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The Equivalence of Ergodicity and Weak Mixing for Infinitely Divisible Processes[1]

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The equivalence of ergodicity and weak mixing for general infinitely divisible processes is proven. Using this result and [9], simple conditions for ergodicity of infinitely divisible processes are derived. The notion of codifference for infinitely divisible processes is investigated, it plays the crucial role in the proofs but it may be also of independent interest.

KEY WORDS: Stationary process; ergodicity; weak mixing; infinitely divisible process.

1. INTRODUCTION

Let $\mathbf{T} = \mathbf{R}$ or $\mathbf{Z}$. A real-valued stationary stochastic process $(X_t)_{t \in \mathbf{T}}$ is said to be ergodic if

$$T^{-1} \int_0^T P(A \cap S^t B)\, dt \xrightarrow[T \to \infty]{} P(A)\, P(B) \tag{1.1}$$

weakly mixing if

$$T^{-1} \int_0^T |P(A \cap S^t B) - P(A)\, P(B)|\, dt \xrightarrow[T \to \infty]{} 0 \tag{1.2}$$

mixing if

$$P(A \cap S^t B) \xrightarrow[t \to \infty]{} P(A)\, P(B) \tag{1.3}$$

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where $(S^t)_{t \in \mathbf{T}}$ is the group of shift transformations induced by $(X_t)_{t \in \mathbf{T}}$ and $A, B \in \mathscr{F}_X$ (the σ-field generated by the process). In the case $\mathbf{T} = \mathbf{Z}$, the integrals in Eqs. (1.1) and (1.2) are replaced by sums $\sum_{t=0}^{T-1}$.

It is obvious that Eqs. (1.3) $\Rightarrow$ (1.2) $\Rightarrow$ (1.1). In ergodic theory there are examples of flows $(S^t)_{t \in \mathbf{T}}$ which are weakly mixing but not mixing and ergodic but not weakly mixing (see [2, 7]). Usually, the weak mixing property is much closer to mixing than to ergodicity (see Proposition 1). However, in the case of stationary Gaussian processes, weak mixing and ergodicity coincide (see [2]). Podgórski [8] has shown that the same is true for symmetric stable processes; this result was later extended to symmetric semistable processes by Kokoszka and Podgórski [4]. In a recent paper Cambanis *et al.* [1] have proven the equivalence of ergodicity and weak mixing for all stationary symmetric infinitely divisible processes. Since it is usually much easier to characterize weak mixing than ergodicity, these results are important in obtaining definitive conditions for ergodicity. Cambanis *et al.* [1] characterized ergodic properties of stationary symmetric infinitely divisible processes in terms of the so-called dynamical functional.

In this paper we prove the equivalence of weak mixing and ergodicity for general (non-necessarily symmetric) stationary infinitely divisible processes (Theorem 1). We investigate the codifference function for infinitely divisible processes (Section 2) and use it to characterize ergodic properties of stationary processes (Proposition 4 and Theorem 2). Since the codifference depends only on two-dimensional marginal distributions of the process (or even less, see Eq. (2.1)), as opposed to a dynamical functional which requires all finite-dimensional distributions, our conditions for ergodicity and mixing seem to be easily verifiable. Finally, we would like to mention that the method of proof of Theorem 1 does not rely on the usual symmetrization technique (which does not seem to be applicable here, see Remark 4) but rather on a harmonic analysis of the codifference and its components.

For the sake of completeness, Eqs. (1.2) and (1.3) are not equivalent for stationary infinitely divisible processes; an appropriate example was constructed by Gross and Robertson [3].

1.1. Notation and Basic Facts

A random vector X is said to be *infinitely divisible* (i.d.) if for every positive integer n, X has the same distribution as the sum of n identically distributed and independent random vectors.

A stochastic process is i.d. if all its finite-dimensional marginal distributions are i.d.

The following lemma, due to Koopman and von Neumann [5], characterizes Cesaro convergence for positive, bounded sequences and functions. For a proof the reader may consult, e.g., [7]. Recall that the *density* of a subset D of a positive half-line is equal to $\lim_{C\to\infty}(|D\cap[0,C]|/C)$, if the limit exists, where $|\cdot|$ denotes Lebesgue measure. In the case when D is a subset of positive integers we define its density as $\lim_{n\to\infty}(|D\cap\{1,..., n\}|/n)$.

