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On Kelvin Transformation

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We prove that in the Euclidean space of arbitrary dimension the inversion of the isotropic stable Lévy process killed at the origin is, after an appropriate change of time, the same stable process conditioned in the sense of Doob by the Riesz kernel. Using this identification we derive and explain transformation rules for the Kelvin transform acting on the Green function and the Poisson kernel of the stable process and on solutions of Schrödinger equation based on the fractional Laplacian. The Brownian motion and the classical Laplacian are included as a special case.

KEY WORDS: Inversion; Kelvin transform; isotropic stable Lévy process; Brownian motion; Doob conditional process, Riesz kernel; Green function; Schrödinger equation; Laplace transform; resolvent.

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1. INTRODUCTION

In what follows d is an arbitrary natural number and $|\cdot|$ denotes the Euclidean norm in $\mathbf{R}^d$. Inversion with respect to the unit sphere is the transformation T of $\mathbf{R}^d_0 = \mathbf{R}^d \setminus \{0\}$ defined as

$$Tx = \frac{x}{|x|^2}.$$

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We denote by $\{X_t, \mathbf{P}_x\}$ the *standard*[6] rotation invariant (isotropic) α-stable Lévy process in $\mathbf{R}^d$ (i.e. with homogeneous independent increments) with exponent $\alpha \in (0, 2]$ and characteristic function

$$\mathbf{E}_x e^{i\xi\cdot(X_t-x)} = e^{-t|\xi|^\alpha}, \quad x, \xi \in \mathbf{R}^d, \ t \geqslant 0. \tag{1}$$

Here $\mathbf{E}_x$ is the expectation for the process starting from x and $\cdot$ denotes the usual inner product. Let $\{X_t^0\}$ be our stable process killed upon hitting the origin. The process is regarded as a Markov process on $\mathbf{R}_0^d$. Our main result is the following theorem.

Theorem 1. The process $\left\{TX^0_{\theta(t)}, t \geqslant 0\right\}$ has the same law as the process $\{X_t^0, t \geqslant 0\}$ conditioned in the sense of Doob by the function $h(x) = |x|^{\alpha-d}$. Here θ is the inverse function of the additive functional $A(t) = \int_0^t |X_s^0|^{-2\alpha} ds$.

In the case of $\alpha = 2 < d$ the result is given in Ref. 37 Ch. VIII (3.17) along with further references (for $\alpha = 2 = d$ we refer to, e.g. Ref. 44 Remarque 1). One of the consequences of Theorem 1 is the following transformation rule for the Green function G_D of $\{X_t\}$ for all $\alpha \in (0, 2]$.

Theorem 2. For every open set $D \subset \mathbf{R}_0^d$

$$G_{TD}(x, y) = |x|^{\alpha-d}|y|^{\alpha-d} G_D(Tx, Ty), \quad x, y \in \mathbf{R}_0^d. \tag{2}$$

Here $TD = \{Tx : x \in D\}$. In the case of the ball, (2) was obtained for $\alpha < 2$ in Ref. 13 (Section 8) by using the explicit form of the Poisson kernel of the ball for our processes[7,38]; see also Ref. 1 and 13 for the classical case $\alpha = 2$. For general D the statement is apparently new even for $\alpha = 2$. However, we should note that there exist well-known analytic techniques based on a characterization of the Green function, which may be used to prove the theorem. We refer the reader to, e.g., Ref. 36. Section 4.4 for $d = \alpha = 2$ and Ref. 18, Corollary to Theorem 2.6 for $\alpha = 2$ and $d = 1, 2, \ldots$ (see also below).

Our interest in Kelvin transform comes primarily from the fact that it reduces potential-theoretic problems pertaining to the point at infinity (and unbounded domains) to those at the origin (and bounded domains), see, e.g., Ref. 1 or 25 for the classical case $\alpha = 2$. This advantage is especially appealing in considerations related to the boundary Harnack principle and asymptotics of harmonic functions at the boundary of domains in $\mathbf{R}^d$, see Refs. 2, 9 and 11 for the case of $\alpha < 2$ and Ref. 13 for $\alpha \leqslant 2$. The reader interested in other recent developments in the potential theory of the Riesz kernel and our isotropic stable processes for $\alpha < 2$ may consult Refs. 15, 17, 29, 35 and the references given there.

We like to note that the case of general D and $0 < \alpha \leqslant 2$ in Theorem 2 can be also studied within the framework of analytic potential theory by means of a transformation formula for Riesz potentials (80), an approach used before to calculate the Poisson kernel of the ball Refs. 38, 31, (Appendix), 7 and 5, pp. 191–207. We refer the reader to Ref. 10 for details (see also Section 5 below). We note, however, that[(10)] exhibited certain difficulties of the *recurrent case* $\alpha \geqslant d$ when $0 \in \partial D$. These difficulties are not only of technical character as they correspond to certain peculiarities in formulas for the Green function and harmonic measure of unbounded domains on the real line.[(7,35)]

The present paper develops the formalism of the Kelvin transform including the recurrent case. Our study shows that, except for the inversion, conditioning and a time change, it is the killing of the underlying isotropic stable process at the origin that allows for a unified treatment of the recurrent case and the *transient* case $\alpha < d$. The properties of the Kelvin transform are derived here in a broader perspective by lifting the calculations from the level of the Riesz potential kernel (the approach of Ref. 10) to the level of the related operator semigroup and the stochastic process. As we mentioned, our results apply in particular to the Brownian motion, Newtonian kernel and Laplace operator, which correspond to the case $\alpha = 2$.

For clarity we note that if $\alpha \leqslant d$ then the origin is a polar set for our process and killing it at the origin has no effect on the process and its Green function other than regarding $\mathbf{R}_0^d$ as a new state space.

We employ standard theory of resolvents (of semigroup) to prove Theorem 1 rather than use the corresponding infinitesimal generators.[(5,7,23,31,32)] Our experience with using the infinitesimal generators for identification of the processes in Theorem 1 indicates essential difficulties in dealing with their domains in the case $\alpha \geqslant 1$.

