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# THE NON-EUCLIDEAN SYMMETRY OF ESCHER'S PICTURE 'CIRCLE LIMIT III'*

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**Abstract**—*Of all Escher's pictures with a mathematical background, the most sophisticated is his 1959 woodcut,* Circle Limit III, *which uses four colours in addition to black and white. Queues of fishes of each colour are swimming along white arcs that cut the peripheral circle at a certain angle. After discussing the kind of symmetry that is involved and the underlying regular tessellations (so cleverly disguised), the author explains why the above-mentioned angle is not 90° but 80°.*

## I. INTRODUCTION

I first met Escher [1] in September 1954, when an exhibition of his work was sponsored by the International Congress of Mathematicians, meeting that year in Amsterdam. Throughout the previous 17 years he had been making designs in which a drawing of some animal (such as a fish or a reptile or a bird) is repeated as on wallpaper, with two remarkable innovations: the basic unit (usually a single animal, or one half of a symmetrical animal or two different animals juxtaposed) is repeated not only by translations but also by other isometries (or *congruent transformations*): rotations, reflections or glide-reflections [2]; and the replicas ingeniously fit together so that there are no interstices. In the language of mathematics (a subject in which Escher resolutely claimed to be 'absolutely innocent of training or knowledge'), the basic unit is a *fundamental region* for a *symmetry group*.

In a letter of December 1958 he wrote: 'Did I ever thank you for sending me . . . "A Symposium on Symmetry"? I was so pleased with this booklet and proud of the two reproductions of my plane patterns!

'Though the text of your article on "Crystal Symmetry and its Generalizations" [3] is much too learned for a simple, self-made plane pattern-man like me, some of the text-illustrations and especially Figure 7, page 11, gave me quite a shock.

'Since a long time I am interested in patterns with "motives" getting smaller and smaller till they reach the limit of infinite smallness. The question is relatively simple if the limit is a point in the centre of a pattern. Also a line-limit is not new to me, but I was never able to make a pattern in which each "blot" is getting smaller gradually from a centre towards the outside circle-limit, as shows your Figure 7 [reproduced here as Fig. 1]. I tried to find out how this figure was geometrically constructed, but I succeeded only in finding the centres and radii of the largest inner-circles. If you could give me a simple explanation how to construct the following circles, whose centres approach gradually from the outside till they reach the limit, I should be immensely pleased and very thankful to you! Are there other systems besides this one to reach a circle-limit?

'Nevertheless I used your model for a large woodcut (of which I executed only a sector of 120° in wood, which I printed 3 times). I am sending you a copy of it. . . .'.

This was his picture 'Circle Limit I', concerning which he wrote on another occasion [4]: 'The largest animal figures are now located in the centre, and the limit of the infinitely many and infinitely small is

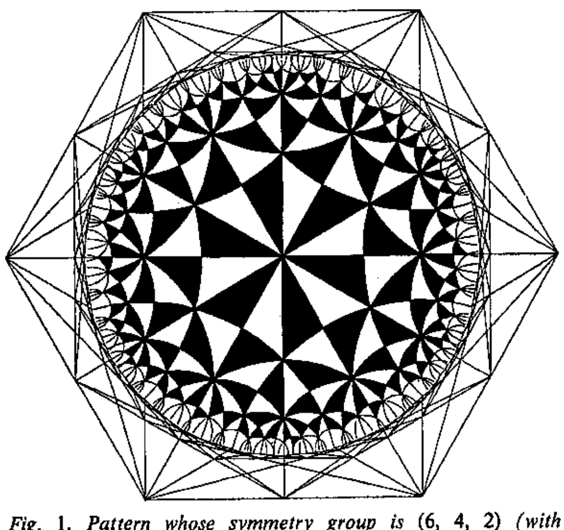

*Fig. 1. Pattern whose symmetry group is* (6, 4, 2) *(with scaffolding). Two adjacent triangles (one white and one black) form a fundamental region.*

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reached at the circular edge. The skeleton of this configuration, apart from the three straight lines passing through the centre, consists solely of arcs with increasingly shorter radii the closer they approach the limiting edge. In addition, they all intersect it at right angles. This woodcut *Circle Limit I*, being a first attempt, displays all sorts of shortcomings. Not only the shape of the fish, still hardly developed from rectilinear abstractions into rudimentary animals, but also their arrangement and relative position, leave much to be desired. . . . There is no continuity, no "traffic flow", no unity of colour in each row.'