Lemma 1. Let $f\colon \mathbf{R}^+\to\mathbf{R}$ [$f\colon \mathbf{N}\to\mathbf{R}$, respectively] be nonnegative and bounded. A necessary and sufficient condition for

$$\lim_{T\to\infty} T^{-1}\int_0^T f(t)\,dt=0 \qquad \left[\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(n)=0\right]$$

is that there exists a subset D of density one in $\mathbf{R}^+$ [in $\mathbf{N}$] such that

$$\lim_{t\to\infty,\, t\in D} f(t)=0$$

Applying Lemma 1 to Eq. (1.2) we get the following well-known characterization of weak mixing.

Proposition 1. A stochastic process is weakly mixing if and only if for any $A, B\in\mathscr{F}_X$ there exists D, a subset of the density one in $\mathbf{T}$, such that

$$\lim_{t\to\infty,\, t\in D} P(A\cap S^tB)=P(A)\,P(B)$$

We will need the following fact (see Ref. 6, Thm. 3.2.3).

Lemma 2. Let μ be a finite measure on $\mathbf{R}$ [$[0, 2\pi)$, respectively]. Then

$$\lim_{T\to\infty} T^{-1}\int_0^T \hat{\mu}(t)\,dt=\mu(\{0\}) \qquad \left[\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\hat{\mu}(n)=\mu(\{0\})\right]$$

where $\hat{\mu}$ denotes the Fourier transform of μ.

We will also use the following lemma.

Lemma 3. Let $(X_t)_{t\in\mathbf{T}}$ be a stationary i.d. process and Q_{0t} be the Lévy measure of $\mathscr{L}(X_0, X_t)$. Then, for every $\delta>0$, the family of finite measures $(Q_{0t}|_{K_\delta^c})_{t\in\mathbf{T}}$ is weakly relatively compact and

$$\lim_{\delta\to 0}\sup_{t\in\mathbf{T}}\int_{K_\delta}|xy|\,Q_{0t}(dx, dy)=0, \tag{1.4}$$

where $K_\delta=\{(x, y): x^2+y^2\leqslant\delta^2\}$.

Proof. To prove the first part of the lemma it is enough to show that the family $(\mathscr{L}(X_0, X_t))_{t \in \mathbf{T}}$ is weakly relatively compact. Indeed, by stationarity, for every $a > 0$ and $t \in \mathbf{T}$, we have

$$\begin{aligned} P((X_0, X_t) \notin K_a) &\leqslant P(|X_0| > 2^{-1/2}a) + P(|X_t| > 2^{-1/2}a) \\ &= 2P(|X_0| > 2^{-1/2}a) \end{aligned}$$

which implies

$$\lim_{a \to \infty} \sup_{t \in \mathbf{T}} P((X_0, X_t) \notin K_a) = 0$$

To prove the second part, again using stationarity, we get

$$\begin{aligned} \int_{K_\delta} |xy| \, Q_{0t}(dx, dy) &\leqslant 2^{-1} \int_{\{|x| \leqslant \delta\}} x^2 Q_{0t}(dx, dy) + 2^{-1} \int_{\{|y| \leqslant \delta\}} y^2 Q_{0t}(dx, dy) \\ &= \int_{\{|x| \leqslant \delta\}} x^2 Q_0(dx) < \varepsilon \end{aligned}$$

for any positive ε, if only δ is small enough. This yields Eq. (1.4) and completes the proof. □

2. CODIFFERENCE

In the past several ideas of generalizing Gaussian covariance have been considered. In the case of symmetric α-stable processes, the following two notions were studied: the *covariation* $[X_1, X_2]$, which requires $\alpha > 1$, and the *codifference* $\tau(X_1, X_2)$, which is well-defined for all α (see Ref. 10, Sections 2.7 and 2.10, respectively, and references therein). When $\alpha < 2$, the covariation is linear with respect to the first variable (but not with respect to the second one), the codifference is not linear with respect to any of variables but is nonnegative definite instead. Both notions coincide with the covariance when $\alpha = 2$ (the Gaussian case) but neither one of them describes the distribution of a stable non-Gaussian process as the covariance does for a Gaussian process.

In this paper we study the codifference for arbitrary i.d. process which, as we show, can be used to measure independence. Since the codifference is always nonnegative definite (Proposition 2 next), we may apply some methods of harmonic analysis in the investigation of i.d. processes.