The paper is organized as follows. In Section 2 we give preliminaries on the Kelvin transform, the isotropic stable Lévy processes, the corresponding harmonic functions and Doob's conditioning. In Section 3 we prove Theorem 1 in the easy transient case. In Section 4 and 5 we give a proof for $\alpha > d = 1$ and $\alpha = d$, respectively. The reader interested only in the prevailing transient case may skip these sections because Section 3 exhibits all the basic algebra. In Section 6 we conclude the proof of Theorem 1 and we prove Theorem 2 along with the well-known preservation property for harmonic functions. In Section 7 we give several new and known transformation rules for the Kelvin transform acting on

Green potentials, harmonic measure and solutions of Schrödinger equation related to our process. At the end of Section 7 we test our formalism against certain well-known but striking phenomena of the point recurrent potential theory ($\alpha > d = 1$). In Section 8, we give some general comments and additional references.

Below we will only consider Borel measures and functions and only integrals which are well defined as Lebesgue integrals, e.g. *absolutely convergent or nonnegative*. In inequalities, c denotes a generic multiplicative constant i.e. a positive real number whose value does not depend on the variables in other factors of a given product. For example see (19).

2. PRELIMINARIES

The following fundamental property of inversion can be verified by using the inner product:

$$|Tx - Ty| = \frac{|x-y|}{|x||y|}, \quad x, y \in \mathbf{R}_0^d. \tag{3}$$

Note that we adopt no convention on the values of T at the origin and infinity.

Let $B = B(Q, r) = \{x \in \mathbf{R}^d : |x - Q| < r\}$, where $Q \in \mathbf{R}_0^d$ and $0 < r < |Q|$. (3) yields that $TB = B(S, \rho)$, where

$$S = \frac{Q}{|Q|^2 - r^2} \quad \text{and} \quad \rho = \frac{r}{|Q|^2 - r^2}. \tag{4}$$

One consequence of (4) is that the Jacobian of T satisfies

$$|\det(JT(y))| = |y|^{-2d}. \tag{5}$$

In what follows $0 < \alpha \leqslant 2$. Let u be a function on $\mathbf{R}_0^d$. The *Kelvin transform* of u is the function $\mathcal{K}u$ on $\mathbf{R}_0^d$ defined by

$$\mathcal{K}u(y) = |y|^{\alpha-d} u(Ty) = |y|^{\alpha-d} u(y/|y|^2). \tag{6}$$

Note that in (6) and below we often drop α and d from our notation. The choice of $\alpha = 2$ yields the classical Kelvin transform, of which (6) is a generalization. Of course, $T \circ T$ and $\mathcal{K} \circ \mathcal{K}$ are identity operators on their respective domains of definition. It will be convenient to define the *dual Kelvin transform* $\tilde{\mathcal{K}}$ as adjoint operation, acting on measures ν on $\mathbf{R}_0^d$:

$$\int_{\mathbf{R}_0^d} f(y) \tilde{\mathcal{K}}\nu(dy) = \int_{\mathbf{R}_0^d} \mathcal{K}f(y)\nu(dy) = \int_{\mathbf{R}_0^d} |y|^{\alpha-d} f(Ty)\nu(dy). \tag{7}$$

If $\nu(dx) = g(x)dx$, then a change of variable and (5) yield

$$\tilde{\mathcal{K}}\nu(dy) = |y|^{-2\alpha}\mathcal{K}g(y)\,dy. \tag{8}$$

Generally, we can express $\tilde{\mathcal{K}}\nu$ in terms of $\mu \circ T^{-1}$, where $\mu \circ T^{-1}(dy) = \mu(T^{-1}(dy))$ for a measure μ on $\mathbf{R}_0^d$. Indeed, since

$$\int f(y)\nu \circ T^{-1}(dy) = \int f(Ty)\nu(dy), \tag{9}$$

therefore

$$\tilde{\mathcal{K}}\nu = |y|^{d-\alpha}\left(\nu \circ T^{-1}\right) = \left(|y|^{\alpha-d}\nu\right) \circ T^{-1}. \tag{10}$$

As usual, for a function g and a measure μ, $g\mu$ is the measure defined by $\int f(y)(g\mu)(dy) = \int f(y)g(y)\mu(dy)$, in which way we understand (10).

The isotropic stable process $\{X_t\}$ determined by (1) is a well-known object.[6] For $0 < \alpha < 2$ it is a symmetric Lévy process in $\mathbf{R}^d$ with Lévy measure ν, zero shift and no Gaussian component.[41] Here $\nu(dy) = \mathcal{A}(d, -\alpha)|y|^{-d-\alpha}\,dy$ and $\mathcal{A}(d, \gamma) = \Gamma((d-\gamma)/2)/(2^\gamma \pi^{d/2}|\Gamma(\gamma/2)|)$, see also (17). For each $t > 0$, the P_0 distribution of X_t has the density function given by inverse Fourier transform and (1):

$$p_t(y) = (2\pi)^{-d}\int_{\mathbf{R}^d} \exp(-t|\xi|^\alpha)e^{-i\xi\cdot y}\,d\xi. \tag{11}$$

(11) implies the *scaling property*

$$p_t(y) = t^{-d/\alpha}p(t^{-1/\alpha}y), \quad t > 0,\ y \in \mathbf{R}^d, \tag{12}$$

and continuity (smoothness) of $p_t(y)$ in (t, y) for $t > 0$. The transition density of $\{X_t\}$ is

$$p(t; x, y) = p_t(y - x). \tag{13}$$

For $\alpha = 1$ we have $p_t(y) = c_d t(|y|^2 + t^2)^{-(d+1)/2}$, $t > 0$, i.e. the Cauchy semigroup on $\mathbf{R}^d$[8,42,45]. The asymptotics of p_t is known precisely, in particular

$$p_1(y) \leqslant c\left(1 \wedge |y|^{-d-\alpha}\right), \quad y \in \mathbf{R}^d, \tag{14}$$

where $\wedge$ denotes the minimum, see, e.g., Ref. 24.

