Optimization Theory

Applied Mathematics

Laboratory assignments

NLP16. Par, Inc. is a small manufacturer of golf equipment and supplies whose management has decided to move into the market for medium- and high- priced golf bags. The steps involved in manufacturing a golf bag, management determined that each golf bag will require the following operations: cutting and dyeing the material, sewing,, finishing (inserting umbrella holder, club separators, etc.), inspection and packaging. The time required for each of these operations is given in the table below:

Department	Standard bag	Deluxe bag
Cutting and Dyeing	0.7	1
Sewing	0.5	0.85
Finishing	1	0.65
Inspection and Packaging	0.1	0.25

After studying departmental workload projections, the director of manufacturing estimates that 630 hours for cutting and dyeing, 600 hours for sewing, 708 hours for finishing, and 135 hours for inspection and packaging will be available for the production of golf bags during the next three months. In addition, if the workers are paid for extra hours, the time available in each department can be increased (independently for each) by at most 20%. Finally, the accounting department has estimated that after getting into account the demands for the two types of bags and the production costs, the profit (in USD) from selling S standard bags will be $80S - 0.1S^2$, while for D deluxe bags it will be $15D - 0.2D^2$. If any department works overtime, the profit must be decreased by the time this department works multiplied by \$60. How many standard and how many deluxe bags should be produced by Par, Inc.? Should any production department work overtime? Find the answers by maximizing the total profit subject to the time constraints (plus nonnegativity constraints on D and S). Apply the Frank-Wolfe method with stepsize computed using Armijo rule to solve the problem. Compare the result you have obtained to that obtained using built-in quadratic programming software.

NLP17. (Based on [MOR17]) Consider three water consuming firms that belong to the same corporation. Each firm makes a product and water is needed in the process of making that product. Let the amounts of water consumed by each firm in kiloliters per hour be denoted by x_i, $i = 1, 2, 3$ and the quantities produced by each firm by q_i, $i = 1, 2, 3$. The net benefits $B_i(q_i, x_i)$ from the production as functions of the quantities produced and the amounts of water used are given for each firm:

$$B_1(q_1, x_1) = (12 - q_1)q_1 - 3q_1^{1.3} - 0.2x_1,$$

$$B_2(q_2, x_2) = (20 - 0.5q_2)q_2 - 5q_2^{1.2} - 0.2x_2,$$

$$B_3(q_3, x_3) = (28 - 2.5q_3)q_3 - 6q_3^{1.15} - 0.2x_3,$$

where the first part is the profit from production, the second is the cost of production and the last is the cost of the water used. The available amount of water is 20kl/h. Moreover, the binding contracts for the firms require that at least 2 units of product 1, at least 3.5 of product 2 and at least 1 of product 3 must be produced. Find the allocation of water to three firms that maximizes the total net benefit of the corporation from their production subject to the minimal production and water availability constraints. Do it applying the Frank-Wolfe method with stepsize computed using Armijo rule to solve the problem.

NLP18. (Based on [BM68]) An aircraft producer is seeking for a design of the airframe with the lowest production cost satisfying several functional constraints. The cost is given by

$$c(x_1, x_2, x_3) = 250x_1^{0.3}x_2^{0.25}x_3^{0.1} + 20(x_1 + x_2)^{0.5}$$

with x_1 denoting airframe weight, x_2 – propeller weight and x_3 – the length of the airframe. The functional constraints are the following: The length of the airframe must be between 80 and 160 meters. The relations between the airframe's length and weight and between the weights of the airframe and that of the propeller which have to be satisfied are the following: $50x_3 \le x_1 \le 75x_3$ and $2x_1 \le x_2 \le 3x_1$. Finally, a mass fraction constraint of the form

$$0.45 \le \frac{0.5x_1 + 20000}{0.5x_1 + x_2 + 20000} \le 0.6$$

is required to obtain good performance of the aircraft.

Write a program minimizing the airframe production cost subject to all the constraints. Your procedure should make use of the Frank-Wolfe method with stepsizes in the subsequent steps computed using golden section.