## II. THE HYPERBOLIC PLANE

Replying to Escher's letter, I told him that Fig. 1 is one of infinitely many such patterns in which a Euclidean or non-Euclidean plane is tessellated by black and white triangles, each having angles

$$\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$$

at its three vertices ($\pi$ means 180°). One vertex is surrounded by $p$ black and $p$ white triangles, another by $q$ and $q$, the third by $r$ and $r$. The symmetry group of such a pattern includes (and usually coincides with) a group that is denoted by $(p, q, r)$, because it is generated by rotations of periods $p, q, r$ about the vertices of any one of the triangles.

The spherical cases $\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1\right)$ are:

$$(p, 2, 2), (3, 3, 2), (4, 3, 2), (5, 3, 2).$$

The corresponding patterns are formed by great circles decomposing the sphere into spherical triangles.

The Euclidean cases $\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1\right)$ are:

$$(3, 3, 3), \quad (4, 4, 2), \quad (6, 3, 2).$$

Here straight lines of the Euclidean plane form ordinary triangles (equilateral in the first case).

The hyperbolic cases $\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1\right)$

are infinite in variety. In one of Poincaré's models, the hyperbolic plane appears as the interior of a bounding circle $\Omega$ drawn in the Euclidean plane. Angles are represented faithfully but distances are distorted, the points of $\Omega$ itself being infinitely far away.

In Fig. 1, the cluster of 6+6 triangles in the middle fills a regular hexagon. Four replicas of this hexagon fit together at each vertex, making altogether an infinite *regular* tessellation $\{6, 4\}$ (6 specifying hexagon, 4 the number of 'faces' at a vertex). Every $(p, q, 2)$ is the rotational symmetry group of such a regular tessellation $[p, q]$. (The regular tessellation is symmetrical also by reflections that interchange black and white.)

In this Poincaré model, the 'straight' lines of the hyperbolic plane appear as arcs of circles orthogonal to $\Omega$. The exterior of $\Omega$ is not part of the hyperbolic plane, but its points are the centres of these circles. Whenever three or more hyperbolic lines pass through a point, the corresponding circles orthogonal to $\Omega$ have a second common point outside $\Omega$; they are coaxial and so their centres lie on a Euclidean line. Whenever a circle belongs to two such coaxial pencils, its centre is determined as the point of intersection of the two corresponding lines of centres; such lines form a 'scaffolding' which can be used to construct the tessellation.

Escher's sketch-books show that he diligently pursued these ideas before completing 'Circle Limits II, III, IV'. In contrast to his criticism of 'Circle Limit I' (quoted above), he wrote: 'In the coloured woodcut *Circle Limit III* most of these defects have been eliminated. We now have only "through traffic" series: all the fish of the same series have the same colour and swim after each other, head to tail, along a circular arc from edge to edge. The nearer they get to the centre, the larger they become. Four colours are needed for each series to be in complete contrast with its surroundings. As all these strings of fish shoot up like rockets from infinitely far away, perpendicularly from the boundary, and fall back again whence they came, not one single component ever reaches the edge. Outside there is "absolute nothingness". And yet this round world cannot exist without the emptiness around it, not only because "inside" presupposes "outside", but also because it is out there in the "nothingness" that the scaffolding lies, determining with geometric precision the centres of the circular arcs which form the skeleton.'

In his commentary [5], Bruno Ernst wrote: 'The best of the four is *Circle Limit III*, dated 1959 . . . . The network [skeleton] for this is a slight variation on the original one. In addition to arcs placed at right angles to the circumference (as they ought to be), there are also some arcs that are not so placed.'

In Part VI we shall see why *all* the white arcs 'ought' to cut the circumference at the same angle, namely 80° (which they do, with remarkable accuracy). Thus Escher's work, based on his intuition, without any computation, is perfect, even though his poetic description of it (*'loodrecht uit de limiet'*, 'perpendicularly from the boundary') was only approximate.

