage of T. 2	Surplus of T. 1	Surplus of T. 2
Olsztyn	4	1	0	0
Ełk	0	4	2	0
Suwałki	3	2	0	0
Mrągowo	0	0	2	5
Łomża	0	6	4	0
Ostrołęka	8	0	0	6
Szczytno	0	0	1	7
Augustów	0	2	4	0

Write a linear program to compute the plan of moving the cranes between locations which minimizes transportation costs. The plan should take into account that:

- Cost of transportation of Type 1 crane is proportional to the distance (you can take the distances between towns from https://www.distancecalculator.net/).
- Cost of transportation of Type 2 crane is 160% of that for the Type 1 crane.
- Type 1 crane can be replaced by a Type 2 crane but not the other way round.

Hint: The variables should correspond to numbers of Type 1 cranes being moved between locations and numbers of all the cranes moved between locations.

LP22. Suppose a large restaurant working 7 days a week wants to hire waiters. Based on past experience, the number of waiters needed on each day of the week can be given as follows:

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Demand	10	8	7	10	14	17	20

There are three types of waiters: full-time waiters work for 5 consecutive days and then rest for the next two; part-time waiters work for 3 consecutive days and then rest for the next 4 days, week-end waiters work only on saturday and sunday. Each worker repeats his weekly working pattern indefinitely. Full-time waiters are paid \$14 per day of work, part-time waiters are paid \$16 per day of work, while week-end waiters are paid \$12 per day of work.

(a) Provide the waiters' work-plan for the week minimizing the cost for the restaurant. The plan must satisfy all the demand constraints.

(b) The restaurant considers increasing the payments of the full-time waiters by \$1 per day and decreasing it by \$1 for the other two groups. Will it change the amount the restaurant has to pay? Will it change the work-plan for the waiters?

LP23. (Based on [NPTEL14]) Consider the situation where there are eight projects but only three people bid for them. Person B does not bid for project 2. Find the assignment of people to projects that maximizes the total number of projects completed subject to the constraints that the total cost does not exceed \$100000 and that each person is assigned to at least two and at most four projects, satsifying additional time constraint which requires that all the projects are finished in 20 days (given that any two projects cannot be done simultaneously by the same person). The costs in \$1000 for each assignment are given in the table below:

	Person A	Person B	Person C
Project 1	12	15	8
Project 2	16	x	7
Project 3	14	15	21
Project 4	13	14	9
Project 5	17	13	11
Project 6	22	24	16
Project 7	13	12	10
Project 8	9	6	12

The times (in days) required to finalize each project if a given person is chosen to do it are given in the table below:

	Person A	Person B	Person C
Project 1	5	6	7
Project 2	4	x	6
Project 3	10	9	7
Project 4	4	7	9
Project 5	7	6	9
Project 6	11	4	7
Project 7	10	7	9
Project 8	6	4	9

Solve the problem using integer programming formulation of the problem.

LP24. (Based on [NPTEL14]) Consider an airline that has demand for baggage handlers in a domestic airport. The hourly requirements for 24 hours starting from midnight are 7, 5, 3, 1, 2, 6, 7, 9, 10, 13, 16, 19, 15, 10, 10, 13, 18, 20, 23, 22, 15, 9, 8 and 7 respectively. There are 6 shifts for handlers: the first shift starts at midninght. The next shifts start at 4AM, 8AM, 12, 4PM and 8PM. Each shift works for 8 consecutive hours. Each shift works for 8 consecutive hours. The demand is the same for all days. Using an integer programming model find the minimum number of handlers required that meets the demand and hour at which the first shift should start so that this minimum number of handlers is minimized.

Suppose there is a possibility of increasing the working time of each shift from 8 to at most 10 hours (the working hours for different shifts may vary). Does it make sense to increase the working time for any shift if handlers are paid 120% for their extra hours?

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