Optimization Theory

Applied Mathematics
Final Exam
Group O

Theoretical Questions (2 points each)

1. Suppose we want to find the minimum of the function $f : \mathbb{R}^n \to \mathbb{R}$ subject to $x \in C$. What conditions on f and C guarantee that such a minimum exists?
2. What is the geometric interpretation of the descent direction chosen by the Newton method? What is that of the descent direction chosen by the steepest descent?
3. How can we read the optimal solution of the dual problem from the final Simplex tableau of the primal problem?
4. What does the barrier method need the first phase for? How is it done?
5. What can we say about the Lagrange multipliers corresponding to the inequality constraints?

Problems (6 points each)

6. Suppose we want to find the minimizer (x^*, y^*) of the function $f(x, y) = x^{2k} + y^{2l}$ with $k, l \in \mathbb{Z}^+$, $k \neq l$. What is this minimizer? Show that Newton's method with optimal stepsize converges to (x^*, y^*) in a finite number of iterations iff $x_0 = 0$ or $y_0 = 0$.
7. Solve the linear program

$$\begin{array}{ll} \text{maximize} & 3x_1 - x_2 \\ \text{subject to} & x_2 \geq 2, \\ & -x_1 + x_2 \leq 1, \\ & x_1 + x_2 \leq 4, \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$

   using the Simplex method. Make a graph of the feasible set and indicate on it which vertices are visited by the Simplex procedure in its successive steps (and in what order).
8. Solve the problem

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & 3x_1 + x_2 \leq 10, \\ & 2x_1 + 6x_2 \leq 27, \\ & x_1 \geq 0, x_2 \geq 0, x_1, x_2 \in \mathbb{Z} \end{array}$$

   using the Branch and Bound procedure. Solve each linear-programming problem encountered graphically.
9. Suppose a producer of windshield wipers knows the demands for his products for six months from january to june, d_i, $i = 1, \ldots, 6$. Based on that he wants to plan his production in these months. He has the following possibilities:
   - He may produce up to 1000 wipers per month on his main production line at the cost of \$3 per wiper.
   - He may put into operation the second production line at additional cost of \$8000. If it's running, then a number not bigger than 500 additional wipers can be produced at the cost of \$4 per wiper.
   - He may store up to 200 wipers produced at the monthly cost of \$0.20 per wiper.
   - If he's still not able to satisfy the monthly demand in a month, he has to pay the penalty of \$5000 plus \$10 per each wiper he didn't provide. This rule is applied for each month separately.

   Write (without solving) the integer program allowing to find the production plan minimizing the producer's cost.
10. Using Karush-Kuhn-Tucker conditions find the minimum of $f(x_1, x_2) = 2x_2 - x_1^3$ subject to $x_2 \leq (x_1 - 1)^2$, $x_1^2 + x_2^2 \leq 1$.

Good luck!
Piotr Więcek

Optimization Theory

Applied Mathematics

Final Exam

Group T

Theoretical Questions (2 points each)

1. How can we check that a point x is a local minimizer of a twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$?
2. What are line search algorithms used for? Give at least two examples of line search algorithms.
3. What special cases (cases when no solution is returned) are covered by the Simplex algorithm? How are they identified by the algorithm?
4. Can Newton's method be directly applied to solve some constrained optimization problems? If the answer is affirmative, precise to what kind of constrained problems.
5. How do we define the augmented Lagrangian for the problem "minimize $f(x)$ subject to $h(x) = 0$"? Where is the augmented Lagrangian used and why Lagrangian cannot be used there instead?

Problems (6 points each)

6. Suppose we want to find the minimizer (x^*, y^*) of the function $f(x, y) = x^{2k} + y^{2l}$ with $k, l \in \mathbb{Z}^+$, $k \neq l$. What is this minimizer? Show that Newton's method with optimal stepsize converges to (x^*, y^*) in a finite number of iterations iff $x_0 = 0$ or $y_0 = 0$.
7. Solve the linear program
$$\begin{array}{ll} \text{maximize} & -2x_1 + x_2 \\ \text{subject to} & x_1 \geq 2, \\ & x_1 - x_2 \leq 1, \\ & x_1 + x_2 \leq 5, \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$
using the Simplex method. Make a graph of the feasible set and indicate on it which vertices are visited by the Simplex procedure in its successive steps (and in what order).
8. Solve the problem
$$\begin{array}{ll} \text{maximize} & 3x_1 + x_2 \\ \text{subject to} & x_1 + 3x_2 \leq 10, \\ & 6x_1 + 2x_2 \leq 33, \\ & x_1 \geq 0, x_2 \geq 0, x_1, x_2 \in \mathbb{Z} \end{array}$$
using the Branch and Bound procedure. Solve each linear-programming problem encountered graphically.
9. Suppose a producer of windshield wipers knows the demands for his products for six months from january to june, d_i, $i = 1, \ldots, 6$. Based on that he wants to plan his production in these months. He has the following possibilities:
   - He may produce up to 1000 wipers per month on his main production line at the cost of \$3 per wiper.
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   Write (without solving) the integer program allowing to find the production plan minimizing the producer's cost.
10. Using Karush-Kuhn-Tucker conditions find the minimum of $f(x_1, x_2) = x_1^3 - 2x_2$ subject to $x_2 \leq (x_1 + 1)^2$, $x_1^2 + x_2^2 \leq 1$.

Good luck!

Piotr Więcek