We define the *codifference* $\tau(X_1, X_2)$ of jointly i.d. real random variables X_1 and X_2 as follows

$$\tau(X_1, X_2) := \log Ee^{i(X_1 - X_2)} - \log Ee^{iX_1} - \log Ee^{-iX_2} \tag{2.1}$$

where log denotes the continuous branch of the logarithmic function which takes on value 0 at 1. We will now give more explicit form of $\tau(X_1, X_2)$. Namely, if $\mathbf{X} = (X_1, X_2)$ has two-dimensional i.d. distribution then, by the Lévy-Khintchine formula, there exist $\mathbf{a} \in \mathbf{R}^2$, a covariance matrix $\Sigma = [\sigma_{ij}]_{i,j=1,2}$, and a Lévy measure Q in $\mathbf{R}^2$ such that for every $\theta \in \mathbf{R}^2$,

$$Ee^{i\langle \theta, \mathbf{X}\rangle} = \exp\left[i\langle \theta, \mathbf{a}\rangle - \tfrac{1}{2}\langle \Sigma\theta, \theta\rangle + \int (e^{i\langle \theta, \mathbf{x}\rangle} - 1 - i\langle \theta, k(\mathbf{x})\rangle)\, Q(d\mathbf{x})\right] \tag{2.2}$$

where

$$k(\mathbf{x})_i = \begin{cases} x_i & \text{if } |x_i| \leqslant 1 \\ 1 & \text{if } x_i > 1 \\ -1 & \text{if } x_i < -1 \end{cases}$$

Substituting $(1, -1)$, $(1, 0)$ and $(0, -1)$ for θ in Eq. (2.2) we get, respectively,

$$\log Ee^{i(X_1 - X_2)} = i(a_1 - a_2) - \tfrac{1}{2}[\sigma_{11} + \sigma_{22} - 2\sigma_{12}] + \int_{\mathbf{R}^2} [e^{i(x_1 - x_2)} - 1 - i(k(\mathbf{x})_1 - k(\mathbf{x})_2)]\, Q(d\mathbf{x})$$

$$\log Ee^{iX_1} = ia_1 - \tfrac{1}{2}\sigma_{11} + \int_{\mathbf{R}^2} [e^{ix_1} - 1 - ik(\mathbf{x})_1]\, Q(d\mathbf{x})$$

and

$$\log Ee^{-iX_2} = -ia_2 - \tfrac{1}{2}\sigma_{22} + \int_{\mathbf{R}^2} [e^{-ix_2} - 1 + ik(\mathbf{x})_2]\, Q(d\mathbf{x})$$

Substituting these formulas into Eq. (2.1) we obtain

$$\tau(X_1, X_2) = \sigma_{12} + \int_{\mathbf{R}^2} (e^{ix_1} - 1)\overline{(e^{ix_2} - 1)}\, Q(d\mathbf{x}) \tag{2.3}$$

Proposition 2. Let $(X_t)_{t \in \mathbf{T}}$ be an arbitrary i.d. process. Then the function

$$\mathbf{T} \times \mathbf{T} \ni (s, t) \to \tau(X_s, X_t) \in \mathbf{C}$$

is nonnegative definite.

By Eq. (2.3),

$$\tau(X_s, X_t) = \sigma_{st} + \int_{\mathbf{R}^2} (e^{ix} - 1)\overline{(e^{iy} - 1)}\, Q_{st}(dx, dy)$$

Since $(s, t) \to \sigma_{st}$ is nonnegative definite, it is enough to show that the second term has this property. The latter is a consequence of the uniqueness of Lévy measures and follows from the following more general lemma.

Lemma 4. Let $(X_1,..., X_n)$ be an i.d. random vector, Q_{ij} be the Lévy measure of $\mathscr{L}(X_i, X_j)$, and let $g\colon \mathbf{R} \to \mathbf{C}$ be such that $|g(x)| \leqslant C \min(|x|, 1)$. If $a_{ij} = \int_{\mathbf{R}^2} g(x)\, \overline{g(y)}\, Q_{ij}(dx, dy)$, then the matrix $[a_{ij}]_{i,j=1,...,n}$ is nonnegative definite.