## III. A REGULAR COMPOUND TESSELLATION

Looking carefully at 'Circle Limit III' (Fig. 2, cf. colour plate), we see that the mathematically significant points are of three types, say $P$, $Q$, $R$, each occurring infinitely many times:

$P$, where the right fins of 4 fish come together (two colours alternating) as at the centre of the whole picture;

*Q*, where the left fins of 3 fish come together (using 3 of the 4 colours);
*R*, where the mouths of 3 fish meet the tail-tips of 3 others (again using 3 of the 4 colours).

The points of type *P* are the vertices of triangles that, from the standpoint of hyperbolic geometry, are equilateral, with a point of type *Q* or *R* at the centre of each triangle (see Fig. 3). Each point of type *P* belongs to 8 such triangles, so we have altogether a tessellation {3, 8}: congruent equilateral triangles, 8 at each vertex. Unlike the Euclidean equilateral triangle, whose angle is 60°, this hyperbolic triangle has an angle of 45°, allowing 8 replicas to fill up the whole 360° at their common vertex.

The points of types *Q* and *R*, being the centres of the triangles, are together the vertices of the dual (or *reciprocal*) tessellation {8, 3}: congruent regular octagons, 3 at each vertex (see Figs. 4 and 5).

Among the points of type *P*, all the $\binom{4}{2} = 6$ pairs of the 4 colours occur with the same frequency. At the centre of the whole picture the two colours are green and yellow. We have to proceed a long way out before finding the complementary pair: blue and brown. Having found one such point, we soon see that there are 8 of them, forming a regular octagon. Figure 3 shows the way 16 faces of {3, 8} fit together to fill up this large octagon. The angle at each vertex of the octagon, being the angle of a triangular face of {3, 8}, is 45°: one-eighth of a whole turn. It follows that each of these 8 blue-brown *P*-points is surrounded by 8 such octagons. (Seven of them are too small to be seen without a microscope, although from the hyperbolic standpoint they are all the same size.) All the vertices of this infinite tessellation {8, 8} are blue-brown *P*-points. The remaining five pairs of colours yield five more {8, 8}s. The superposition of these six {8, 8}s, called a *compound* tessellation, is denoted by the symbol

$$\{3, 8\}\ [6\{8, 8\}]\ \{8, 3\},$$

indicating that it is a compound of 6 tessellations {8, 8} whose vertices together coincide with the vertices of a {3, 8}, while their faces have the same centres as the faces of the dual {8, 3}. Thus, in effect, Escher anticipated by five years my discovery [6] of this hyperbolic compound, which is analogous [7] to Kepler's *stella octangula*: the finite compound

$$\{4, 3\}\ [2\ \{3, 3\}]\ \{3, 4\}$$

of 2 tetrahedra {3, 3} whose 8 vertices coincide with those of a cube {4, 3} while their 8 faces lie in the same planes as the faces of the reciprocal octahedron {3, 4}.

