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Arithmetic Brownian motion subordinated by tempered stable and inverse tempered stable processes

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In the last decade the subordinated processes have become popular and have found many practical applications. Therefore in this paper we examine two processes related to time-changed (subordinated) classical Brownian motion with drift (called arithmetic Brownian motion). The first one, so called normal tempered stable, is related to the tempered stable subordinator, while the second one – to the inverse tempered stable process. We compare the main properties (such as probability density functions, Laplace transforms, ensemble averaged mean squared displacements) of such two subordinated processes and propose the parameters' estimation procedures. Moreover we calibrate the analyzed systems to real data related to indoor air quality.

1. Introduction

Processes based on the Brownian motion were considered in many aspects and have found various practical applications [1–8]. But the assumption of normality for the observations seems not to be reasonable in the number of examined phenomenon. Therefore in many Gaussian models the Brownian motion is replaced by its various modifications. One of the simplest modification is the extension of Gaussian by another distribution, for example the α-stable one. Processes based on the stable distribution are very useful in modeling data that exhibit fat tails. For example, the classical Ornstein–Uhlenbeck process was extended to the stable case and analyzed in Refs. [9,10] as a suitable model for financial data description, see also Ref. [11]. Another possibility of modification for Brownian-type processes is the introduction of time-changed Brownian models. This extension is related to replacement of real time in Brownian systems by a non-decreasing Lévy process (called a subordinator), that in this case plays a role of random (operational) time. The new process is called subordinate. The idea of subordination was introduced in 1949 by Bochner [12] and expounded in his book in Ref. [13]. The theory of subordinated processes is also explored in detail in Ref. [14]. The subordinated processes were studied in many areas of interest, for example in finance [15–18], physics [19–22], ecology [23] and biology [24].

The subordinated processes based on the diffusive Brownian motion were considered in many disciplines. The Brownian motion subordinated by the gamma process, so called variance gamma, is analyzed in Ref. [25] in the context of option price modeling. The other applications and main characteristics of such system can be found in Ref. [26]. Subordination of Brownian motion by the inverse Gaussian process is called the normal inverse Gaussian (NIG) and was considered for example in Ref. [27] and proposed for modeling turbulence and financial data. Applications of NIG processes to asset returns are also shown in Ref. [28] while analysis of real environmental data by using NIG distribution is presented in Ref. [29]. Let us mention that the Brownian motion driven by a strictly increasing Lévy subordinator is also a Lévy process, moreover when the subordinator is a temporally homogeneous Markov process then the subordinate has also this property, [14].

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Another possibility of subordination is the replacement of real time in Brownian diffusion by inverse subordinators, and processes that arise after this transformation exhibit properties of anomalous diffusion. In the domain of anomalous diffusion the typical approach is based on continuous time random walk (CTRW), [30,31], and the subordinated Lévy process can be treated as a limit in the distribution of CTRW, [32]. The key issue in the framework of CTRW as well as in the subordination technique is the waiting-times distribution corresponding to observed constant time periods [33]. In the last decade the anomalous diffusion processes were analyzed by a various number of authors in many disciplines. For example the subordinated Brownian motion driven by an inverse α-stable subordinator was considered in Refs. [11,32,34,35], the inverse tempered stable subordinator was examined in Refs. [33,36–38], while the inverse gamma process was mentioned in Ref. [39]. The general case of Lévy processes that can play the role of inverse subordinators were explored for example in Refs. [40–42]. In general, the anomalous diffusion models are used for systems that exhibit properties not observable in diffusive processes, such as nonlinear in time mean squared displacement or visible constant time periods.

In this paper we examine two processes related to subordinated Brownian motion. The first one, so called normal tempered stable [43,44], is a Brownian motion with drift (called arithmetic Brownian motion, ABM) driven by a tempered stable subordinator, while the second one is an ABM subordinated by an inverse tempered stable subordinator. The tempered stable processes are extensions of the α-stable Lévy systems but possess also the properties of Gaussian models, therefore in the last few years they have become popular and very useful in the description of many real data, [10,44,45]. We compare the main statistical properties of the ABM driven by a tempered stable and an inverse tempered stable subordinator. Moreover in two considered cases we propose the parameters' estimation procedures and validate them. In order to illustrate the theoretical results we calibrate the examined processes to real data related to indoor air quality.

The rest of the paper is organized as follows: In Section 2 we introduce the ABM and tempered stable subordinator. We present the main properties of those processes and define the time-changed ABM driven by the tempered stable process. For this system we also examine the main statistical characteristics, such as the probability density function, Laplace transform and ensemble averaged mean squared displacement that can be an useful tool for distinction between diffusion and anomalous diffusion models. In this section we propose also the estimation procedure based on the distance between theoretical and empirical Laplace transforms and validate it. In Section 3 we examine the inverse tempered stable subordinator and its main statistical properties as well as define the ABM driven by the inverse tempered stable process. In this section we also present the estimation procedure for unknown parameters. In Section 4 we model real data sets related to indoor air quality using the mentioned subordinated processes. The last section contains conclusions.

2. Arithmetic Brownian motion with tempered stable subordinator

2.1. Arithmetic Brownian motion

The arithmetic Brownian motion (ABM) is a process $\{X(t),\ t \geq 0\}$ defined by Refs. [33,46]:

$$\mathrm{d}X(t) = \beta \mathrm{d}t + \mathrm{d}B(t), \tag{1}$$

where $\{B(t),\ t \geq 0\}$ is a classical Brownian motion. The solution of Eq. (1) takes the form

$$X(t) = X(0) + \beta t + B(t). \tag{2}$$

In the further analysis we assume $X(0) = 0$ with probability one and in this case the process defined above has Gaussian distribution with mean βt and variance t. It is called also Brownian motion with drift, [33] and first of all was used to description of stock prices [1,47]. It is an extension of the classical Brownian motion therefore its modifications have found many other practical applications, like diffusion in liquids modeling, [48] and description of hydrology time series [49].