We assume, as we may, that sample paths of X_t are right-continuous and have left limits a.s. As usual, $\mathbf{P}_x$ denotes the distribution of the process starting from x and $\mathbf{E}_x$ is the corresponding expectation. $\{X_t, \mathbf{P}_x\}$ is strong Markov with respect to the so-called standard filtration (in fact it is a Feller process), see Refs. 4, 40, and 41 for details. The scaling (12) lifted to the level of the process reads

$$\mathbf{P}_x\{X_t, t \geqslant 0\} = \mathbf{P}_{kx}\{k^{-1}X_{tk^{-\alpha}}, t \geqslant 0\}, \quad x \in \mathbf{R}^d, k > 0, \tag{15}$$

which identifies *distributions* of the two processes relative to measures $\mathbf{P}_x$ and $\mathbf{P}_{kx}$, respectively. For C^2 functions u with compact support the generator of the process is the fractional Laplacian(3,12,45):

$$\Delta^{\alpha/2}u(x) = \mathcal{A}(d, -\alpha) \int_{\mathbf{R}^d} \frac{u(x+y) - u(x) - \mathbf{1}_{|y|<1} \nabla u(x) \cdot y}{|y|^{d+\alpha}} \, dy, \quad 0 < \alpha < 2. \tag{16}$$

The limiting case of $\alpha = 2$ corresponds to the Brownian motion $\{B_{2t}\}$ with Laplacian $\Delta = \sum_{i=1}^d \partial_i^2$ as the generator. We have $\mathcal{F}(\Delta^{\alpha/2}u)(\xi) = -|\xi|^\alpha \mathcal{F}(u)(\xi)$ for such u. Here $\mathcal{F}$ is the Fourier transformation and $0 < \alpha \leqslant 2$. This and (16) follow from the Lévy-Khinchin formula,(3) see also Ref. 28.

For a number $\lambda \geqslant 0$ and a (Borel measurable) function f on $\mathbf{R}^d$ we define as usual the λ-resolvent operator:

$$U_\lambda f(x) = \mathbf{E}_x \int_0^\infty e^{-\lambda t} f(X_t) dt = \int_0^\infty \int_{\mathbf{R}^d} e^{-\lambda t} p(t; x, y) f(y) \, dy dt, \quad x \in \mathbf{R}^d$$

(provided the integrals are well defined, e.g. absolutely convergent or non-negative). It has

$$u_\lambda(y) = \int_0^\infty e^{-\lambda t} p_t(y) \, dt$$

as its kernel in the sense that

$$U_\lambda f(x) = \int_{\mathbf{R}^d} u_\lambda(y - x) f(y) \, dy.$$

The kernel of the 0-resolvent or the potential operator $U = U_0$ is the M. Riesz' kernel(5,6):

$$u(y) = \int_0^\infty p_t(y) \, dt = \mathcal{A}(d, \alpha)|y|^{\alpha-d}, \quad y \in \mathbf{R}^d, \quad \text{if } \alpha < d. \tag{17}$$

By (12) it is easy to verify that $u(y) \equiv \infty$ if $\alpha \geqslant d$. In this case it is appropriate to consider the *compensated kernels*(7,23,30,35):

$$K_\alpha(y) = \int_0^\infty [p_t(y) - p_t(x_0)] \, dt, \tag{18}$$

where $x_0 = 0$ for $\alpha > d = 1, x_0 = 1$ for $\alpha = d = 1$ and $x_0 = (1, 0)$ for $\alpha = d = 2$. By a direct computation we get

$$K_\alpha(y) = c_\alpha |y|^{\alpha-1},\ y \in \mathbf{R}, \qquad \text{if } 1 < \alpha \leqslant 2,$$
$$K_1(y) = \frac{1}{\pi} \ln \frac{1}{|y|},\ y \in \mathbf{R}, \qquad \text{if } \alpha = d = 1,$$

and

$$K_2(y) = \frac{1}{2\pi} \ln \frac{1}{|y|},\ y \in \mathbf{R}^2, \qquad \text{if } \alpha = d = 2,$$

compare Ref. 35. Here $c_\alpha = [2\Gamma(\alpha)\cos(\pi\alpha/2)]^{-1} < 0$. For $\lambda > 0$ and every $\alpha \in (0, 2)$ by (12) and (14) we have

$$u_\lambda(y) \leqslant c \int_0^\infty e^{-\lambda t} \left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|y|^{d+\alpha}} \right) dt = c \int_0^{|y|^\alpha} e^{-\lambda t} \frac{t}{|y|^{d+\alpha}}\, dt + c \int_{|y|^\alpha}^\infty e^{-\lambda t} t^{-\frac{d}{\alpha}}\, dt$$
$$\leqslant c\lambda^{-2}\Gamma(2)|y|^{-d-\alpha} + c\lambda^{-1}|y|^{-d} e^{-\lambda |y|^\alpha} \leqslant c|y|^{-d-\alpha}, \quad y \in \mathbf{R}^d. \tag{19}$$

Of course,

$$u_\lambda \leqslant u. \tag{20}$$

For (open) $B \subset \mathbf{R}^d$ we define $\tau_B = \inf\{t \geqslant 0;\ X_t \notin B\}$, the *first exit time* of B. We say that a function f on $\mathbf{R}^d$ is harmonic for our α-stable process in open set $D \subset \mathbf{R}^d$ (or α-*harmonic*) if

$$f(x) = \mathbf{E}_x[\tau_B < \infty;\ f(X_{\tau_B})], \quad x \in B, \tag{21}$$

for every open bounded set B with closure $\overline{B} \subset D$ (compare Ref. 43). We always assume that the expectation in (21) is absolutely convergent. If (21) holds for $B = D$ then f is called regular harmonic for $\{X_t, \mathbf{P}_x\}$ on D or regular α-*harmonic* on D. Regular α-harmonicity implies plain α-harmonicity because (21) is inherited by subsets of B. This follows easily from the strong Markov property. The relation between these two concepts of being harmonic is reminiscent of that between being a martingale and a closed martingale (see also Refs. 9 and 34). If B is bounded then (12) yields $\tau_B < \infty$ $\mathbf{P}_x$-almost surely for every x $(a.s.)$ and then $\mathbf{E}_x[\tau_B < \infty;\ f(X_{\tau_B})] = \mathbf{E}_x f(X_{\tau_B})$ in (21). Note also that the equality in (21) extends trivially to all $x \in \mathbf{R}^d$.