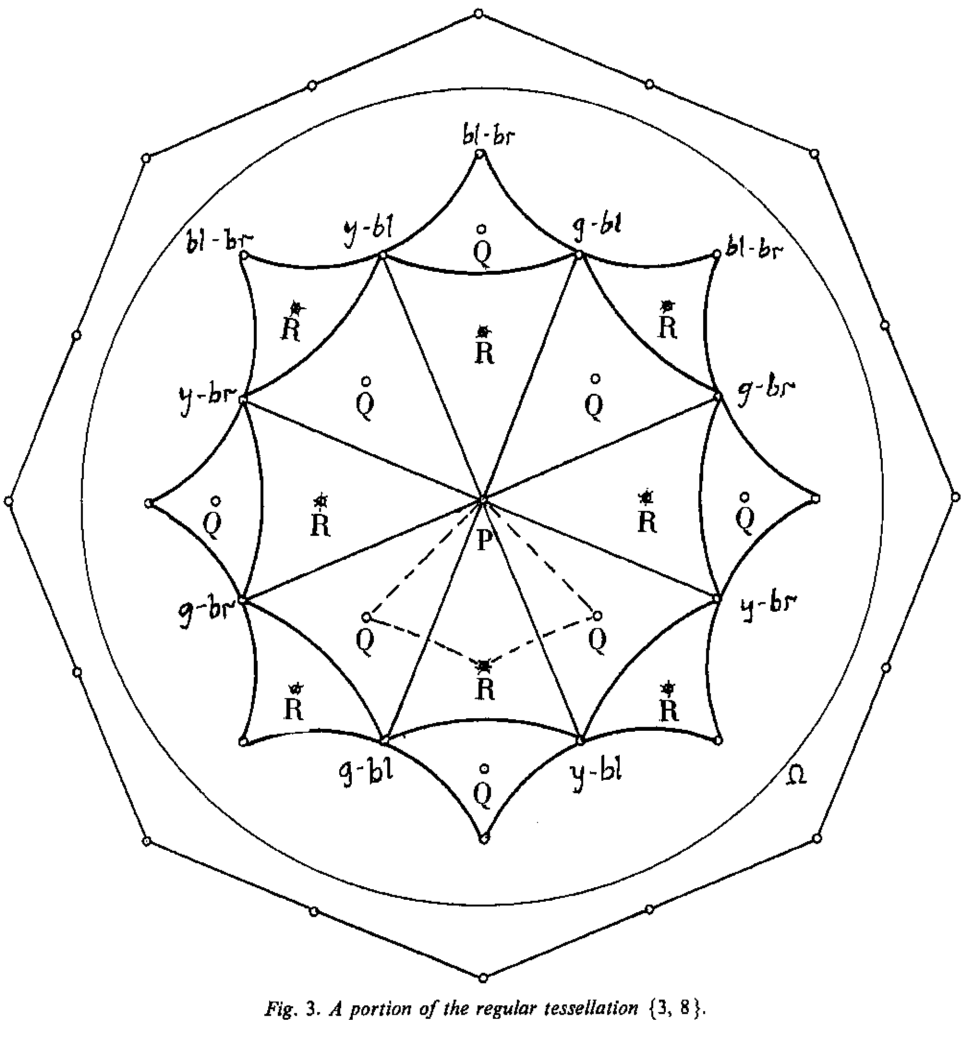


*Fig. 3. A portion of the regular tessellation {3, 8}.*

## IV. GROUPS OF HYPERBOLIC MOTIONS

From the standpoint of hyperbolic geometry, the disc enclosed by the peripheral circle is the whole plane, and the fishes (which fill it) are all congruent. There is an isometry taking any one fish to any other, but the fish itself is not symmetrical. In other words, when colour is disregarded, any fish serves as a fundamental region for the symmetry group of the whole pattern. This symmetry group is denoted by (4, 3, 3) because it is generated by rotations $A, B, C$ through angles $2P$, $2Q$, $2R$ about the vertices of a triangle $PQR$ [8] where, in the present case, $P = \pi/4$, $Q = R = \pi/3$. Such an isosceles triangle $PQR$ is formed by any three neighbouring points of these respective types. Each point of type $P$ is the common vertex of a cluster of eight such triangles filling a face $QRQRQRQR$ of $\{8, 3\}$. Any two adjacent triangles form together a fundamental region for this group (4, 3, 3). For instance, we can use two that share a long side and thus form a *kite* ($PQRQ$ drawn with broken lines in Fig. 3): one quarter of a face of $\{8, 3\}$. We easily see that such a kite has exactly the same area (in the hyperbolic sense) as one fish.