2.2. Tempered stable subordinator

The tempered stable subordinator $\{T(t),\ t \geq 0\}$ is a strictly increasing Lévy process with tempered stable increments, i.e. with the following Laplace transform, [37,38]:

$$\langle \mathrm{e}^{-zT(t)} \rangle = \mathrm{e}^{t(\lambda^{\alpha} - (\lambda+z)^{\alpha})}, \quad \lambda > 0,\ 0 < \alpha < 1. \tag{3}$$

When $\lambda = 0$, then $\{T(t)\}$ becomes totally skewed α-stable Lévy process. The probability density function (pdf) of tempered stable subordinator can be expressed in the following form:

$$f_{T(t)}(x) = \mathrm{e}^{-\lambda x + \lambda^{\alpha} t} f_{U(t)}(x), \tag{4}$$

where $f_{U(t)}(\cdot)$ is a pdf of the process $\{U(t),\ t \geq 0\}$—a totally skewed α-stable Lévy motion with the stability index α, [33,50]. Using tail approximation of stable density, [51], we obtain the following:

$$f_{T(t)}(x) \sim 2\alpha c_{\alpha} \mathrm{e}^{-\lambda x + \lambda^{\alpha} t} t^{\alpha} x^{-(\alpha+1)}, \quad x \to \infty \tag{5}$$

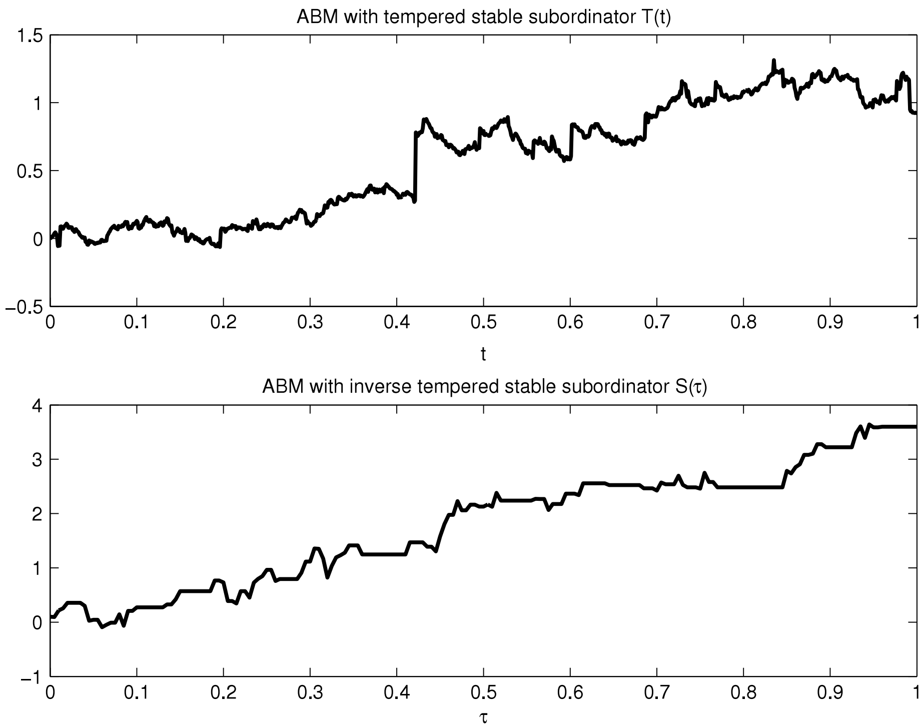

Fig. 1. The sample trajectory of the process $\{Y_T(t),\ t \geq 0\}$ subordinated by tempered stable subordinator (top panel) and process $\{Y_S(t),\ t \geq 0\}$ subordinated by inverse tempered stable subordinator (bottom panel) with parameters $\alpha = 0.8$, $\lambda = 1$ and $\beta = 1$ on the interval [0, 1].

for some constant c_α. Therefore the right tail can be approximated by:

$$1 - F_{T(t)}(x) \sim \mathrm{e}^{-\lambda x + \lambda^\alpha t} \left(\frac{t}{x}\right)^\alpha, \quad x \to \infty. \tag{6}$$

In the above relation $F_{T(t)}(\cdot)$ is the distribution function of $T(t)$. Let us mention that for $\lambda = 0$ (the α-stable case) for fixed t, the above formula reduces to $1 - F_{U(t)}(x) \sim x^{-\alpha}$.

2.3. *ABM with tempered stable subordinator*

The ABM driven by tempered stable subordinator is the process $\{Y_T(t),\ t \geq 0\}$ defined as follows:

$$Y_T(t) = X(T(t)), \tag{7}$$

where $\{X(t)\}$ is ABM defined in (1) and $\{T(t)\}$ is strictly increasing Lévy process with tempered stable increments with the Laplace transform given in (3). We assume processes $\{X(t)\}$ and $\{T(t)\}$ are independent.

Using the form of the solution of Eq. (1) we get:

$$Y_T(t) = B(T(t)) + \beta T(t). \tag{8}$$

The process $\{Y_T(t)\}$ is known in the literature as the normal tempered stable, [43] and the algorithm for generating its sample trajectories in points $t_1, t_2, \ldots, t_n$ proceeds as follows:

1. Simulate the increments of subordinator $\Delta T_i = T(t_i) - T(t_{i-i})$ with the initial value $T_0 = 0$. In order to do this use the simple algorithm presented for example in Ref. [37].
2. Simulate n standard Gaussian random variables $N_1, N_2, \ldots, N_n$.
3. Define $\Delta Y_T(i) = N_i\sqrt{\Delta T_i} + \beta \Delta T_i$ and put $Y_T(t_i) = \sum_{k=1}^{i} \Delta Y_i$.

The sample trajectory of the process $\{Y_T(t)\}$ with parameters $\alpha = 0.8$, $\lambda = 1$ and $\beta = 1$ is presented on the top panel of Fig. 1.

On the basis of the formula for Laplace transform of general subordinated processes given in Ref. [14] we can calculate this characteristics for the process $\{Y_T(t)\}$, namely:

$$\langle \mathrm{e}^{-zY_T(t)} \rangle = \left\langle \mathrm{e}^{-T(t)\left(\beta z - \frac{1}{2}z^2\right)} \right\rangle = \exp\left\{ t\left(\lambda^\alpha - \left(\lambda + \beta z - \frac{1}{2}z^2\right)^\alpha\right)\right\}. \tag{9}$$