If $f \geqslant 0$ is α-harmonic in open bounded D and continuous on the closure $\overline{D}$ of D then f is regular α-harmonic on D. Indeed, let $x \in D, n \to \infty$. Consider open precompact D_n increasing to D. By quasi-left continuity of $\{X_t, \mathbf{P}_x\}$ (4,40) we have $X_{\tau_{D_n}} \to X_{\tau_D}$. Using Fatou's lemma and monotone convergence we conclude that $f(x) = \mathbf{E}_x f(X_{\tau_{D_n}}) \to \mathbf{E}_x f(X_{\tau_D})$.

The α-*harmonic measure*

$$\omega_D^x(A) = \mathbf{P}_x(X_{\tau_D} \in A), \quad x \in \mathbf{R}^d, \ A \subset \mathbf{R}^d,$$

reproduces α-harmonic functions. For example, f is *regular* α-harmonic on D if and only if

$$f(x) = \int_{\mathbf{R}^d} f(y)\omega_D^x(dy), \quad x \in D. \tag{22}$$

The values of f outside $\overline{D}$ are irrelevant in the theory of harmonic functions of the Brownian motion (i.e. 2-harmonic functions) because $\mathbf{R}^d \backslash \overline{D}$ is of zero harmonic measure ω_D^x if $x \in D$. This, however, is an exception and for general Markov processes (with discontinuous trajectories) including our α-stable processes a functions which is harmonic on D need to be defined also on D^c. It is well known that the functions harmonic for the Brownian motion in our sense are precisely those annihilating the Laplacian.[(43)] A similar characterization is valid for α-harmonic functions and $\Delta^{\alpha/2}$ (for every $\alpha \in (0, 2]$), see Ref. 12. We will not use the result in what follows since (21) is better as a definition.

By the strong Markov property, the Riesz potential Ug is regular α-harmonic on $\mathbf{R}^d \backslash \mathrm{supp}(g)$ provided it is absolutely convergent there. Let $g(y) = |B_\epsilon|^{-1}\mathbf{1}_{B_\epsilon}(y)$ where $\epsilon > 0$, $B_\epsilon = \{y \in \mathbf{R}^d : |y| < \epsilon\}$ and $|B_\epsilon|$ is the volume of B_ϵ. There is $c > 1$ such that $cu(x) \geqslant Ug_\epsilon(x) \to u(x)$ as $\epsilon \to 0$. By dominated convergence the function

$$h(x) = |x|^{\alpha - d}$$

is α-harmonic in $\mathbf{R}_0^d$. As a consequence of the first equality in (17) and the semigroup property:

$$p(t+s; x, y) = \int_{\mathbf{R}^d} p(t; x, z)p(s; z, y)\,dz,$$

h is excessive for the semigroup of the process in the usual sense, i.e. for every $x \in \mathbf{R}^d$

$$h(x) \geqslant \int_{\mathbf{R}^d} h(y)p(t; x, y)\,dy \tag{23}$$

and

$$\int_{\mathbf{R}^d} h(y)p(t; x, y)\,dy \uparrow h(x) \quad \text{as } t \downarrow 0. \tag{24}$$

For every $t \geqslant 0$ and $x \neq 0$ we have $\mathbf{P}_x(X_t = 0) = 0$. Let $p^T(t; x, y)$ denote the density function of the $\mathbf{P}_{T_x}$-distribution of TX_t on $\mathbf{R}_0^d$ for $t > 0$. By (5), rotational invariance of $p_t(y)$, (3) and (12) we have

$$p^T(t; x, y) = p_t(Ty - Tx)|y|^{-2d} = p_t\left(\frac{y-x}{|x||y|}\right)|y|^{-2d} \tag{25}$$

$$= |x|^d |y|^{-d} p_{t|x|^\alpha |y|^\alpha}(y-x). \tag{26}$$

Observe that $TX_0 = x$ under $\mathbf{P}_{T_x}$.

We note in passing that by Liouville's theorem[23] translations, dilations, unitary maps and the inversion generate (by composition) all conformal transformations of $\mathbf{R}^d$ $(d > 2)$. All of these were used to get (26) here.

Let $\sigma_0 = \inf\{t \geqslant 0 : X_t = 0\}$ and define

$$X_t^0 = \begin{cases} X_t & \text{if } t < \sigma_0, \\ \partial & \text{if } t \geqslant \sigma_0. \end{cases}$$

The state space for this *stable process killed at the origin* is $\mathbf{R}_0^d \cup \{\partial\}$. Here ∂ is an additional terminal state ("cemetery") with the usual convention that integration over $\{\partial\}$ returns zero for every numerical function. We also make the convention that $T\partial = \partial$ (this will play some role in Section 4). By strong Markov property the transition density of $\{X_t^0\}$ is

$$p^0(t; x, y) = p(t; x, y) - \mathbf{E}_x[p(t - \sigma_0; X_{\sigma_0}, y); \sigma_0 < t]$$

$$= p(t; x, y) - \mathbf{E}_x[p(t - \sigma_0; 0, y); \sigma_0 < t]. \tag{27}$$

Of course,

$$p^0(t; x, y) \leqslant p(t; x, y), \quad x, y \in \mathbf{R}_0^d, \ t > 0. \tag{28}$$

3. TRANSIENT CASE

In the transient case $\alpha < d$, which we consider in this section, we have that

$$\mathbf{P}_x\{X_t = 0 \text{ for some } t \geqslant 0\} = 0, \quad x \neq 0. \tag{29}$$

Equation (29) is an easy consequence of unboundedness of the Riesz kernel at the origin and the mean-value property (21) of $h(x) = |x|^{\alpha - d}$ on precompact subsets of $\mathbf{R}_0^d$ (see also Ref. 6). Indeed, let $0 < \epsilon < M$ and define $V_M = \{x \in \mathbf{R}^d : \epsilon < |x| < M\}$ and $V = \{x \in \mathbf{R}^d : |x| > \epsilon\}$. For every $x \in \mathbf{R}_0^d$ we have

$$h(x) = \mathbf{E}_x h(X_{\tau_{V_M}}) \geqslant \epsilon^{\alpha - d} \mathbf{P}_x\{|X_{\tau_{V_M}}| < \epsilon\}. \tag{30}$$

Thus $\mathbf{P}_x\{\tau_V<\infty\}\leqslant\epsilon^{d-\alpha}h(x)$ and so

$$\mathbf{P}_x\{\sigma_0<\infty\}=0, \quad x\in\mathbf{R}_0^d, \tag{31}$$

which is equivalent to (29).