Proof. Let Q be the Lévy measure of $\mathscr{L}(X_1,..., X_n)$. By the uniqueness of Lévy measures, which implies their consistency, we have

$$a_{ij} = \int_{\mathbf{R}^2} g(x)\, \overline{g(y)}\, Q_{ij}(dx, dy) = \int_{\mathbf{R}^n} g(x_i)\, \overline{g(x_j)}\, Q(dx_1,..., dx_n)$$

$1 \leqslant i, j \leqslant n$. Thus, for every $z_1,..., z_n \in \mathbf{C}$,

$$\begin{aligned}\sum_{i,j} z_i \bar{z}_j a_{ij} &= \sum_{i,j} z_i \bar{z}_j \int_{\mathbf{R}^n} g(x_i)\, \overline{g(x_j)}\, Q(dx_1,..., dx_n)\\ &= \int_{\mathbf{R}^n} \left| \sum_i g(x_i)\, z_i \right|^2 Q(dx_1,..., dx_n) \geqslant 0\end{aligned}$$

which ends the proof. □

The following result is an analogue of the fact that uncorrelated jointly normal random variables are independent.

Proposition 3. Let $(X_1,..., X_n)$ be an i.d. random vector. Assume that, for each i, $1 \leqslant i \leqslant n$, the Lévy measure of $\mathscr{L}(X_i)$ has no atoms in $2\pi\mathbf{Z}$. Then $X_1,..., X_n$ are independent if and only $\tau(X_i, \pm X_j) = 0$, for every $i \neq j$.

Proof. If X_i and X_j are independent then, of course, $Ee^{i(X_i \pm X_j)} = Ee^{iX_i} Ee^{\pm iX_j}$, implying $\tau(X_i, \pm X_j) = 0$. To prove the converse, observe that if $\tau(X_i, \pm X_j) = 0$, then, by Eq. (2.3)

$$\sigma_{ij} + \int_{\mathbf{R}^2} (e^{ix} - 1)\overline{(e^{iy} - 1)}\, Q_{ij}(dx, dy) = 0 \tag{2.4}$$

and

$$-\sigma_{ij} + \int_{\mathbf{R}^2} (e^{ix} - 1)(e^{iy} - 1)\, Q_{ij}(dx, dy) = 0$$

(we use the notation from Lemma 4). Summing these equalities side-by-side and taking the real part on the left-hand side we get

$$\int_{\mathbf{R}^2} (\cos x - 1)(\cos y - 1)\, Q_{ij}(dx, dy)$$

$$= \int_{\mathbf{R}^n} (\cos x_i - 1)(\cos x_j - 1)\, Q(dx_1, ..., dx_n) = 0 \tag{2.5}$$

We will show that Q is concentrated on the axes. Assume, to the contrary, that there is a $\mathbf{y} = (y_1, ..., y_n)$ belonging to the support of Q and such that $y_i y_j \neq 0$, for some $i \neq j$. Then, from Eq. (2.5), we infer that $y_i = 2\pi k$ or $y_j = 2\pi l$ for some nonzero integers k, l, and, either

$$Q(\{\mathbf{x} : x_i = 2\pi k\}) > 0$$

or

$$Q(\{\mathbf{x} : x_j = 2\pi l\}) > 0$$

Since the projection of Q onto the mth axis coincides with the Lévy measure of $\mathscr{L}(X_m)$ on every set not containing zero, $1 \leqslant m \leqslant n$, we infer that either the Lévy measure of $\mathscr{L}(X_i)$ or of $\mathscr{L}(X_j)$ has an atom in $2\pi\mathbf{Z}$. This contradicts our assumption and proves that Q is concentrated on the axes. Using this information and Eq. (2.4) we get $\sigma_{ij} = 0$, for $i \neq j$. It now follows from the Lévy-Khintchine form of the characteristic function that the coordinates of $(X_1, ..., X_n)$ are independent. □

Remark 1. The condition $\tau(X_1, X_2) = 0$ alone does not imply the independence of X_1 and X_2, except when (X_1, X_2) is Gaussian. Indeed, as it is stated in [Ref. 10, p. 104], one can find a symmetric α-stable random vector (X_1, X_2), for any $1 < \alpha < 2$, such that $\tau(X_1, X_2) = 0$ and X_1, X_2 are dependent (notice that Lévy measures of stable random variables are continuous, thus satisfy conditions of Proposition 3). We will give here another easily verifiable example. Consider an i.d. random vector (X_1, X_2) without Gaussian part and with the corresponding Lévy measure Q given by

$$Q = x\delta_{\{\pi/4,\, 7\pi/4\}} + y\delta_{\{7\pi/4,\, \pi/4\}} + z\delta_{\{\pi/4,\, \pi/2\}}$$

where x, y, z are positive solutions of the system of equations

$$(1-\sqrt{2})\,x+(1-\sqrt{2})\,y+z=0$$
$$(1-\sqrt{2})\,x+(\sqrt{2}-1)\,y+(1-\sqrt{2})\,z=0$$