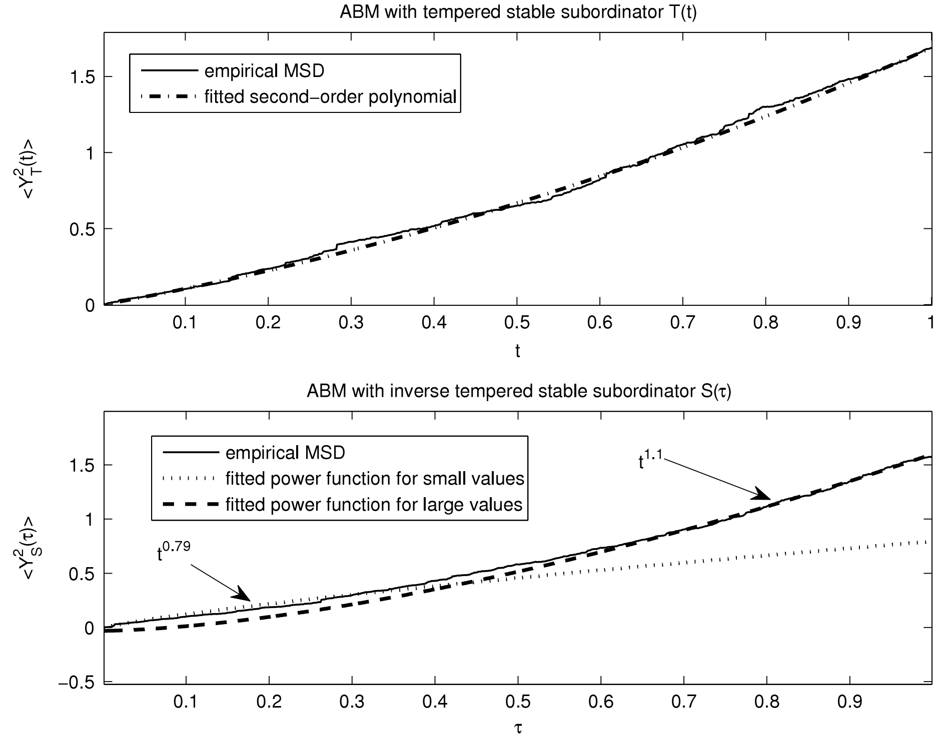

Fig. 2. The ensemble averaged MSD calculated on the basis of 1000 trajectories for the $\{Y_T(t)\}$ (top panel) and $\{Y_S(t)\}$ (bottom panel) processes with $\alpha = 0.8, \lambda = 1$ and $\beta = 0.01$. On the top panel we present the fitted second-order polynomial while on the bottom panel—fitted power functions for small and large values.

The pdf of the process $\{Y_T(t)\}$ defined in (8) can be calculated by using the following formula:

$$f_{Y_T(t)}(x) = \int_0^\infty f_{X(z)}(x) f_{T(t)}(z)\mathrm{d}z, \tag{10}$$

where $f_{X(z)}(\cdot)$ and $f_{T(t)}(\cdot)$ denote the pdfs of the ABM $\{X(t)\}$ and tempered stable subordinator $\{T(t)\}$, respectively. Therefore we obtain:

$$f_{Y_T(t)}(x) = \int_0^\infty \frac{1}{\sqrt{2\pi z}} \mathrm{e}^{-(x-\beta z)^2/2z - \lambda z + \lambda^\alpha t} f_{U(t)}(z)\mathrm{d}z, \tag{11}$$

where $f_{U(t)}(\cdot)$ is the pdf of the totally skewed α-stable Lévy motion $\{U(t),\ t \geq 0\}$ mentioned above.

On the basis of formulas (8) and (9) we can also calculate the main characteristics such as mean and autocovariance function, namely:

$$\langle Y_T(t)\rangle = \langle B(T(t))\rangle + \langle \beta T(t)\rangle = \beta\langle T(t)\rangle = \beta t\alpha\lambda^{\alpha-1} \tag{12}$$

$$\mathrm{cov}(t,s) = \langle Y_T(t), Y_T(s)\rangle - \langle Y_T(t)\rangle\langle Y_T(s)\rangle = \min(s,t)\left(\alpha\lambda^{\alpha-1} + \beta^2\alpha(1-\alpha)\lambda^{\alpha-2}\right). \tag{13}$$

We consider also one of the most popular characteristics of the process that is especially important in real data analysis because it can be a useful tool for identification diffusion and anomalous diffusion models. This characteristics is called mean squared displacement (MSD). Here we consider the ensemble averaged MSD that is defined as a second moment of a given process. Therefore for $\{Y_T(t)\}$ it takes the form, [39]:

$$\langle Y_T^2(t)\rangle = \int_0^\infty x^2 f_{Y_T(t)}(x)\mathrm{d}x, \tag{14}$$

where $f_{Y_T(t)}(\cdot)$ is a pdf of the process $\{Y_T(t)\}$. Let us emphasize the ensemble averaged MSD for real data can be calculated only on the basis of many trajectories, for example when the experiment is repeated many times and the separate measurements are treated as different trajectories of the same process, [52]. For the process $\{Y_T(t)\}$ defined in (8) the ensemble averaged MSD is a polynomial of the second order with respect to t:

$$\langle Y_T^2(t)\rangle = t^2(\beta^2\alpha\lambda^{\alpha-1})^2 + t\left(\alpha\lambda^{\alpha-1} + \beta^2\alpha(1-\alpha)\lambda^{\alpha-2}\right). \tag{15}$$

As we observe, in case of $\beta = 0$, the ensemble averaged MSD scales as t. In Fig. 2 (top panel) we present this characteristics calculated on the basis of 1000 trajectories for the ABM with tempered stable subordinator with $\alpha = 0.8, \lambda = 1$ and $\beta = 0.01$. Moreover we also show the fitted second-order polynomial.