It follows that $\mathbf{P}_x\{X_t^0=X_t$ for all $t\geqslant 0\}=1$ for every $x\in\mathbf{R}_0^d$. We will identify the process $\{X_t^0\}$ with $\{X_t,\mathbf{P}_x,x\in\mathbf{R}_0^d\}$ and $\{TX_t^0\}$ with $\{TX_t\}$. For the latter process we define the resolvent U_λ^T, potential operator U^T and their kernels u_λ^T and u^T in the usual way. By the observation following (26) and a change of variable we obtain

$$\begin{aligned} u^T(x,y) &= \int_0^\infty p^T(t;x,y)\,dt=|x|^d|y|^{-d}\int_0^\infty p_{t|x|^\alpha|y|^\alpha}(y-x)\,dt \\ &= |x|^{d-\alpha}|y|^{-d-\alpha}\int_0^\infty p_s(y-x)\,ds=\frac{|y|^{\alpha-d}}{|x|^{\alpha-d}}u(y-x)|y|^{-2\alpha}. \end{aligned} \tag{32}$$

For our excessive function $h(x)=|x|^{\alpha-d}$ we define after[22] time homogeneous subprobability transition densities on $\mathbf{R}_0^d$ (because the origin is a pole of h, see also Ref. 18)

$$p^h(t;x,y)=h(x)^{-1}p(t;x,y)h(y)=|x|^{d-\alpha}p_t(y-x)|y|^{\alpha-d}.$$

The resulting h-process will be denoted by $\{X_t^h\}$. It may be proved that $X_t^h\to 0$ at the (finite) lifetime of the process[12,18] but we have no need of the result in what follows. The resolvent kernel of the process is

$$u_\lambda^h(x,y)=\int_0^\infty e^{-\lambda t}\frac{|y|^{\alpha-d}}{|x|^{\alpha-d}}p(t;x,y)\,dt=\frac{|y|^{\alpha-d}}{|x|^{\alpha-d}}u_\lambda(y-x), \quad \lambda\geqslant 0. \tag{33}$$

In particular

$$u^T(x,y)=u^h(x,y)|y|^{-2\alpha}.$$

We let

$$A(t)=\int_0^t |X_s^0|^{-2\alpha}\,ds=\int_0^t |TX_s^0|^{2\alpha}\,ds, \quad 0\leqslant t\leqslant\infty. \tag{34}$$

$A(t)$ is finite, continuous and strictly increasing for $0\leqslant t<\infty$ for every trajectory because the trajectories of $\{X_t^0\}$ do not approach the origin in finite time. Clearly, $A'(t)=|X_t^0|^{-2\alpha}$, $t<\infty$. We define $\theta(t)=A^{-1}(t)=\inf\{s: A(s)>t\}$. It is a continuous strictly increasing function of $t\in[0,A(\infty))$ and $\theta(t)=\infty$ if $t\geqslant A(\infty)$ (see Lemma 3 below). We have $\theta'(t)=|X_{\theta(t)}|^{2\alpha}$ if $t<A(\infty)$. We define $Y_t=TX_{\theta(t)}^0$ if $t<A(\infty)$ and we let $Y_t=\partial$ if $A(\infty)\leqslant t<\infty$. It is a Markov (in fact strong Markov) process.[6,43] The resolvent of $\{Y_t\}$ will be denoted $U_\lambda^Y(\lambda\geqslant 0)$. $U^Y=U_0^Y$ will be the potential operator and u_λ^Y, u^Y – the respective integral kernels. Note that Y_t *starts* at $x\in\mathbf{R}_0^d$ when X_t starts at Tx, which explains for the presence of $\mathbf{E}_{Tx}$ in the formulas below (compare (25)). By a change of variable

$$\begin{aligned} U^Y f(x) &= \mathbf{E}_{Tx} \int_0^{A(\infty)} f(TX_{\theta(t)})\, dt \\ &= \mathbf{E}_{Tx} \int_0^{\infty} f(TX_s)|X_s|^{-2\alpha}\, ds = \int_{\mathbf{R}^d} f(y)|y|^{2\alpha} u^T(x,y)\, dy, \quad x \in \mathbf{R}_0^d. \end{aligned} \tag{35}$$

Thus, the kernel of the potential operator of $\{Y_t\}$ is

$$u^Y(x,y) = |y|^{2\alpha} u^T(x,y) = \frac{|y|^{\alpha-d}}{|x|^{\alpha-d}} \frac{\mathcal{A}(\alpha,d)}{|y-x|^{d-\alpha}} = u^h(x,y). \tag{36}$$

We will prove that $U^h = U^Y$ yields that the respective processes are equal in law (compare Ref. 27 II, p. 352). We aim at the equality $U_\lambda^h = U_\lambda^Y$, $\lambda > 0$, which however *cannot* be proved as in (35). In what follows $\lambda > \delta \geqslant 0$ are arbitrary. Assume that a function f on $\mathbf{R}_0^d$ is such that

$$U^h|f|(x) < \infty \text{ for all } x \in \mathbf{R}_0^d. \tag{37}$$

The semigroup property of $\{p^h(t;x,y), t \geqslant 0\}$ yields the *pointwise* resolvent equation:

$$U_\delta^h f(x) - U_\lambda^h f(x) = (\lambda-\delta) U_\lambda^h U_\delta^h f(x) = (\lambda-\delta) U_\delta^h U_\lambda^h f(x), \quad x \in \mathbf{R}_0^d, \tag{38}$$

where each term is an absolutely convergent integral. In particular, for $\delta = 0$ we have

$$U^h f(x) - U_\lambda^h f(x) = \lambda U_\lambda^h U^h f(x) = \lambda U^h U_\lambda^h f(x), \quad x \in \mathbf{R}_0^d. \tag{39}$$