The rotations $A, B, C$ satisfy the relations

$$A^4 = B^3 = C^3 = ABC = 1,$$

which provide an abstract definition or *presentation* for the group (4, 3, 3). Since $A = (BC)^{-1}$, this is a redundant presentation: the group is generated just as well by $B$ and $C$ alone, that is, by rotations through 120° about any two neighbouring points of types $Q$ and $R$. With two such generators, the group has the presentation

$$B^3 = C^3 = (BC)^4 = 1.$$

The symmetry group (4, 3, 3) has a subgroup (2, 2, 2, 2, 2, 2) of index 12, which preserves colour. In other words, we must travel out so far from the middle of the picture, before finding a repetition of both shape and colour, that the fundamental region for the colour-preserving symmetry group has 12 times the area of a fish (that is, three times the area of a face of $\{8, 3\}$). This number 12 can be computed by observing that there are not only four different colours but three ways, for each coloured

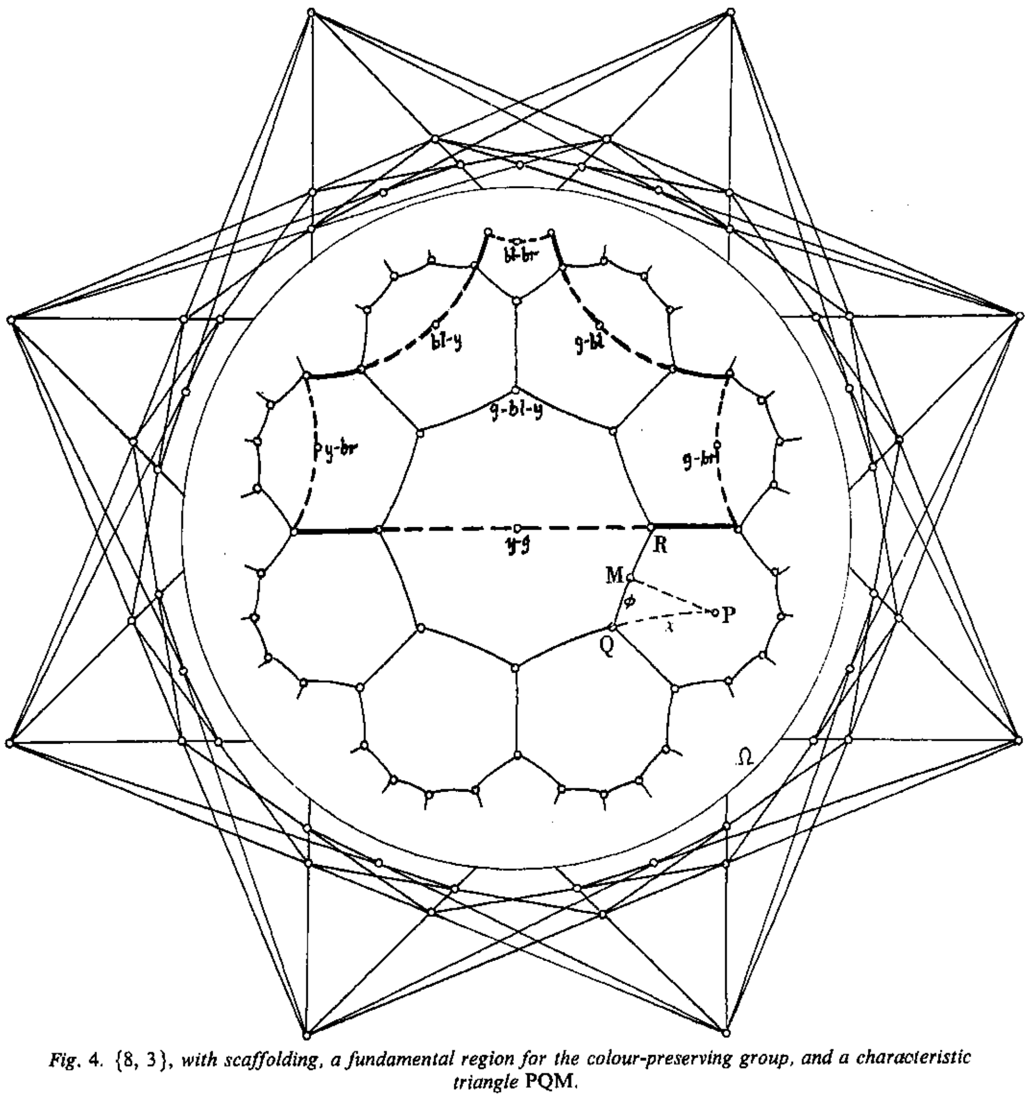


*Fig. 4.* {8, 3}, *with scaffolding, a fundamental region for the colour-preserving group, and a characteristic triangle* PQM.

