Optimization Theory

Applied Mathematics

Laboratory assignments

NLP11. Find the minimum variance portfolio consisting of stocks given in the table from the problem LP7 which yields at least 4% of expected yearly return. Suppose that you have €1mln at your disposal. Use the Markovitz mean-variance optimization model (it is nicely explained in [DX15], the way the data needed for the model should be derived from the table of total returns is also explained there) to construct the portfolio. Suppose, in addition a risk-free national bond with the yearly rate of return of 2.5% (with zero variance and uncorrelated with any of the stocks) is available. How does it change the optimal portfolio?

NLP12. Find the optimal risky portfolio (that is – a portfolio which maximizes the Sharpe ratio) for the 5 assets given in the excercise LP7 using quadratic programming. Suppose that you have €1mln at your disposal and that risk-free rate is 2.5%. The way to construct the quadratic program giving the optimal risky portfolio given the data about the total returns is given in detail in [DX15].

NLP13. (Based on [BM68]) The standard way to solve regression problems is by minimizing the mean quadratic error. One of the methods of doing it is by using quadratic programming. In case of linear regression this is the problem of minimizing

$$\sum_{i=1}^{m} \alpha_i^2 \tag{1}$$

computed as

$$\alpha_i = b_0 + x_{i1}b_1 + \ldots + x_{in}b_n - y_i, \quad \text{for } i = 1, \ldots, m, \tag{2}$$

which can be written as the following quadratic program:

$$\text{minimize} \sum_{i=1}^{m} \alpha_i^2$$

subject to

$$-y_i + b_0 + x_{i1}b_1 + \ldots + x_{in}b_n + \alpha_i = 0, \quad i = 1, \ldots, m.$$

A good feature of this method is that it allows to introduce constraints on regression coefficients, which cannot be done in most tools for solving regression models.

Suppose that the following data describes the dependence of volume of sales of some candy produced in different sizes and tastes y_i on several factors such as its size x_{i1}, the price x_{i2} and the amount spent on its promotion x_{i3}.

i	y_i	x_{i1}	x_{i2}	x_{i3}
1	99	1	8.5	106
2	89	1	8.2	72
3	103	2	9.5	75
4	86	1	7.4	99
5	91	1	7.0	165
6	95	1	8.0	180
7	100	1	9.5	99
8	135	2	12.3	210
9	112	2	10.8	175
10	81	1	7.0	120
11	89	1	8.2	132
12	97	1	9.0	157
13	113	1	15.7	75
14	51	0.5	3.7	80
15	75	1	6.2	105
16	98	1	10.6	205
17	100	2	11.4	109
18	94	2	7.1	90
19	121	2	13.0	87
20	39	0.5	3.4	150

We assume that the demand for product is elastic with respect to the price and the promotion spending. It means that it approximately satisfies the equation

$$\ln y_i = \beta_0 + \beta_1 \ln x_{i1} + \beta_2 \ln x_{i2} + \beta_3 \ln x_{i3}$$

for some coefficients β_k, $k = 0, \ldots, 3$ such that β_2 and β_3 are at least 1. Find the coefficients β_k minimizing the mean quadratic error for the above formula with the elasticity constraints added using the QP approach described above. Compare them with the results obtained when the elasticity constraints are not imposed. Is the assumption that the demand is elastic with respect to these two factors justified by the data?

NLP14. (Based on [AnLu07]) Write a procedure allowing you to find approximate distance between two ellipsoids in $\mathbb{R}^3$

$$E_i = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{(x - x_i)^2}{a_i^2} + \frac{(y - y_i)^2}{b_i^2} + \frac{(z - z_i)^2}{c_i^2} \leq 1 \right\}, \quad i = 1, 2$$

using the following sequential quadratic programming approach:

(i) Define sets D_1 and D_2 as smallest cuboids containing E_1 and E_2, that is, for $i = 1, 2$,

$$D_i = \left\{ (x, y, z) \in \mathbb{R}^3 : x_i - a_i \leq x \leq x_i + a_i, y_i - b_i \leq y \leq y_i + b_i, z_i - c_i \leq z \leq z_i + c_i \right\};$$

(ii) Find points (x_1^*, y_1^*, z_1^*) on D_1 and (x_2^*, y_2^*, z_2^*) on D_2, which minimize the distance between D_1 and D_2 (this is a quadratic programming problem);

(iii) For each i find a point $(\widehat{x_1}, \widehat{y_1}, \widehat{z_1})$ of intersection of the frontier of the ellipsoid with the straight line going through points (x_1, y_1, z_1) and (x_1^*, y_1^*, z_1^*). Find the plane going through $(\widehat{x_1}, \widehat{y_1}, \widehat{z_1})$, tangent to the ellipsoid.

(iv) For each i improve the set approximating ellipsoid E_i by adding the condition that (x, y, z) is on the internal side of the plane found in (iv) to the constraints defining the set D_i. After this is done, go to (ii).

NLP15. Write a procedure allowing you to find approximate distance between two paraboloids in $\mathbb{R}^3$ $z = f(x, y) = a_1 x^2 + b_1 y^2 + c_1$ (with $a_1, b_1, c_1 > 0$) and $z = g(x, y) = a_2 x^2 + b_2 y^2 + c_2$ (with $a_2, b_2, c_2 < 0$) given by the user. The procedure should make use of the following sequential quadratic programming approach:

(i) Define the set D_1 as the set of points lying above the plane $z = c_1$ and D_2 as the set of those lying below $z = c_2$.

(ii) Find points (x_i^*, y_i^*, z_i^*) in D_i, $i = 1, 2$ which minimize the distance between the two sets (this is a quadratic programming problem);

(iii) Find the plane tangent to the graph of f at $(x_1^*, y_1^*, f(x_1^*, y_1^*))$ and add the condition that (x, y, z) is above this plane to the constraints defining the set D_1.

(iv) Find the plane tangent to the graph of g at $(x_2^*, y_2^*, g(x_2^*, y_2^*))$ and add the condition that (x, y, z) is below this plane to the constraints defining the set D_2.

(v) Go to (ii).

References:

[AnLu07] A. Antoniou, W.-S. Lu, Practical Optimization, Springer Science+Business Media, LLC, 2007

[BM68] J. Bracken, G.P. McCormick, Selected Applications of Nonlinear Proframming, John Wiley & Sons, Inc., 1968

[DX15] Z. Donovan, M. Xu, Quadratic Programming: Applications (lecture notes), available online at http:// community.wvu.edu/ ∼krsubramani/courses/sp15/optfin/lecnotes/QPapps.pdf