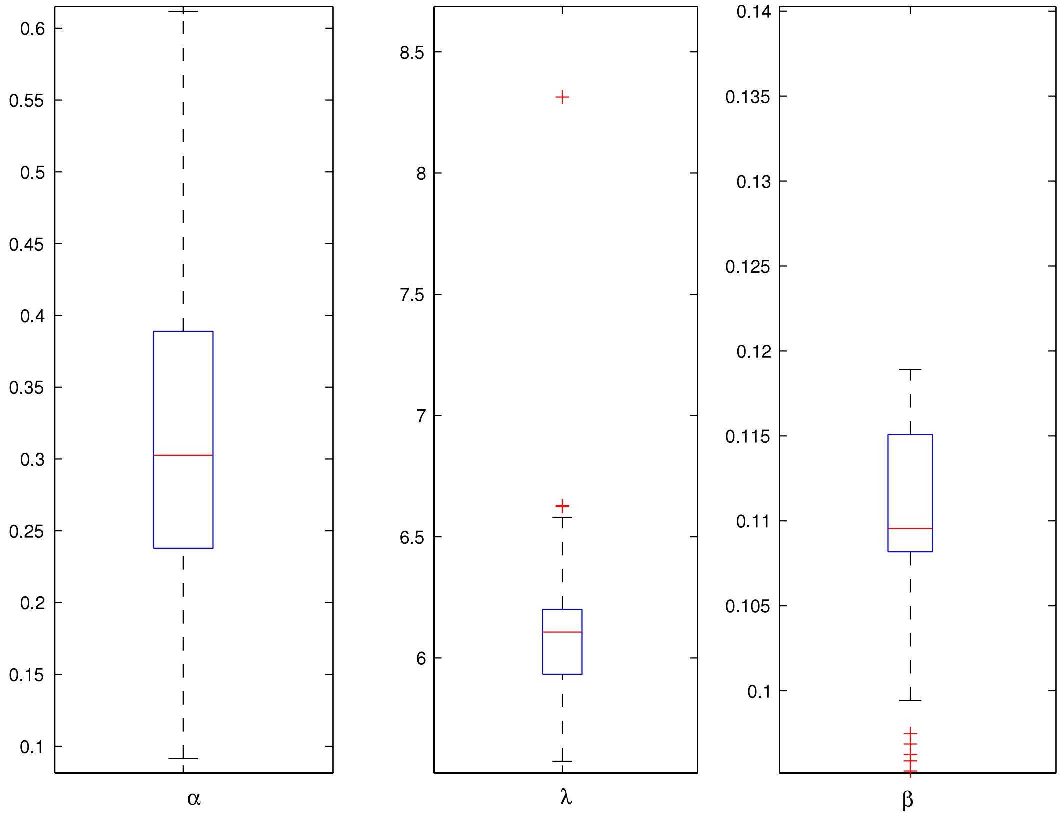

Fig. 3. The boxplots of estimators for parameters of ABM with tempered stable subordinator $\{Y_T(t)\}$. The values were calculated on the basis of 1000 trajectories of length 1000 each. The theoretical values are: $\alpha = 0.26$, $\lambda = 6$, $\beta = 0.11$.

2.4. Estimation procedure

One of the most popular method of estimation the unknown parameters of ABM with tempered stable subordinator is the method of moments. However, we propose here an alternative approach which was also used in Ref. [53] to test tempered stable distribution for a given random sample. The parameters α, λ and β can be estimated by using the formula of the Laplace transform given in (9). The $\{Y_T(t)\}$ is a Lévy process, [14], then its increments are independent and have the same distribution with the following characteristic function:

$$\langle e^{-z(Y_T(t+\Delta)-Y_T(t))}\rangle = \exp\left\{\Delta\left(\lambda^\alpha - \left(\lambda + \beta z - \frac{1}{2}z^2\right)^\alpha\right)\right\}. \tag{16}$$

The estimation scheme for random sample $y_1, y_2, \ldots, y_n$ proceeds as follows:

1. Calculate increments of the observations: $\Delta y_i = y_{i+1} - y_i$ for $i = 1, 2, \ldots, n-1$.
2. For the increments find the empirical Laplace transform using the following formula:

$$\phi(z) = \frac{1}{n-1}\sum_{i=1}^{n-1} e^{-z\Delta y_i}.$$

3. By using the least squares method find the estimators $\hat{\alpha}, \hat{\lambda}, \hat{\beta}$ that satisfy:

$$(\hat{\alpha}, \hat{\lambda}, \hat{\beta}) = \min_{\alpha,\lambda,\beta}\left(\phi(z) - \exp\left\{\lambda^\alpha - \left(\lambda + \beta z - \frac{1}{2}z^2\right)^\alpha\right\}\right)^2.$$

In order to show the efficiency of the described method, in Fig. 3 we present the values of the estimated parameters for simulated process $\{Y_T(t)\}$. To the analysis we take 1000 trajectories of length 1000. The theoretical values are: $\alpha = 0.26$, $\lambda = 6$, $\beta = 0.11$. As we observe, the theoretical values are close to the estimated ones which indicates the procedure works properly.

3. ABM with inverse tempered stable subordinator

In this section we examine the main characteristics of the ABM driven by the inverse tempered stable subordinator. Such kinds of systems were considered in the literature in many aspects, [33,36–39], therefore some properties we present without proofs.

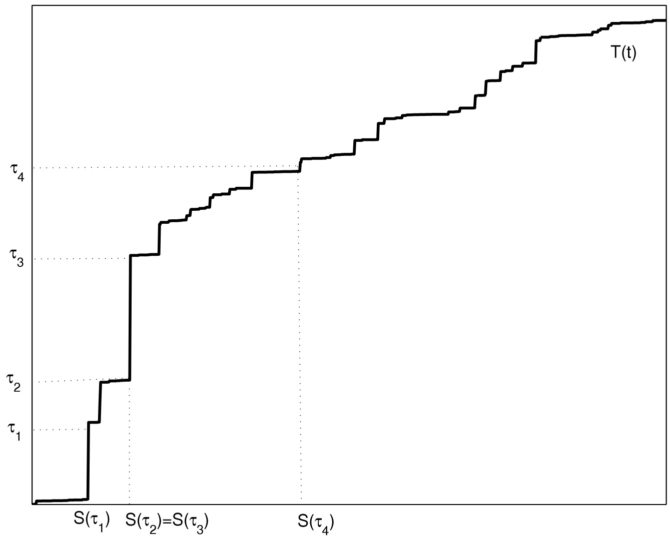

Fig. 4. The relation between the process $\{T(t)\}$ (with parameters $\alpha = 0.26$ and $\lambda = 6$) and its inverse subordinator $\{S(\tau)\}$. Constant periods of the inverse subordinator are related to jumps of the subordinator.

3.1. *Inverse tempered stable subordinator*

The inverse tempered stable subordinator is the process $\{S(\tau),\ \tau \geq 0\}$ defined as follows:

$$S(\tau) = \inf\{t > 0 : T(t) > \tau\}, \tag{17}$$

where $\{T(t),\ t \geq 0\}$ is the tempered stable Lévy process defined via its Laplace transform in (3). In Fig. 4 we present the relation between the process $\{T(t)\}$ and its inverse subordinator $\{S(\tau)\}$. Observe that constant periods of $\{S(\tau)\}$ occur accordingly to the jumps of the process $\{T(t)\}$. To the simulation we take $\alpha = 0.26$ and $\lambda = 6$.

Using the relation $P(S(\tau) \leq x) = P(T(x) > \tau)$ we can calculate the pdf of the process $\{S(\tau)\}$, namely:

$$f_{S(\tau)}(x) = -\frac{\partial}{\partial x} \int_0^\tau f_{T(x)}(u)\mathrm{d}u. \tag{18}$$

Using relation (4) we get:

$$f_{S(\tau)}(x) = -\frac{\partial}{\partial x} \int_0^\tau \mathrm{e}^{-\lambda u + \lambda^\alpha x} f_{U(x)}(u)\mathrm{d}u, \tag{19}$$

where $\{U(t)\}$ is a totally skewed α-stable Lévy process described in the previous section.