Observe that, for example, $x=2-\sqrt{2}$, $y=\sqrt{2}$ and $z=2\sqrt{2}-2$ are such positive solutions. Using Eq. (2.3) it is easy to check that $\tau(X_1, X_2)=0$, but since Q is not concentrated on the axes, X_1, X_2 are dependent.

Remark 2. It is easy to generalize Proposition 3 to arbitrary infinitely divisible random vectors $(X_1,..., X_n)$. Indeed, let C_i denote the set of atoms of the Lévy measure of $\mathscr{L}(X_i)$ (C_i is countable and does not contain zero), and put $Z_i=\{2\pi k/c : k\in \mathbf{Z},\ c\in C_i\}$. Then, for each nonzero number a which does not belong to any of Z_i, $1\leqslant i\leqslant n$, the random vector $(aX_1,..., aX_n)$ satisfies the condition on Lévy measures of Proposition 3. Thus, using this proposition, $X_1,..., X_n$ are independent if and only if $\tau(aX_i, \pm aX_j)=0$ for every $i\neq j$.

We will now return to stationary i.d. processes. If $(X_t)_{t\in\mathbf{T}}$ is stationary, then

$$\tau(X_s, X_t)=\tau(X_0, X_{t-s})$$

hence the function

$$\tau(t):=\tau(X_0, X_t) \tag{2.6}$$

is nonnegative definite and

$$\tau(0)=-\log|Ee^{iX_0}|^2$$

By Bochner's theorem there exists a finite Borel measure ν on $\mathbf{R}$ (or on $[0, 2\pi)$ if $\mathbf{T}=\mathbf{Z}$) such that

$$\tau(t)=\int_{\mathbf{R}} e^{it\lambda}\nu(d\lambda) \tag{2.7}$$

Thus

$$\begin{aligned} Ee^{i(X_0-X_t)}\,|Ee^{iX_0}|^{-2} &= e^{\tau(t)} \\ &= e^{\hat{\nu}(t)}=\widehat{\exp(\nu)}(t) \end{aligned} \tag{2.8}$$

where $\exp(\nu)=\sum_{n=0}^{\infty}(\nu^{*n}/n!)$, $\nu^{*0}=\delta_0$.

In Ref. 9 a simple condition for mixing of i.d. processes is given. Namely, if $(X_t)_{t\in\mathbf{T}}$ is a stationary i.d. process such that Q_0, the Lévy measure of X_0, has no atoms in the set $2\pi\mathbf{Z}$, then this process is mixing if and only if $Ee^{i(X_0-X_t)}\,|Ee^{iX_0}|^{-2}\to 1$, as $t\to\infty$. In terms of the codifference we have the following.

Proposition 4. Let $(X_t)_{t\in\mathbf{T}}$ be a stationary i.d. process such that Q_0, the Lévy measure of X_0, has no atoms in the set $2\pi\mathbf{Z}$. Then the process is mixing if and only if $\tau(t)\to 0$, as $t\to\infty$; it is weakly mixing if and only if there exists a set D of the density one in $\mathbf{T}$ such that $\tau(t)\to 0$, as $t\to\infty$, $t\in D$.

Remark 3. It follows from Ref. 9 (see Corollary 3 and the comment following Theorem 3) that $(X_t)_{t\in\mathbf{T}}$ is weakly mixing if and only if there exists a set D of the density one in $\mathbf{T}$ such that

$$\lim_{t\to\infty,\, t\in D}\sigma_{0t}=0 \qquad \text{and} \qquad \lim_{t\to\infty,\, t\in D}\int_{\mathbf{R}^2}\min(|xy|,1)\,Q_{0t}(dx,dy)=0$$

3. ERGODICITY AND WEAK MIXING

The following is the main result of this paper.

Theorem 1. Let $(X_t)_{t\in\mathbf{T}}$ be a measurable, stationary i.d. process. Then the process is ergodic if and only if it is weakly mixing.