From now on our equalities are supposed to hold for every $x \in \mathbf{R}_0^d$. Recall that $U^h f = U^Y f$. By (39) and the resolvent equation for $\{Y_t\}$, we obtain $U_\lambda^h f + \lambda U_\lambda^h U^h f = U_\lambda^Y f + \lambda U_\lambda^Y U^Y f$, thus

$$(U_\lambda^h - U_\lambda^Y)(I + \lambda U^h) f = 0. \tag{40}$$

Let φ be a continuous function with compact support in $\mathbf{R}_0^d$. We have $U^h|\varphi|(x) < \infty$ on $\mathbf{R}_0^d$ because the kernel u^h of the potential operator U^h is locally integrable function (36). By (39)

$$(I + \lambda U^h)(I - \lambda U_\lambda^h)\varphi = \varphi. \tag{41}$$

Let $f = (I - \lambda U_\lambda^h)\varphi$. We shall verify that f satisfies (37). Indeed,

$$U^h|(I - \lambda U_\lambda^h)\varphi|(x) \leqslant U^h|\varphi|(x) + \lambda U^h|U_\lambda^h \varphi|(x).$$

We only need to estimate the second term. From (33), (17), (19) and (20) we get

$$u_\lambda^h(x,y) = \frac{|y|^{\alpha-d}}{|x|^{\alpha-d}} u_\lambda(y-x) \leqslant c \frac{|y|^{\alpha-d}}{|x|^{\alpha-d}} \left(|y-x|^{\alpha-d} \wedge |y-x|^{-\alpha-d} \right).$$

There are numbers $0 < m < M < \infty$ such that supp $\varphi \subset \{m < |y| < M\}$. We have

$$\begin{aligned} |U_\lambda^h \varphi(x)| &\leqslant c||\varphi||_\infty |x|^{d-\alpha} \int_{m<|y|<M} |y|^{\alpha-d} \left(|y-x|^{\alpha-d} \wedge |y-x|^{-\alpha-d} \right) dy \\ &\leqslant c \left(|x|^{d-\alpha} \wedge |x|^{-2\alpha} \right). \end{aligned}$$

Thus,

$$\begin{aligned} U^h |U_\lambda^h \varphi|(x) &\leqslant c \int \frac{|y|^{\alpha-d}}{|x|^{\alpha-d}} |y-x|^{\alpha-d} \left(|y|^{d-\alpha} \wedge |y|^{-2\alpha} \right) dy \\ &= c|x|^{d-\alpha} \int |y-x|^{\alpha-d} \left(1 \wedge |y|^{-\alpha-d} \right) dy < \infty, \quad x \in \mathbf{R}_0^d. \end{aligned}$$

By (40) and (41) we have $U_\lambda^h \varphi(x) - U_\lambda^Y \varphi(x) = 0, x \in \mathbf{R}_0^d$. Let $g^h(t) = \int \varphi(y) p^h(t; x, y)\, dy$ and $g^Y(t) = \int \varphi(y) p^Y(t; x, dy), t \geqslant 0$, where $p^Y(t; x, \cdot)$ is the distribution of Y_t when $\{Y_t\}$ starts at x. Clearly, $U_\lambda^h(x)$ is the Laplace transform of g^h and $U_\lambda^Y(x)$ is the Laplace transform of g^Y. By uniqueness of Laplace transform [5,21,45] for every $x \in \mathbf{R}_0^d$ we conclude that $\int p^h(t; x, y)\varphi(y)\, dy = \int \varphi(y) p^Y(t; x, dy)$ for almost every $t \geqslant 0$. The class of considered functions φ is countably dense in

$$C_o(\mathbf{R}_0^d) = \{ f \in C(\mathbf{R}_0^d) \colon \lim_{x \to 0} f(x) = 0 \text{ and } \lim_{|x| \to \infty} f(x) = 0 \},$$

thus one-dimensional distributions of the two processes are identical for almost every and so (by right continuity of trajectories) for every $t \geqslant 0$. By Markov property the processes are identical in distribution.

For completeness we give the following result which implies that the lifetime of the process $\{Y_t\}$ is finite for a.e. trajectory (see Ref. 44 for more detailed information when $\alpha = 2 < d$).

Lemma 3. $A(\infty) < \infty$ almost surely for every starting point $x \in \mathbf{R}_0^d$.

Proof. Let $B = \{x \in \mathbf{R}^d \colon |x| < 1\}$ and define the *last* exit time from B: $\Lambda_B = \sup\{s \geqslant 0 \colon |X_s| < 1\}$. Fix $x \in \mathbf{R}_0^d$. Note that $|X_s| \to \infty$ as $s \to \infty$ ($\mathbf{P}_x - a.s.$). This follows from the decay of h at infinity by a similar argument as we used before to prove (29) . In particular, $\Lambda_B < \infty$ ($\mathbf{P}_x - a.s$). Clearly,

$$A(\infty) = A(\Lambda_B) + \int_{\Lambda_B}^{\infty} |X_s^0|^{-2\alpha} ds$$

and $A(\Lambda_B) < \infty$ ($\mathbf{P}_x - a.s$) because X_s does not approach the origin in finite time by (31) and quasi-left continuity. We have

$$\mathbf{E}_x \int_{\Lambda_B}^{\infty} |X_s^0|^{-2\alpha} ds \leqslant \mathbf{E}_x \int_0^{\infty} |X_s|^{-2\alpha} \mathbf{1}_{B^c}(X_s) ds$$
$$= \int_{B^c} \mathcal{A}(d,\alpha)|y-x|^{\alpha-d}|y|^{-2\alpha} dy < \infty.$$