Since $f_{U(x)}(u) = \frac{1}{x^{1/\alpha}} f_{U(1)}(u/x^{1/\alpha})$, [54], then we obtain:

$$\begin{aligned}
f_{S(\tau)}(x) &= -\frac{\partial}{\partial x} \int_0^\tau \mathrm{e}^{-\lambda u + \lambda^\alpha x} \frac{1}{x^{1/\alpha}} f_{U(1)}(u/x^{1/\alpha})\mathrm{d}u \\
&= -\int_0^\tau \mathrm{e}^{-\lambda u + \lambda^\alpha x} \left[\lambda^\alpha \frac{1}{x^{1/\alpha}} f_{U(1)}(u/x^{1/\alpha}) - \frac{1}{\alpha x^{1/\alpha+1}} f_{U(1)}(u/x^{1/\alpha}) - \frac{u}{\alpha x^{2/\alpha+1}} f'_{U(1)}(u/x^{1/\alpha}) \right] \mathrm{d}u \\
&= \left(\frac{1}{\alpha x} - \lambda^\alpha \right) \int_0^\tau \mathrm{e}^{-\lambda u + \lambda^\alpha x} \frac{1}{x^{1/\alpha}} f_{U(1)}(u/x^{1/\alpha})\mathrm{d}u + \int_0^\tau \mathrm{e}^{-\lambda u + \lambda^\alpha x} \frac{u}{\alpha x^{2/\alpha+1}} f'_{U(1)}(u/x^{1/\alpha})\mathrm{d}u \\
&= \left(\frac{1}{\alpha x} - \lambda^\alpha \right) \int_0^{\tau/x^{1/\alpha}} \mathrm{e}^{-\lambda u x^{1/\alpha} + \lambda^\alpha x} f_{U(1)}(u)\mathrm{d}u + \frac{1}{\alpha x} \int_0^{\tau/x^{1/\alpha}} \mathrm{e}^{-\lambda u x^{1/\alpha} + \lambda^\alpha x} u f'_{U(1)}(u)\mathrm{d}u \\
&= \frac{1}{\alpha x} \mathrm{e}^{\lambda^\alpha x} \int_0^{\tau/x^{1/\alpha}} \mathrm{e}^{-\lambda u x^{1/\alpha}} (f_{U(1)}(u) + u f'_{U(1)}(u))\mathrm{d}u - \lambda^\alpha \int_0^{\tau/x^{1/\alpha}} \mathrm{e}^{-\lambda u x^{1/\alpha} + \lambda^\alpha x} f_{U(1)}(u)\mathrm{d}u \\
&= \frac{1}{\alpha x} \mathrm{e}^{\lambda^\alpha x} \int_0^{\tau/x^{1/\alpha}} (\mathrm{e}^{-\lambda u x^{1/\alpha}} u f_{U(1)}(u))' + \lambda \mathrm{e}^{-\lambda u x^{1/\alpha}} x^{1/\alpha} u f_{U(1)}(u)\mathrm{d}u - \lambda^\alpha \int_0^{\tau/x^{1/\alpha}} \mathrm{e}^{-\lambda u x^{1/\alpha} + \lambda^\alpha x} f_{U(1)}(u)\mathrm{d}u.
\end{aligned}$$

As a final result we obtain:

$$f_{S(\tau)}(x) = \frac{1}{\alpha x} \tau f_{T(x)}(\tau) + \lambda \frac{1}{\alpha x} \int_0^\tau u f_{T(x)}(u)\mathrm{d}u - \lambda^\alpha \int_0^\tau f_{T(x)}(u)\mathrm{d}u. \tag{20}$$

In case of $\lambda = 0$ we have:

$$f_{S(\tau)}(x) = \frac{1}{\alpha x}\tau f_{U(x)}(\tau), \tag{21}$$

that coincides with the result presented in Ref. [54] for the α-stable case. For large τ the density given in (20) tends to:

$$f_{S(\tau)}(x) \sim \frac{1}{\alpha x}\tau f_{T(x)}(\tau) + \lambda \frac{1}{\alpha x}\langle T(x)\rangle - \lambda^{\alpha}. \tag{22}$$

Since $\langle T(x)\rangle = \alpha x \lambda^{\alpha-1}$, then we have

$$f_{S(\tau)}(x) \sim \frac{1}{\alpha x}\tau f_{T(x)}(\tau) \quad \text{for large } \tau. \tag{23}$$

Similarly, for small τ we get:

$$f_{S(\tau)}(x) \sim \frac{1}{\alpha x}\tau f_{T(x)}(\tau). \tag{24}$$

On the basis of Eq. (20) we can calculate also the Laplace transform of $\{S(\tau)\}$, namely:

$$\langle \mathrm{e}^{-zS(\tau)}\rangle = \int_0^{\infty} \mathrm{e}^{-zx} f_{S(\tau)}(x)\mathrm{d}x. \tag{25}$$

However such derivation requires numerical approximations.

3.2. ABM driven by inverse tempered stable subordinator

The ABM driven by the inverse tempered stable subordinator is defined as follows:

$$Y_S(\tau) = X(S(\tau)) = \beta S(\tau) + B(S(\tau)), \tag{26}$$

where $\{X(t)\}$ is the classical ABM explored in Section 2. The simulation procedure of the process $\{Y_S(\tau),\ \tau \geq 0\}$ is completely described in Ref. [37]. The sample trajectory of the process is presented in Fig. 1 (bottom panel). To the simulation we take the following values of the parameters: $\alpha = 0.8$, $\lambda = 1$ and $\beta = 1$. Let us remind that on the top panel we present the sample trajectory of the ABM with tempered stable subordinator $\{T(t)\}$, namely $\{Y_T(t)\}$. For $\{Y_S(\tau)\}$ we observe constant time periods that are typical for subdiffusive processes. This behavior is not visible for $\{Y_T(t)\}$.