Proof. Any weakly mixing process is ergodic. We will show the converse for i.d. processes. Let $(X_t)_{t\in\mathbf{T}}$ be i.d. stationary and ergodic and let

$$\tau(t)=\sigma_{0t}+\int_{\mathbf{R}^2}(e^{ix}-1)\overline{(e^{iy}-1)}\,Q_{0t}(dx,dy) \tag{3.1}$$

be its codifference. In view of Eq. (2.7), $\tau=\hat{\nu}$, and since both terms on the right-hand side of Eq. (3.1) are nonnegative definite, there exist finite Borel measures ν_G and ν_P on $\mathbf{R}$ [on $[0,2\pi)$ if $\mathbf{T}=\mathbf{Z}$] such that

$$\tau(t)=\widehat{\nu_G}(t)+\widehat{\nu_P}(t) \tag{3.2}$$

Thus $\nu=\nu_G+\nu_P$. The ergodicity of the process implies

$$T^{-1}\int_0^T Ee^{i(X_0-X_t)}\,dt \xrightarrow[T\to\infty]{} |Ee^{iX_0}|^2 \tag{3.3}$$

(see, e.g., [2, p. 19]). Hence, by Eq. (3.3) and Lemma 2,

$$T^{-1} \int_0^T Ee^{i(X_0 - X_t)} |Ee^{iX_0}|^{-2} \, dt$$

$$= T^{-1} \int_0^T \widehat{\exp(\nu)}(t) \, dt \xrightarrow[T \to \infty]{} \exp(\nu_G + \nu_P)(\{0\}) = 1$$

Thus

$$\nu_G(\{0\}) + \nu_P(\{0\}) = 0 \qquad \text{and} \qquad (\nu_G + \nu_P) * (\nu_G + \nu_P)(\{0\}) = 0 \quad (3.4)$$

From Eq. (3.4), $\nu_G * \nu_G(\{0\}) = 0$, therefore

$$T^{-1} \int_0^T \sigma_{0t}^2 \, dt = T^{-1} \int_0^T (\widehat{\nu_G}(t))^2 \, dt \xrightarrow[T \to \infty]{} \nu_G * \nu_G(\{0\}) = 0$$

Since σ_{0t} is real, by Lemma 1, there exists a set D of the density one in $\mathbf{T}$, such that

$$\lim_{t \to \infty,\, t \in D} \sigma_{0t} = 0 \tag{3.5}$$

Now, by Eqs. (3.1), (3.2), (3.4) and Lemma 2 we have

$$T^{-1} \int_0^T \int_{\mathbf{R}^2} (e^{ix} - 1)(e^{-iy} - 1) \, Q_{0t}(dx, dy) \, dt$$

$$= T^{-1} \int_0^T \widehat{\nu_P}(t) \, dt \xrightarrow[T \to \infty]{} \nu_P(\{0\}) = 0$$

Taking the real part of the first term here we get

$$T^{-1} \int_0^T \int_{\mathbf{R}^2} [(\cos x - 1)(\cos y - 1) + \sin x \sin y] \, Q_{0t}(dx, dy) \, dt \xrightarrow[T \to \infty]{} 0 \tag{3.6}$$

By stationarity and Lemma 4, the functions

$$t \to \int_{\mathbf{R}^2} (\cos x - 1)(\cos y - 1) \, Q_{0t}(dx, dy)$$

and

$$t \to \int_{\mathbf{R}^2} \sin x \sin y \, Q_{0t}(dx, dy)$$

are nonnegative definite. Therefore, there exist finite Borel measures λ_1 and λ_2 such that Eq. (3.6) can be rewritten as

$$T^{-1} \int_0^T \hat{\lambda}_1(t) \, dt + T^{-1} \int_0^T \hat{\lambda}_2(t) \, dt \xrightarrow[T \to \infty]{} 0$$

implying $\lambda_1(\{0\}) = \lambda_2(\{0\}) = 0$. Hence

$$T^{-1} \int_0^T \int_{\mathbf{R}^2} (\cos x - 1)(\cos y - 1) \, Q_{0t}(dx, dy) \, dt \xrightarrow[T \to \infty]{} \lambda_1(\{0\}) = 0 \quad (3.7)$$