fish, to distribute the remaining three colours among its neighbours. This group **(2, 2, 2, 2, 2, 2)** is generated by half-turns about all the points of type $P$ or, more economically, by half-turns about just six such points involving different pairs of the four colours. Figure 4 shows a convenient choice of the six points: the centres (blue-yellow, yellow-green, green-blue) of three octagons having a common vertex (green–blue–yellow) and the centres (green–brown, blue-brown, yellow–brown) of three further octagons, each adjacent to two of the first three. These six points $P$ are the mid-points of the sides of an equiangular hexagon $QQQQQQ$ (drawn in heavy lines) that serves as a fundamental region for the group. The siz points $Q$ lie on a circle (not drawn) and, from the hyperbolic standpoint, the sides of the hexagon are of two lengths, occurring alternately. Thus, the hexagon is more symmetrical than it seems to be: it has the symmetry of an equilateral triangle.

The six half-turns satisfy the relations

$$T_1^2 = T_2^2 = T_3^2 = T_4^2 = T_5^2 = T_6^2$$
$$= T_1\,T_2\,T_3 T_4\,T_5\,T_6 = 1,$$

which provide a presentation for the group (2, 2, 2, 2, 2, 2), whose Euclidean counterpart

$$\mathbf{p2} = (2, 2, 2, 2)$$

is generated by half-turns about the mid-points of the four sides of a rectangle (or any other quadrangle) [9], with the presentation

$$T_1^2 = T_2^2 = T_3^2 = T_4^2 = T_1\,T_2\,T_3\,T_4 = 1.$$

## V. THE ARCS ALONG WHICH ESCHER'S FISHES SWIM

Figure 5 shows a portion of the {8, 3} with heavy arcs . . .*RRRR*. . . and . . .*QQQQQ*. . . enclosing a zigzag . . .*QRQRQRQRQ* . . .. From the hyperbolic standpoint, the lines of this zigzag, being edges of {8, 3}, are straight, and of equal length, say $2\phi$. From considerations of symmetry, we see that the mid-points of these edges lie on a (hyperbolic) straight line $a$ (circular arc orthogonal to $\Omega$; see Fig. 6). However, the arcs . . .*RRRR*. . . and . . .*QQQQQ*. . . are not orthogonal to $\Omega$. It is important to note that the perpendicular distance to $a$ from any point on either of these arcs is constant, say $\delta$. The pair of arcs, being the locus of points distant $\delta$ from $a$, is called an *equidistant-curve* with *axis a* and *altitude* $\delta$. The zigzag is called a *Petrie polygon* of {8, 3}; every two consecutive edges belong to a face, but no three belong to the same face.

The equidistant-curve, like a Euclidean hyperbola, has two congruent branches. The $R$ branch appears in Escher's picture (Fig. 2; cf. colour plate) as a white line along the backs of a row of fishes chasing one another. The $Q$ branch was not drawn by Escher, though we can see the points of type $Q$ that would be strung out along it.

His white lines seem at first sight to form a tessellation of triangles and quadrangles all of