NLP19. In the classic inventory control model we are supposed to decide how much of several products should be ordered to be further sold. Let us assume that there are 4 types of product that we are selling and their quantities to be ordered x_i $i = 1, \ldots, 4$ are to be determined. In order to find these values we minimize the cost

$$c(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4} \left(h_i x_i + \frac{c_i d_i}{x_i} \right)$$

subject to the available space constraint

$$\sum_{i=1}^{4} b_i x_i \leq B.$$

Here h_i is annual inventory holding cost for item of type i, c_i is the delivery cost for item of type i, d_i is the annual demand for item of type i, b_i is the space item of type i takes in storage and B is the total space that we have at our disposal. In addition we assume that in order to maintain fluidity of commerce we need to order at least 10 units of each type. Write a procedure allowing you to find the optimal orders of products of each type. Assume that the values of parameters of the model are as follows: $h_1 = 10$, $h_2 = h_3 = 15$, $h_4 = 20$, $c_1 = 25$, $c_2 = c_3 = 30$, $c_4 = 40$, $d_1 = 45000$, $d_2 = 15000$, $d_3 = 30000$, $d_4 = 20000$, $b_1 = b_2 = 8.5$, $b_3 = 12$, $b_4 = 4$, $B = 4000$. The procedure is supposed to make use of the Frank-Wolfe method with the stepsize chosen with Armijo rule.

NLP20. (Based on [BM68]) Consider a mixture of m chemical elements. It has been predetermined that the m different types of atoms can combine chemically to produce n compounds, where the monotonic atom is regarded for our purpose as a possible compound. Define: x_j = the number of moles of compound j present in the mixture at equilibrium,
x = the total number of moles in the mixture, where $x = \sum_{j=1}^{n} x_j$,
a_{ij} = the number of atoms of element i in a molecule of compound j,
b_i = the number of atomic weights of element i in the mixture.
The composition of the mixture at equilibrium is determined in such a way that it minimizes the total free energy of the mixture given by

$$\sum_{j=1}^{n} x_j \left[c_j + \ln \left(\frac{x_j}{x} \right) \right], \quad \text{where } c_j = \left(\frac{F^0}{RT} \right)_j + \ln P,$$

where $(F^0/RT)_j$ is the modal standard (Gibbs) free energy function for the jth compound, which may be found in tables, and P is the total pressure in atmospheres. In addition the balance relationships

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad \text{for } i = 1, \ldots, m,$$

$$x_j \geq 0 \quad \text{for } j = 1, \ldots, n$$

need to be satisfied.

Compute the optimal composition of the mixture for the following data: There are several compounds that can be used to produce $\frac{1}{2}N_2H_4 + \frac{1}{2}O_2$ ($b_1 = 2$, $b_2 = 1$, $b_3 = 1$) at pressure 750psi:

j	Compound	$(F^0/RT)_j$	c_i	a_{1j} (H)	a_{2j} (N)	a_{3j} (O)
1	H	-10.021	-6.089	1		
2	H_2	-21.096	-17.164	2		
3	H_2O	-37.986	-34.054	2		1
4	N	-9.846	-5.914		1	
5	N_2	-28.653	-24.721		2	
6	NH	-18.918	-14.986	1	1	
7	NO	-28.032	-24.100		1	1
8	O	-14.640	-10.708			1
9	O_2	-30.594	-26.662			2
10	OH	-26.111	-22.179	1		1

Use the Frank-Wolfe method with stepsize chosen using Armijo rule to solve the problem. Compare the results you've obtained to those given in [BM68].

NLP21. A chain of stores wants to build a distribution center to provide its retail outlets with articles to sell. The data available are the locations of the retail outlets (x_i, y_i), $i = 1, \ldots, 8$ and the amounts of articles sold at each outlet w_i, $i = 1, \ldots, 8$ as well as the locations of production centers (x_i^P, y_i^P), $i = 1, 2$. Find the location for the distribution center (x_0, y_0), which minimizes the weighted sum of Euclidean distances from the center to the outlets (with w_i serving as weights) and keeps the distances from each of the production centers below (or equal to) 200. The data for the problem is given in the tables below.