The Laplace transform of the ABM driven by inverse tempered stable subordinator $\{S(\tau)\}$ can be calculated by using the following formula:

$$\langle \mathrm{e}^{-zY_S(\tau)}\rangle = \left\langle \mathrm{e}^{-S(\tau)\left(\beta z - \frac{1}{2}z^2\right)}\right\rangle = \int_0^{\infty} \mathrm{e}^{-\left(\beta z - \frac{1}{2}z^2\right)x} f_{S(\tau)}(x)\mathrm{d}x, \tag{27}$$

where $f_{S(\tau)}(\cdot)$ is given in (20). The pdf $f_{Y_S(\tau)}(x)$ of the process $\{Y_S(\tau)\}$ satisfies the generalized Fokker–Planck equation:

$$\frac{\partial f_{Y_S(\tau)}(x)}{\partial \tau} = \left[-\beta\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial^2 x}\right]\Phi(f_{Y_S(\tau)}(x)), \tag{28}$$

where Φ is an operator defined in Ref. [33]. On the other hand we can use the formula for the pdf of subordinated processes given in Ref. [14]:

$$f_{Y_S(\tau)}(x) = \int_0^{\infty} f_{X(z)}(x) f_{S(\tau)}(z)\mathrm{d}z. \tag{29}$$

Therefore we obtain the following:

$$f_{Y_S(\tau)}(x) = \int_0^{\infty} \frac{1}{\sqrt{2\pi z}}\mathrm{e}^{-(x-\beta z)^2/2z} f_{S(\tau)}(z)\mathrm{d}z. \tag{30}$$

The mean of the process $\{Y_S(\tau)\}$ is given by, Ref. [39]:

$$\langle Y_S(\tau)\rangle = \beta\langle S(\tau)\rangle = \beta\int_0^{\tau} \mathrm{e}^{-\lambda u}u^{\alpha-1}E_{\alpha,\alpha}((\lambda u)^{\alpha})\mathrm{d}u, \tag{31}$$

where $E_{\alpha,\beta}(z)$ is a generalized Mittag-Leffler function defined as follows, [55]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}. \tag{32}$$

According to Ref. [36], for small τ, $\langle S(\tau)\rangle \sim \tau^{\alpha}$, while for large values of τ the mean tends to τ. The asymptotic behavior of $\langle S(\tau)\rangle$ is also discussed in Ref. [39]. Moreover the ensemble averaged MSD takes the form:

$$\langle Y_S^2(\tau)\rangle = \langle(\beta S(\tau) + B(S(\tau)))^2\rangle = \langle \beta^2 S(\tau)\rangle + 2\beta\langle S(\tau)B(S(\tau))\rangle + \langle B^2(S(\tau))\rangle. \tag{33}$$

Since $\langle S(\tau)B(S(\tau))\rangle = 0$ and $\langle B^2(S(\tau))\rangle = S(\tau)$, [41], then we have

$$\langle Y_S^2(\tau)\rangle = \beta^2\langle S^2(\tau)\rangle + \langle S(\tau)\rangle, \tag{34}$$

where

$$\langle S^2(\tau)\rangle = \int_0^\infty x^2 f_{S(\tau)}(x)\mathrm{d}x.$$

According to Ref. [41], the second moment of the process $\{Y_S(\tau)\}$ is bounded by exponential function, i.e.

$$\langle Y_S^2(\tau)\rangle \le c\mathrm{e}^{\tau}, \tag{35}$$

for some constant c, therefore for small τ $\langle S^2(\tau)\rangle$ is bounded. Using relation (24) we get:

$$\begin{aligned}\langle S^2(\tau)\rangle &\sim \int_0^\infty x^2 \frac{1}{\alpha x}\tau f_{T(x)}(\tau)\mathrm{d}x = \int_0^\infty \frac{x}{\alpha x^{1/\alpha}}\tau \mathrm{e}^{-\lambda\tau+\lambda^\alpha x} f_{U(1)}(\tau/x^{1/\alpha})\mathrm{d}x \\ &= \int_0^\infty \frac{\tau^{2\alpha}}{x^{2\alpha}}\mathrm{e}^{-\lambda\tau+\lambda^\alpha(\tau/x)^\alpha} f_{U(1)}(x)\mathrm{d}x \sim \tau^{2\alpha}.\end{aligned}$$

On the other hand for large τ we have:

$$\langle S^2(\tau)\rangle \sim \tau^{2\alpha}\mathrm{e}^{-\lambda\tau}\int_0^\infty x^{-2\alpha}\mathrm{e}^{\lambda^\alpha(\tau/x)^\alpha} f_{U(1)}(x)\mathrm{d}x. \tag{36}$$

Therefore as a final result we obtain that the ensemble averaged MSD for the process $\{Y_S(\tau)\}$ satisfies:

$$\begin{aligned}&\langle Y_S^2(\tau)\rangle \sim c_1\tau^{2\alpha} + c_2\tau^{\alpha}, \quad \text{when } \tau\to 0\\ &\langle Y_S^2(\tau)\rangle \sim d_1\tau^{2\alpha}\mathrm{e}^{-\lambda\tau}\int_0^\infty x^{-2\alpha}\mathrm{e}^{\lambda^\alpha(\tau/x)^\alpha} f_{U(1)}(x)\mathrm{d}x + d_2\tau, \quad \text{when } \tau\to\infty\end{aligned}$$

for some constants c_1, c_2, d_1, d_2. Let us mention, c_1 and d_1 depend on β in such way that for $\beta = 0$, $c_1 = d_1 = 0$, therefore for $\beta = 0$ the ensemble averaged MSD behaves like τ^α for small values of τ and like τ for large values of this parameter, [36].

The ensemble averaged MSD calculated on the basis of 1000 trajectories for the process $\{Y_S(\tau),\ \tau \ge 0\}$ we present on the bottom panel of Fig. 2. To the simulation we take $\alpha = 0.8$, $\lambda = 1$ and $\beta = 0.01$. We show also the fitted power function for small and large values of arguments. Let us mention, on the top panel we present the ensemble averaged MSD for the ABM with tempered stable subordinator $\{Y_T(t),\ t \ge 0\}$ with fitted polynomial of order 2.

3.3. *Estimation procedure*

The estimation procedure for parameters α, β and λ of ABM with inverse tempered stable subordinator is described in Ref. [33]. The idea was also proposed in Refs. [11,56] for the case of an α-stable inverse subordinator. We only mention here that it is based on the decomposition of the time series into two vectors. The first one is responsible for the lengths of constant time periods typical for the subdiffusive processes, while the second one appears after removing the constant periods. According to the theory, in our case the first vector constitutes an independent identically distributed sample from tempered stable distribution. One possible method of estimating the parameters α and λ is the method of moments which was used in Ref. [33] for the financial data with subdiffusive behavior. However in this case it is better to consider another approach because the random sample that describes the lengths of constant time periods contains only the natural values. Similarly as in the α-stable case [56] here we propose to use the method based on the behavior of the tail for tempered stable distribution which behaves like $\mathrm{e}^{-\lambda x}x^{-\alpha}$. Therefore we estimate the parameters α and λ by fitting this function to the empirical tail using the least squares method. The second vector that arises after removing constant time periods, is related to the external process $\{X(t)\}$, therefore in the case of ABM the differenced series has Gaussian distribution with mean β. Therefore the parameter is estimated as a mean of the differenced series. In order to present the efficiency of the presented method we simulate 1000 trajectories (each of length 1000) of ABM with inverse tempered stable subordinator $\{Y_S(\tau)\}$ with $\alpha = 0.4$, $\lambda = 0.2$ and $\beta = 0$. Next, we estimate the parameters using the presented scheme and in Fig. 5 we show the results. As we observe, the fitted parameters are close to the theoretical values.