This ends the proof. □

4. POINTWISE RECURRENT CASE

In this section $\alpha > d = 1$. It is no longer true that $\{X_t^0\}$ has the same law as $\{X_t\}$ because

$$\mathbf{P}_x\{\sigma_0 < \infty\} = 1, \quad x \in \mathbf{R}. \tag{42}$$

Indeed, let $\lambda > 0$ and note that u_λ is now bounded and continuous in $\mathbf{R}$ and has the maximum at the origin, see (12). By considering the process $\{X_t, \mathbf{P}_x\}$ killed with constant rate λ, as at the beginning of Section 3 we see that u_λ is harmonic for this process in $\mathbf{R}_0$. Put simply:

$$u_\lambda(x) = \mathbf{E}_x u_\lambda(X_{\tau_B}) e^{-\lambda \tau_B}, \quad x \in \mathbf{R},$$

for open B precompact in $\mathbf{R}_0 = \mathbf{R}\setminus\{0\}$. When B *increases* to $\mathbf{R}_0$ then $\tau_B \uparrow \sigma_0$ and (by quasi-left continuity) $u_\lambda(X_{\tau_B})e^{-\lambda\tau_B} \to u_\lambda(X_{\sigma_0})e^{-\lambda\sigma_0}$ on the set $\{\sigma_0 < \infty\}$. Of course, $u_\lambda(X_{\tau_B})e^{-\lambda\tau_B} \to 0$ whenever $\sigma_0 = \infty$. By bounded convergence theorem we obtain regular harmonicity on $\mathbf{R}_0$:

$$u_\lambda(x) = \mathbf{E}_x\{\sigma_0 < \infty; u_\lambda(X_{\sigma_0})e^{-\lambda\sigma_0}\} = u_\lambda(0)\mathbf{E}_x e^{-\lambda\sigma_0}, \quad x \in \mathbf{R}. \tag{43}$$

We let $x \in \mathbf{R}$ and $\epsilon \to 0^+$. By scaling (15) (see also (55) below), continuity of u_λ and (43)

$$\mathbf{P}_x(\sigma_0 = \infty) = \mathbf{P}_{\epsilon x}(\sigma_0 = \infty) < 1 - u_\lambda(\epsilon x)/u_\lambda(0) \to 0.$$

This proves our claim.

We will focus on properties of $\{X_t^0\}$. The state space for $\{X_t^0\}$ and $\{TX_t^0\}$ is $\mathbf{R}_0$ augmented by ∂. Let

$$G_\lambda(x,y)=\int_0^\infty e^{-\lambda t}p^0(t;x,y)dt, \quad \lambda\geqslant 0.$$

$G_\lambda(x,y)$ is the kernel of the λ-resolvent G_λ of $\{X_t^0\}$:

$$\begin{aligned}G_\lambda f(x) &= \mathbf{E}_x\left[\int_0^\infty e^{-\lambda t}f(X_t^0)dt\right]=\mathbf{E}_x\left[\int_0^{\sigma_0} e^{-\lambda t}f(X_t)dt\right]\\ &=\int_{\mathbf{R}_0} f(y)G_\lambda(x,y)dy,\end{aligned}$$

and $G(x,y)=G_0(x,y)$ is the kernel of the potential operator $G=G_0$ of $\{X_t^0\}$ i.e. it is the Green function of $\{X_t\}$ for $\mathbf{R}_0$. Of course,

$$G_\lambda(x,y)\leqslant G(x,y). \tag{44}$$

By (28) and (19)

$$G_\lambda(x,y)\leqslant c|y-x|^{-1-\alpha}. \tag{45}$$

By (18)

$$\lim_{\lambda\to 0+}[u_\lambda(y-x)-u_\lambda(0)]=K_\alpha(y-x).$$

The next result was proved before in Ref. 35, p. 379 by using a related argument.

Lemma 4. For all $x,y\in\mathbf{R}_0$ we have

$$G(x,y)=K_\alpha(y-x)-K_\alpha(x)-K_\alpha(y)=c_\alpha(|y-x|^{\alpha-1}-|y|^{\alpha-1}-|x|^{\alpha-1}).$$

Proof. Let $x,y\in\mathbf{R}$ and $\lambda>0$. (27) yields

$$G_\lambda(x,y) = u_\lambda(y-x)-\int_0^\infty e^{-\lambda t}\mathbf{E}_x[p(t-\sigma_0;0,y);\sigma_0<t]dt.$$

Let $\mu^x_{\sigma_0}$ be the distribution of σ_0 with respect to $\mathbf{P}_x$. By a change of variable and (43)

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}\mathbf{E}_x[p(t-\sigma_0;0,y);\sigma_0<t]\,dt\\
&=\int_0^\infty e^{-\lambda t}\int_{[0,t)} p(t-u;0,y)\mu^x_{\sigma_0}(du)\,dt\\
&=\int_{[0,\infty)}\mu^x_{\sigma_0}(du)\int_u^\infty e^{-\lambda t}p(t-u;0,y)dt\\
&=\int_{[0,\infty)}e^{-\lambda u}\mu^x_{\sigma_0}(du)\int_0^\infty e^{-\lambda v}p(v;0,y)dv\\
&=\mathbf{E}_x e^{-\lambda\sigma_0}u_\lambda(y)=\frac{u_\lambda(x)u_\lambda(y)}{u_\lambda(0)}.
\end{aligned}$$

By letting $\lambda\to 0+$ we get

$$\begin{aligned}
G(x,y) &\leftarrow G_\lambda(x,y)=u_\lambda(y-x)-u_\lambda(x)u_\lambda(y)/u_\lambda(0)\\
&= [u_\lambda(y-x)-u_\lambda(0)]-[u_\lambda(x)-u_\lambda(0)][u_\lambda(y)-u_\lambda(0)]/u_\lambda(0)\\
&\quad -[u_\lambda(x)-u_\lambda(0)]-[u_\lambda(y)-u_\lambda(0)]\\
&\to K_\alpha(y-x)-K_\alpha(x)-K_\alpha(y).
\end{aligned}$$