Outlet number	x_i	y_i	w_i
1	-19	65	4200
2	37	180	2100
3	149	10	1500
4	87	235	6500
5	-52	-12	4000
6	10	-64	1200
7	25	-11	1500
8	45	-121	3000

Prod. center number	x_i^P	y_i^P
1	105	90
2	-48	20

Your solution should make use of the gradient projection method with stepsizes computed using Armijo rule.

NLP22. Suppose a telecommunication company tries to find the location for a transmission tower. The tower is to cover the area including several towns in the Cincinnati region. The data given below enumerates them together with their locations on the grid (the S-N and W-E distances are given in miles)

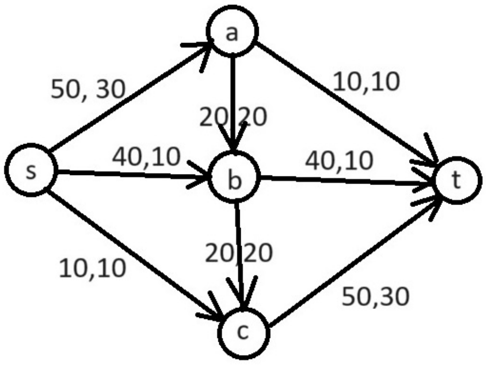
Town	x_i	y_i
Cincinnati	17	14
Florence	10	10
Covington	12	16
Evendale	12	22
Fairfax	13	17
Milford	19	19
Northgate	7	22

The quality of the transmission depends crucially on the distance between the tower and any mobile. Therefore, the goal of the company is to put the tower in a spot that minimizes the maximum Euclidean distance between the tower and any of the towns that are supposed to be covered. This is a minmax problem so it can be transformed into a constrained minimization problem just like in exercise 9 on the problem set 3. Formulate and solve this problem using the barrier method.

NLP23. One of many ways to chose optimal portfolio is to maximize the expected yearly return subject to the constraint on the probability of a given loss. If we assume that the price changes of the stocks are Gaussian, we may pose this problem as a convex optimization problem. The way to do it is explained on pages 22–23 in [SVBL98]. Find the maximum-return portfolio consisting of stocks given in the table from the problem LP7 which keeps the probability of loss exceeding 20% below $\beta = 0.1$. Suppose that you have €1mln at your disposal. It is nicely explained in [DX15] how the data needed for the model should be derived from the table of total returns given in the problem LP7.

Solve the convex optimization problem that you will obtain using the barrier method.

NLP24. Suppose you are given a communication network depicted on the picture below.



You want to find the way for the cars travelling from node s to node t which minimizes their expected time of travel. All the existing arcs (i, j), together with minimal travelling times $t_{ij}^{\min}$ (in minutes) and the capacities of the arcs c_{ij} (in thousands of cars) are given on the picture (the first number is the time, the second is the capacity). The actual travel time for each link (i, j) depends on the amount of traffic x_{ij} given in thousands of cars according to the formula

$$t_{ij}(x_{ij}) = t_{ij}^{\min} + 10\frac{x_{ij}}{1 - x_{ij}/c_{ij}}.$$

Your problem is thus to find the flow through each arc in the network that minimizes the function

$$f(x) = \sum_{(i,j)} t_{ij} x_{ij}$$

subject to the capacity constraints $0 \le x_{ij} \le c_{ij}$ for each existing arc and flow conservation constraints (whatever enters a node must leave it) for each node. Suppose the number of cars entering node s is the same as the number leaving node t and equal to 6000.

Formulate the problem and solve it using the barrier method. Do not bother that the solution may feature fractional numbers of cars.

References:

[BM68] J. Bracken, G.P. McCormick, Selected Applications of Nonlinear Proframming, John Wiley & Sons, Inc., 1968

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[DX15] Z. Donovan, M. Xu, Quadratic Programming: Applications (lecture notes), available online at http://community.wvu.edu/ ~krsubramani/courses/sp15/optfin/lecnotes/QPapps.pdf

[MOR17] M. Or Rashid, Optimization of Non-Linear Programming Problem and its Application in Mathematical Modelling, M.Phil. Thesis, University of Chittagong, 2017

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