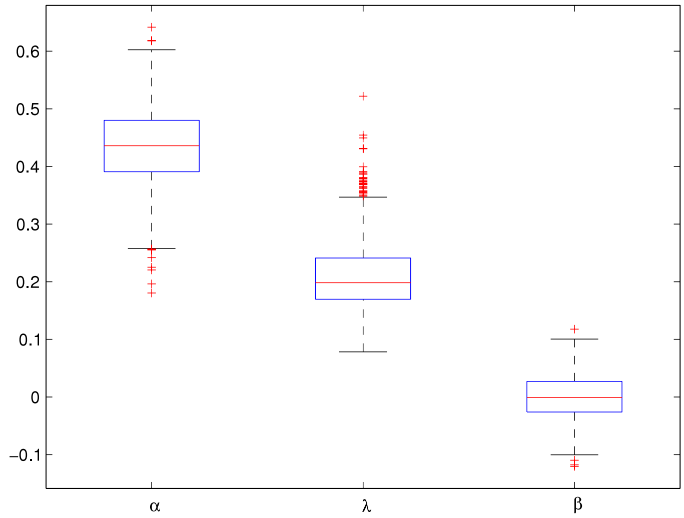

Fig. 5. The boxplots of the values of estimators for parameters of ABM with inverse tempered stable subordinator $\{Y_S(\tau)\}$. The values are calculated on the basis of 1000 trajectories of length 1000. The theoretical values are: $\alpha = 0.4$, $\lambda = 0.2$, $\beta = 0$.

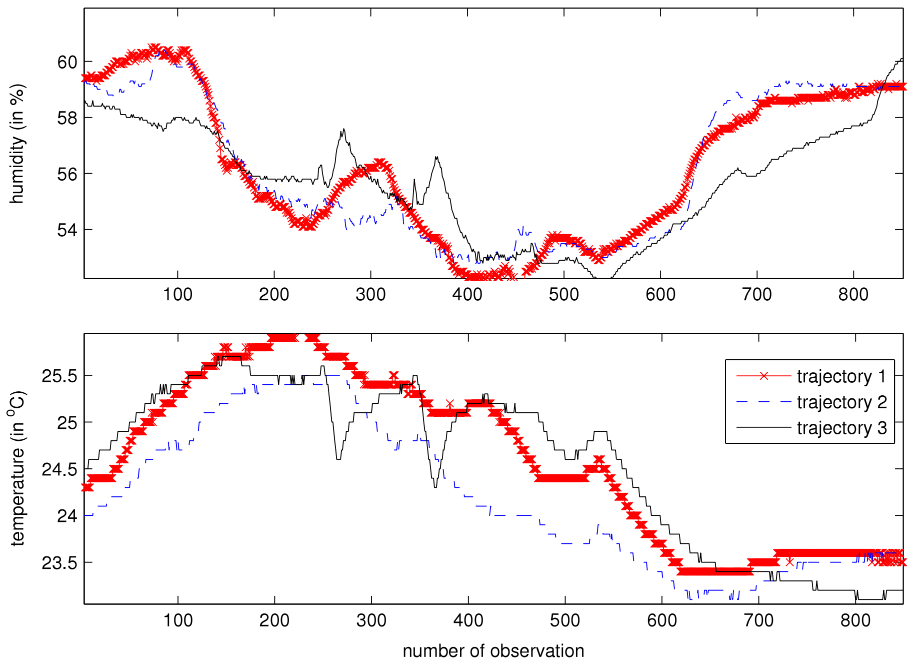

Fig. 6. The humidity (top panel) and temperature (bottom panel) measured by three sensors between 9:48 and 23:59 on 12.08.2010.

4. Applications

In this section we examine real data sets that describe humidity (in %) and temperature (in °C) of the indoor air in some open space of a huge company. For simplicity we denote humidity as DATA1 and temperature—as DATA2. Those two analyzed quantities were measured by three sensors placed in the open space. Therefore for DATA1 and DATA2 we have three paths (trajectories). The humidity and temperature was quoted per minute. To the analysis we take data from 9:48–23:59 on 12.08.2010 (851 observations of each trajectory). In Fig. 6 we present paths of humidity (top panel) and data related to temperature (bottom panel).

As we observe on the bottom panel of Fig. 6, three sets related to DATA2 exhibit behavior typical to subdiffusive process $\{Y_S(\tau)\}$ described in the previous section, namely the constant time periods. This behavior is not visible for DATA1 (top panel). In the first step we examine the empirical ensemble averaged MSD calculated on the basis of three trajectories (of DATA1 and DATA2), see Fig. 7. To the ensemble averaged MSD of DATA1 we fit the second-order polynomial (top panel), while for small and large values of the MSD of DATA2—the power functions. In both cases we use the least squares method. On the basis of three trajectories we cannot make exact statistical inferences, but even in this case, as we observe, the

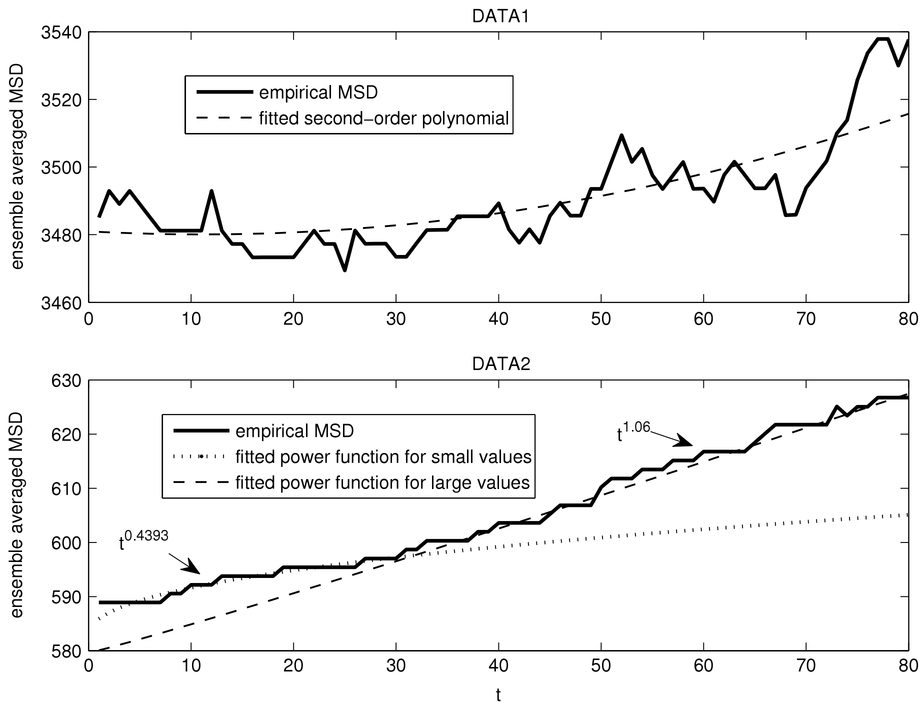

Fig. 7. The ensemble averaged MSD of humidity (top panel) and temperature (bottom panel) together with the fitted second-order polynomial (for DATA1) and power function for small and large arguments (for DATA2).