Define $R_T(dx, dy) = T^{-1} \int_0^T Q_{0t}(dx, dy) \, dt$. Since, by Lemma 3, the family of finite measures $(Q_{0t}|_{K_\delta^c})_{t \in \mathbf{T}}$ is weakly relatively compact for every $\delta > 0$, the same is true for the family $(R_T|_{K_\delta^c})_{T > 0}$. By Eq. (3.7),

$$\int_{\mathbf{R}^2} (\cos x - 1)(\cos y - 1) \, R_T(dx, dy) \xrightarrow[T \to \infty]{} 0 \quad (3.8)$$

We will show that

$$\int_{\mathbf{R}^2} \min(|xy|, 1) \, R_T(dx, dy) \xrightarrow[T \to \infty]{} 0 \quad (3.9)$$

To this end, let $T_n \to \infty$, $T_n \in \mathbf{T}$. Using the diagonalization procedure we can find a subsequence (T'_n) of (T_n) and a measure R on $\mathbf{R}^2 \backslash \{0\}$ such that

$$R_{T'_n}|_{K_\delta^c} \Rightarrow R|_{K_\delta^c}, \qquad \text{as} \quad n \to \infty$$

for every $\delta > 0$. By Eq. (3.8) and the fact that $(\cos x - 1)(\cos y - 1) \geqslant 0$, we get

$$\int_{K_\delta^c} (\cos x - 1)(\cos y - 1) \, R(dx, dy) = 0$$

for every $\delta > 0$, implying that R is concentrated on the set of lines

$$\{(x, y) : x \in 2\pi\mathbf{Z} \text{ or } y \in 2\pi\mathbf{Z}\}$$

By the stationarity of the process, the projections of Q_{0t} onto the first and the second axis (excluding zero) are equal to Q_0, the Lévy measure of $\mathscr{L}(X_0)$; the same is true for R_T and so for R.

Let us assume for a moment that Q_0 has no atoms in the set $2\pi\mathbf{Z}$. Then R must be concentrated on the axes in $\mathbf{R}^2$. Hence, for every $\delta > 0$ such that $R(\{(x, y) : x^2 + y^2 = \delta^2\}) = 0$,

$$\limsup_{n \to \infty} \int_{\mathbf{R}^2} \min(|xy|, 1)\, R_{T'_n}(dx, dy)$$

$$\leqslant \limsup_{n \to \infty} \int_{K_\delta^c} \min(|xy|, 1)\, R_{T'_n}(dx, dy)$$

$$+ \sup_{T > 0} T^{-1} \int_0^T \int_{K_\delta} |xy|\, Q_{0t}(dx, dy)\, dt$$

Since the last quantity can be made arbitrarily small by Eq. (1.4) of Lemma 3, Eq. (3.9) follows. From Eq. (3.9) and Lemma 1 we infer that there exists a set D of density one in $\mathbf{T}$ such that

$$\lim_{t \to \infty,\, t \in D} \int_{\mathbf{R}^2} \min(|xy|, 1)\, Q_{0t}(dx, dy) = 0 \tag{3.10}$$

Equations (3.5) and (3.10) imply weak mixing by Remark 3.

Now we will remove the assumption that Q_0 has no atoms in $2\pi\mathbf{Z}$. Let $\{x_n\}$ be the sequence of atoms of Q_0. The set $Z = \{2\pi k/x_n : n \geqslant 1,\, k \in \mathbf{Z}\}$ is countable hence there exists a nonzero $a \in \mathbf{R}\backslash Z$. Consider the process $(aX_t)_{t \in \mathbf{T}}$ and let Q_0^a be the Lévy measure of aX_0. It is easy to verify that Q_0^a has no atoms in the set $2\pi\mathbf{Z}$; since the process $(aX_t)_{t \in \mathbf{T}}$ is also ergodic, it is weakly mixing by virtue of the first part of the proof. This implies that $(X_t)_{t \in \mathbf{T}}$ is weakly mixing as well. The proof is complete. □

Corollary 1. An infinitely divisible stationary process $(X_t)_{t \in \mathbf{T}}$ is ergodic if and only if its symmetrization is ergodic.