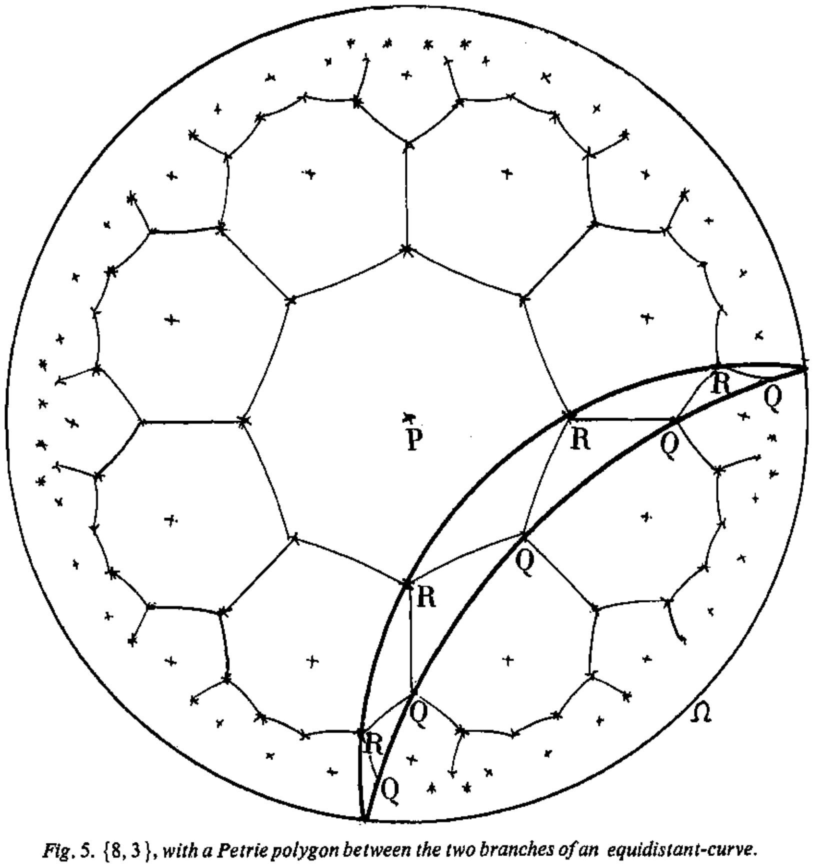


*Fig. 5. {8, 3}, with a Petrie polygon between the two branches of an equidistant-curve.*

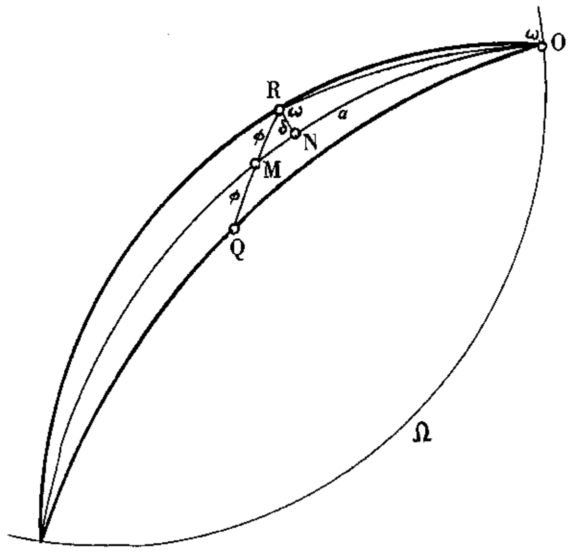


*Fig. 6. The angle of parallelism $\omega = \Pi(\delta)$.*

whose angles are 60°. But this is absurd, as a triangle with angles 60° would be Euclidean, not hyperbolic. The above discussion explains the paradox by showing that Escher's 'lines' are not straight: each is one branch of an equidistant-curve.

## VI. THE ANGLES AT THE PERIPHERAL CIRCLE

The angle (about 80°) at which such an arc . . .*RRR*. . . cuts the peripheral circle $\Omega$ (representing the set of all points at infinity, such as $O$) appears in Fig. 6 as the angle $\omega$ at $O$. It reappears at $R$ as the angle between two arcs orthogonal to $\Omega$ (representing straight lines of the hyperbolic plane), namely, $RN$ (which, being orthogonal also to the arcs $a$ and . . .*RRR*. . ., represents an altitude $\delta$ of the equidistant-curve) and $RO$. In the hyperbolic geometry, these arcs $RN$ and $RO$, perpendicular and parallel (respectively) to the axis $a$ of the equidistant-curve, form the so-called *angle of parallelism* for the distance $RN = \delta$ ; in Lobachevsky's notation [10]

$$\omega = \Pi(\delta) = 2 \operatorname{arc\,tan} e^{-\delta}. \qquad (1)$$

The point marked $P$, in the lower half of Fig. 4, is the centre of a face of $\{8, 3\}$. Suppose this octagon has edge $2\phi = QR$ and circum-radius $PQ = \chi$. Then $M$, the mid-point of $QR$, forms with $PQ$ a right-angled triangle whose two acute angles are $\pi/8$ at $P$, $\pi/3$ at $Q$. Two of the classical formulae of hyperbolic trigonometry [11] yield:

$$\cosh \phi \sin(\pi/3) = \cos(\pi/8) \qquad (2)$$

$$\cosh \chi = \cot(\pi/8) \cot(\pi/3). \qquad (3)$$

Similarly, in Fig. 6, $MRN$ is a right-angled triangle with $MR = \phi$, $RN = \delta$, and angle $\pi/3$ at $R$. Another one of the classical formulae yields

$$\tanh \delta = \tanh \phi \cos(\pi/3) = \tfrac{1}{2} \tanh . \phi$$

By equation (1), $\tanh \delta = \cos \omega$, and by equation (2),

$$\cosh^2 \phi = (2+\sqrt{2})/3,$$

$$\tanh^2 \varphi = 1 - \operatorname{sech}^2 \phi = 1 - 3/(2+\sqrt{2}) = (3-2\sqrt{2})/\sqrt{2},$$

$$\tanh \phi = (\sqrt{2} - 1)/\sqrt[4]{2} = 2^{1/4} - 2^{-1/4}.$$

Hence, $\tanh \delta = (2^{1/4} - 2^{-1/4})/2$ and

$$\omega = \Pi(\delta) = \operatorname{arc\,cos} (2^{1/4} - 2^{-1/4})/2 = \operatorname{arc\,cos} 0{\cdot}17417 = 79°58'.$$

Escher's integrity is revealed in the fact that he drew this angle correctly even though he apparently believed that it 'ought' to be 90°.

## VII. A QUASI-REGULAR TESSELLATION OF TRIANGLES AND QUADRANGLES

Escher's points of type $R$ could, of course, have been joined in pairs by *straight* line-segments in the sense of hyperbolic geometry, that is, by arcs of circles *orthogonal* to the peripheral circle. The result, lacking the 'traffic flow' which he admired in '*Circle Limit III*', is a quasi-regular tessellation of triangles and quadrangles such that every edge belongs to one triangle and one quadrangle. Having just *half* the vertices of $\{8, 3\}$, it is $h\{8, 3\}$ in the notation of [12]. Since its vertices are the images of $R$ under the group generated by reflections in the sides of the triangle $PQR$, it is $3 \mid 3\,4$ in the notation of [13]. We shall soon see that the angles of the triangles and quadrangles are not 60° but 48° 41′ and 71° 19′, respectively.

According to equation (3), the circum-radius $\chi$ of an octagonal face of $\{8, 3\}$ is given by

$$\cosh \chi = (\sqrt{2}+1)/\sqrt{3}.$$

A quadrangular face of $h\{8, 3\}$ is a 4-gon having this same circum-radium $\chi$; therefore, by equation (3) with 4 in place of 8, its angle $2\theta_4$ is given by

$$(\sqrt{2}+1)/\sqrt{3} = \cot(\pi/4) \cot \theta_4 = \cot \theta_4,$$

$$2\theta_4 = 2 \operatorname{arc\,tan} (\sqrt{6} - \sqrt{3}) = 71° \, 19'.$$

Since each vertex of $h\{8, 3\}$ is surrounded by 3 triangles and 3 quadrangles, the angle $2\theta_3$ of a triangle is simply

$$2\theta_3 = 120° - 2\theta_4 = 120° - 71° \, 19' = 48° \, 41'.$$

This quasi-regular tessellation of triangles (of angle 48° 41′) and quadrangles (of angle 71° 19′) has for its vertices all Escher's points of type $R$. There is, of course, a congruent $h\{8, 3\}$ whose vertices, being the remaining half of the vertices of $\{8, 3\}$, are all the points of type $Q$. In each case Escher shows only the vertices, not the edges; therefore the angles $2\theta_3$ and $2\theta_4$ cannot be directly measured in his picture.

**Editor's Note**—See Note by J. C. Rush in this issue [*Leonardo* **12**, 48 (1979)].

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