# Theory and Methods of Optimization 

Embedded Robotics

Problem set 1

1. ([Be99]) Show that 2-dimensional function $f(x, y)=\left(x^{2}-4\right)^{2}+y^{2}$ has two global minima and one stationary point, which is neither a local maximum nor a local minimum.
2. Find all the local minima and all the local maxima of the 3-dimensional function

$$
f(x, y, z)=x^{3}-2 x^{2}+y^{2}+z^{2}-2 x y+x z-y z+3 z
$$

3. (Based on [JB03]) A bus company must provide drivers for buses. The schedule varies from hour to hour because of customer demand as shown in the figure. Time 0 on the figure represents midnight, and times are shown with a 24 -hour clock starting at midnight. This is only an example schedule. You are to write a model for a general problem with parameters describing the demand for the each four-hour period.


There are three classes of drivers: part time drivers who work a four hour shift, full time drivers who work a continuous eight hour shift, and split shift drivers who work four hours, are off four hours, and then return for another four hours of work. You are to write your model in terms of these parameters. The parameters take on specific values for a specific instance of the problem.
The hourly wages for drivers from each group are: $\$ 8$ for part-time drivers, $\$ 10$ for full-time drivers and $\$ 13$ for split-time drivers. Full and split time drivers are drawn from the same pool of persons. There are 15 part-time drivers available and 20 full-/split-time drivers available. An additional pool of 15 drivers can be obtained to work full or split shifts at a premium wage of 1.5 times the wage of the regular drivers. A driver of any class can work at most one shift per day, although a shift started at the end of the day may continue into the next day. Write (without solving) the LP model that determines the number of drivers of each type to hire in each time period in order to satisfy demand. The goal is to minimize total wage cost.
4. The factory produces 3 products A, B and C from ingredients X, Y and Z. A unit of product A is then sold for $\$ 2$, a unit of B for $\$ 3$, while a unit of C for $\$ 5$. Each product can be sold in fractional amounts. A unit of material $X$ costs $\$ 1$, a unit of $Y, \$ 2$, while a unit of $Z, \$ 3$. To produce a unit of product C a mix of $\frac{1}{6}$ of material $\mathrm{X}, \frac{1}{3}$ of Y and $\frac{1}{3}$ of Z is necessary. To produce a unit of B a mix of $\frac{1}{2}$ of X with $\frac{1}{4}$ of Z is needed, while to produce $\mathrm{A}-\frac{1}{2}$ of X must be mixed with $\frac{1}{3}$ of Y . The processing of $\mathrm{X}, \mathrm{Y}$ and Z to obtain $\mathrm{A}, \mathrm{B}$ and C does not cost anything. The demand for the product B is limited to 1500 units, while that for product C to 700 units. Suppose also that the owner of the factory can spend at most $\$ 10000$ for the ingredients. Write (without solving) a linear program maximizing the profit of the factory (the value of the products sold minus the value of the ingredients bought) subject to all the constraints. Write also its matrix form (with constraints in form of inequalities).
5. Consider the following linear programming problem with 3 variables $x_{1}, x_{2}, x_{3}$ unconstrained with respect to their sign:

$$
\begin{array}{ll}
\text { Maximize } & 2 x_{1}-3 x_{2}+x_{3} \\
\text { subject to } & 2 x_{1}-x_{2}+6 x_{3} \leq 13 \\
& x_{1}-x_{3} \leq 4 \\
& 5 x_{1}+3 x_{2}+x_{3} \leq 9
\end{array}
$$

Rewrite it equivalently using 4 nonnegative variables.
6. Consider the following (non-linear) problem:

$$
\begin{array}{ll}
\text { Minimize } & \max \left\{\left|5 x_{1}-2 x_{2}\right|,\left|2 x_{2}-x_{1}\right|\right\} \\
\text { subject to } & x_{1}+3 x_{2} \leq 2 \\
& -4 x_{1}+x_{2} \leq 3 \\
& x_{1} \geq 0, x_{2} \geq 0
\end{array}
$$

Transform it into an equivalent linear program by introducing 1 additional variable (plus changing the objective function and adding some constraints).
7. Consider the linear program:

$$
\begin{array}{ll}
\text { Maximize } & \alpha x_{1}+2 x_{2}+x_{3}-4 x_{4} \\
\text { subject to } & x_{1}+x_{2}-x_{4}+2 \Delta \\
& 2 x_{1}-x_{2}+3 x_{3}-2 x_{4}=5+7 \Delta \\
& x_{1} \geq 0, x_{2} \geq 0, x_{3} \geq 0, x_{4} \geq 0
\end{array}
$$

where $\alpha, \Delta \in \mathbb{R}$ are viewed as parameters.
(a) Using the fact that the two first constraints are equalities, rewrite the problem using only 2 variables.
(b) Using graphical interpretation of the problem, identify when the problem is unbounded (has no optimal solution) and when it has an optimal solution. What is it then (there will be several cases depending on the value of $\Delta)$ ?
8. ([BHM77]) Consider the linear program:

$$
\begin{array}{ll}
\text { Maximize } & -3 x_{1}+6 x_{2} \\
\text { subject to } & 5 x_{1}+7 x_{2} \leq 35 \\
& -x_{1}+2 x_{2} \leq 2 \\
& x_{1} \geq 0, x_{2} \geq 0
\end{array}
$$

(a) Solve this problem graphically. Are there multiple optimal solutions?
(b) Solve it using the simplex method. How the fact that there are multiple optimal solutions can be determined using the final simplex tableau?
9. Solve the linear program

$$
\begin{array}{ll}
\operatorname{maximize} & -x_{1}+2 x_{2} \\
\text { subject to } & -x_{1}+x_{2} \leq-1, \\
& x_{1}+x_{2} \leq 3 \\
& x_{1} \geq 0, x_{2} \geq 0
\end{array}
$$

using the Simpex method. Make a picture of the feasible set and show the steps of Simplex on the picture.
10. Solve the linear program

$$
\begin{array}{ll}
\operatorname{maximize} & 3 x_{1}+x_{2} \\
\text { subject to } & x_{1}-x_{2} \leq-1 \\
& x_{1} \geq 1 \\
& x_{1}+2 x_{2} \leq 8 \\
& x_{1} \geq 0, x_{2} \geq 0
\end{array}
$$

using the simpex method (you will need to use both phases). Draw the graphical interpretation of the problem. At each step of the simplex procedure, draw the movements that you have made on the picture.
11. Consider the (non-linear) problem

$$
\begin{array}{ll}
\text { Maximize } & \frac{2 x_{1}-x_{2}+3}{x_{1}+2 x_{2}+2} \\
\text { subject to } & x_{1} \leq 1, \\
& x_{2} \geq 1, \\
& 2 x_{1}+x_{2} \leq 2 \\
& x_{1} \geq 0, x_{2} \geq 0
\end{array}
$$

Introduce 3 new variables which will enable you to transform it into a linear program. Solve it using the simplex method. Compute the values of the original variables corresponding to the solution obtained.
12. Write the dual problem of the linear program from exercise 10. Read its optimal solution from the final simplex tableau of that problem. Check that it satisfies all the constraints of the dual problem and that it gives the value of the dual objective function equal to the optimal value of the primal problem.

## References:

[Be99] D.P. Bertsekas, Nonlinear Programming, Athena Scientific, Belmont, MA: 1999.
[BHM77] S.P. Bradley, A.C. Hax, T.L. Magnanti, Applied Mathematical Programming, Addison-Wesley Publishing Company, 1977
[JB03] P.A. Jensen, J.F. Bard, Operations Research Models and Methods, John Wiley and Sons, 2003