Proof. Clearly it is enough to prove the corollary only for the Poissonian part of $(X_t)_{t \in \mathbf{T}}$. Let $(Y_0, Y_t) = (X_0, X_t) - (X'_0, X'_t)$ be the symmetrization of (X_0, X_t). If Q_{0t} denotes the Lévy measure of $\mathscr{L}(X_0, X_t)$, then $\bar{Q}_{0t}$, the Lévy measure of $\mathscr{L}(Y_0, Y_t)$, is given by the formula: for every Borel set A in $\mathbf{R}^2$

$$\bar{Q}_{0t}(A) = Q_{0t}(A) + Q_{0t}(-A)$$

Now notice that the second condition of Remark 3 is satisfied by Q_{0t} if and only if it is satisfied by $\bar{Q}_{0t}$. Theorem 1 concludes the proof. □

Remark 4. One may wonder whether the usual symmetrization method could be used to prove Corollary 1 directly (and then one can deduce Theorem 1 from the result of Ref. 1, where the symmetric case was studied), but this method holds some major obstacles on the way. The most basic of these is, if $a_n \to a$ in Cesaro sense, then not necessarily $|a_n|^2 \to |a|^2$ in Cesaro sense. Another problem with this approach is that, for a measure preserving transformation S, the fact that S is ergodic does not imply that $S \times S$ is ergodic (if it were, S would be weakly mixing by [Ref. 7, Thm. 6.1] and there would be nothing to prove).

Theorem 2. Let $(X_t)_{t \in \mathbf{T}}$ be an i.d. processes such that the Lévy measure of X_0 has no atoms in $2\pi\mathbf{Z}$. Then $(X_t)_{t \in \mathbf{T}}$ is ergodic (weakly mixing) if and only if one of the following conditions holds:

(i) $\lim_{T \to \infty} T^{-1} \int_0^T E e^{i(X_0 - X_t)}\, dt = |E e^{iX_0}|^2$,

(ii) $\lim_{T \to \infty} T^{-1} \int_0^T |\tau(t)|\, dt = 0$,

(iii) $\lim_{T \to \infty} T^{-1} \int_0^T |E e^{i(X_0 - X_t)}|^2\, dt = |E e^{iX_0}|^4$,

(iv) $\lim_{T \to \infty} T^{-1} \int_0^T |\tau(t) + \tau(-t)|\, dt = 0$,

there exists a set D of density one in $\mathbf{T}$ such that

(v) $\lim_{t \to \infty,\, t \in D} E e^{i(X_0 - X_t)} = |E e^{iX_0}|^2$,

(vi) $\lim_{t \to \infty,\, t \in D} \tau(t) = 0$,

(vii) $\lim_{t \to \infty,\, t \in D} (\tau(t) + \tau(-t)) = 0$,

(viii) $\lim_{t \to \infty,\, t \in D} \sigma_{0t} = 0$ and
$\lim_{t \to \infty,\, t \in D} \int_{\mathbf{R}^2} \min(|xy|, 1)\, Q_{0t}(dx, dy) = 0$.

Proof. Condition (i) is just Eq. (3.3). Thus (i) is equivalent to ergodicity of $(X_t)_{t \in \mathbf{T}}$ by the virtue of the proof of Theorem 1. Since τ is bounded, (ii) is equivalent to (vi) by Lemma 1. We have just shown that (i) is equivalent to ergodicity of $(X_t)_{t \in \mathbf{T}}$, therefore (iii) is equivalent to ergodicity of its symmetrization. However, by Corollary 1, ergodicity of the original process and of its symmetrization are equivalent. (iv) is a version of (ii) for the symmetrization of the process and is equivalent to (vii).

The remaining equivalences are easy consequences of Proposition 4, Remark 3, Theorem 1, and Corollary 1. □

Remark 5. As we have already mentioned it in the proof of Theorem 1, if the Lévy measure of $\mathscr{L}(X_0)$ has atoms in $2\pi\mathbf{Z}$, then there is a nonzero number a such that the Lévy measure of $\mathscr{L}(aX_0)$ has no atoms in $2\pi\mathbf{Z}$. But the ergodic properties of the rescaled process and of the original one are the same. This simple remark makes Theorem 2 applicable for arbitrary i.d. processes.

Remark 6. Condition (viii) of Theorem 2 does not require any assumptions on the Lévy measure (see Remark 3).

Remark 7. If $(X_t)_{t\in\mathbf{T}}$ is given by the stochastic integral, $(X_t)_{t\in\mathbf{T}} =^{d} (\int_S f_t(s)\, \Lambda(ds))_{t\in\mathbf{T}}$, then Theorems 4 and 5 of Ref. 9 provide simple conditions for mixing and weak mixing in terms of the family $\{f_t\}_{t\in\mathbf{T}}$. This is probably the most convenient case of an i.d. process in which ergodic properties of the process can be easily verified.

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