Table 1
Estimated parameters of the ABM with a tempered stable subordinator for DATA1.

DATA1	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$
Trajectory 1	0.23	6.1	0.11
Trajectory 2	0.29	5.9	0.12
Trajectory 3	0.24	6.1	0.12

Table 2
Estimated parameters of the ABM with inverse tempered stable subordinator for DATA2.

DATA2	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$
Trajectory 1	0.41	0.13	−0.0085
Trajectory 2	0.3936	0.11	−0.0039
Trajectory 3	0.4906	0.129	−0.012

ensemble averaged MSD for DATA1 behaves like second-order polynomial for all arguments that suggests behavior similar to this observed in Fig. 2 (top panel). The empirical MSD of the temperature that we observe on bottom panel of Fig. 7 clearly suggests subdiffusive behavior.

In the next step of our analysis we estimate the parameters of the suggested processes, i.e. ABM with a tempered stable subordinator for DATA1 and ABM driven by an inverse tempered stable subordinator for DATA2. In both cases we use the procedures presented in Sections 2 and 3, respectively. In Table 1 we present the estimated parameters for DATA1.

Moreover, in Fig. 8 we present the empirical Laplace transform for each of the three trajectories of DATA1 and a theoretical one given in (9). In order to confirm that the tempered stable subordinator is better than the stable one we show also the Laplace transform for the process $\{Y_T(t)\}$ with $\lambda = 0$ (the other parameters we estimate using the same method as this presented in Section 2). Moreover we test also the Gaussian and α-stable distributions for a differenced series of DATA1 using the methods based on the distance between empirical and theoretical distribution functions, [57]. Those tests reject the hypothesis of the stable or Gaussian behavior of the observed data.

After analysis of the humidity we start the examination of temperature. The ensemble averaged MSD indicates the data can be considered as a subdiffusive process. Therefore we propose to use the ABM with inverse tempered stable subordinator described in Section 3. Let us mention, the α-stable subordinator is not appropriate in this case because of the behavior of ensemble averaged MSD that in stable case behaves like τ^{α} for all values of arguments. The estimated values of α, λ and β parameters for three trajectories we present in Table 2.

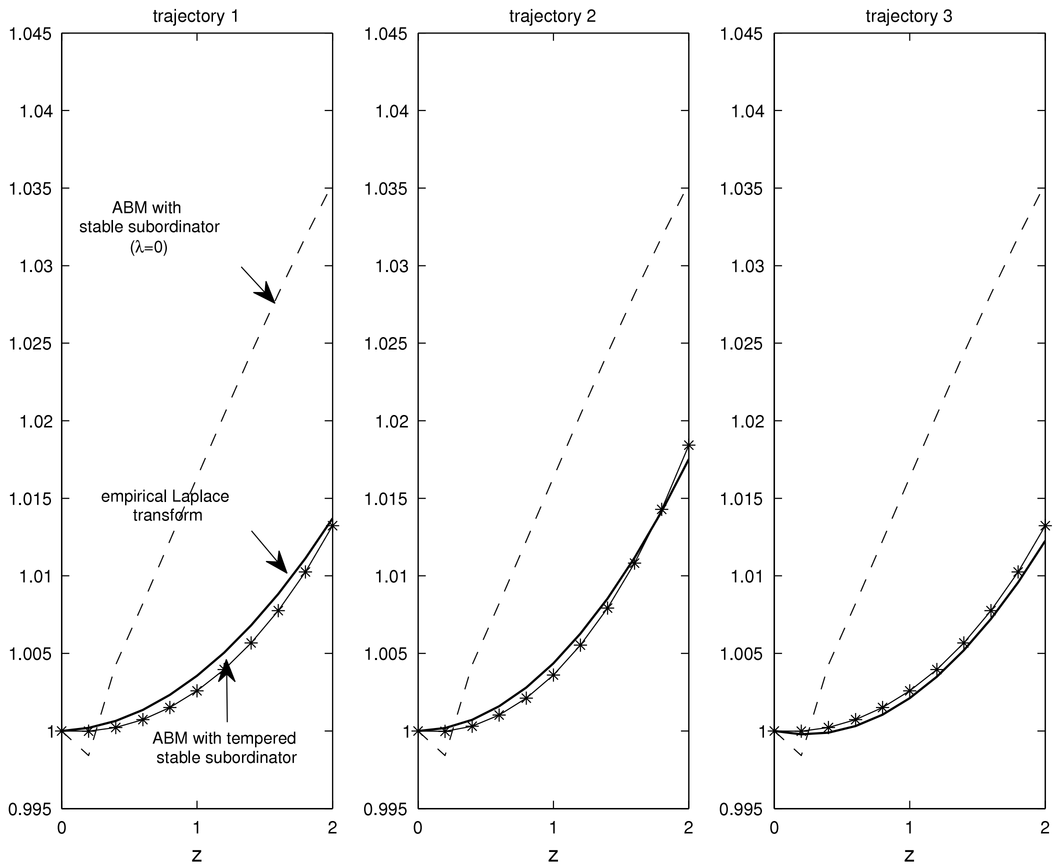

Fig. 8. The empirical Laplace transform (solid line), theoretical Laplace transform for ABM with tempered stable subordinator (star line) and theoretical Laplace transform for ABM with stable subordinator (dashed line) for three trajectories of DATA1.

5. Conclusions

In this paper we have examined two processes related to subordinated ABM driven by tempered stable type systems. The first one, so called normal tempered stable, arises after subordination of ABM by a strictly increasing tempered stable process, while the second one is a result of subordination of ABM with an inverse tempered stable model. We have compared the main characteristics of such systems, like Laplace transforms, asymptotic behavior of pdf as well as the ensemble averaged MSD, that can be an useful tool for identification a proper model. We have described also the estimation procedures for the parameters of considered processes and validated them. Finally, we have analyzed the real data related to indoor air quality using the presented methodology.

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