This ends the proof. □

For clarity we note that $c_\alpha=[2\Gamma(\alpha)\cos(\pi\alpha/2)]^{-1}<0$ and $|y-x|^{\alpha-1}\leqslant |x|^{\alpha-1}+|y|^{\alpha-1}$, thus $K_\alpha(y-x)-K_\alpha(x)-K_\alpha(y)\geqslant 0$ for $x,y\in\mathbf{R}$. The derivative of $G(x,y)$ with respect to y is positive precisely for $y\in(0,x)$, hence

$$G(x,y)\leqslant G(x,x)=-2c_\alpha|x|^{\alpha-1}. \tag{46}$$

Observe that $G(x,y)$ extends to a continuous function on $\mathbf{R}\times\mathbf{R}$, by letting $G(0,y)=0$.

By the strong Markov property, Gg is harmonic for the killed process $\{X^0_t\}$ on $\mathbf{R}_0\backslash\mathrm{supp}(g)$ provided it is absolutely convergent there. Naturally, the definition of harmonicity on open $D\subset\mathbf{R}_0$ with respect to $\{X^0_t\}$ is

$$f(x)=\mathbf{E}_x f(X^0_{\tau_B}),\quad x\in B, \tag{47}$$

for every open bounded set B with closure $\bar{B}\subset D$. This coincides with α-harmonicity on $D\subset\mathbf{R}_0$ because $\mathbf{P}_x(X_{\tau_B}=0)=0$ for every point x in such B, but we have no need in proving here this assertion. Using the usual approximation argument $G(x,y)$ is harmonic for $\{X^0_t\}$ in $x\in\mathbf{R}_0\backslash\{y\}$. Since $\lim_{y\to\pm\infty}G(x,y)=-c_\alpha|x|^{\alpha-1}$, by Fatou's lemma, (46) and a limiting procedure $h(x)=|x|^{\alpha-1}$ is also harmonic for $\{X^0_t\}$ in $\mathbf{R}_0$.

As in Section 3 we condition the process $\{X_t^0\}$ by $h(x)=|x|^{\alpha-1}$ and the resulting h-process will be denoted $\{X_t^h\}$. It is determined by time homogeneous transition densities

$$p^{0,h}(t;x,y)=|x|^{1-\alpha}p_t^0(x,y)|y|^{\alpha-1}.$$

It is important to notice that

$$h(x)=\int_{\mathbf{R}} h(y)p(t;x,y)\,dy, \quad x\in \mathbf{R}_0, \tag{48}$$

so that the distribution of $\{X_t^h\}$ on the path space is a probability rather than subprobability measure (compare (23)). Indeed, by (47) and continuity of h

$$h(x)=\mathbf{E}_x h(X^0_{\tau_B}), \quad x\in B,$$

for *every* open bounded $B\subset \mathbf{R}_0$. By strong Markov property this further yields

$$h(x)=\mathbf{E}_x h(X^0_{\tau_B\wedge t}), \quad x\in B,$$

for every (deterministic) $t\geqslant 0$. Letting $B=\{y\in\mathbf{R}\colon 0<|y|<n\}$ and $n\to\infty$ and using (14) and bounded convergence theorem we obtain (48).

The resolvent kernel of $\{X_t^h\}$ is

$$\begin{aligned} u_\lambda^h(x,y) &= \int_0^\infty e^{-\lambda t}|x|^{1-\alpha}p^0(t;x,y)|y|^{\alpha-1}\,dt \\ &= |x|^{1-\alpha}G_\lambda(x,y)|y|^{\alpha-1}, \quad \lambda\geqslant 0. \end{aligned} \tag{49}$$

In particular

$$u^h(x,y)=u_0^h(x,y)=\frac{|y|^{\alpha-1}}{|x|^{\alpha-1}}G(x,y).$$

For later convenience we observe that by (46)

$$u^h(x,y)\leqslant 2|c_\alpha||y|^{\alpha-1}. \tag{50}$$

We now compute the potential kernel $u^T(x,y)$ of the inverted process $\{T(X_t^0)\}$. The transition density of $\{TX_t^0\}$ is

$$p^{0,T}(t;x,y)=p^0\left(t;\frac{1}{x},\frac{1}{y}\right)\frac{1}{|y|^2},$$

compare (25). Thus

$$\begin{aligned} u^T(x,y) &= \int_0^\infty p^{0,T}(t;x,y)\,dt = \int_0^\infty p^0(t;1/x,1/y)\frac{1}{y^2}dt \\ &= G(1/x,1/y)\frac{1}{y^2} = c_\alpha\left(|1/y-1/x|^{\alpha-1} - |1/y|^{\alpha-1} - |1/x|^{\alpha-1}\right)\frac{1}{y^2} \\ &= \frac{|y|^{\alpha-1}}{|x|^{\alpha-1}}G(x,y)|y|^{-2\alpha}. \end{aligned} \tag{51}$$

In (51) we used (3), which however is very simple in dimension 1. This calculation also indicates a more prosaic way of proving (32).

In the present context the additive functional (34) is well defined, finite and strictly increasing for $0 \leqslant t < \sigma_0$. By our convention on ∂ we have

$$A(t) = \int_0^t |X_s^0|^{-2\alpha} ds = \int_0^{t\wedge\sigma_0} |X_s|^{-2\alpha} ds. \tag{52}$$

We let, similarly as in Section 3, $\theta(t) = A^{-1}(t)$ for $0 < t < A(\sigma_0)$, where $A(\sigma_0) = \int_0^{\sigma_0} |X_s|^{-2\alpha} ds$ (see Lemma 5 below in this respect). We define $Y_t = T(X_{\theta(t)})$ if $t < A(\sigma_0)$ and we let $Y_t = \partial$ if $A(\sigma_0) \leqslant t < \infty$. It is a strong Markov process with respect to the standard filtration. The resolvent operator of this time-changed process will be denoted $U_\lambda^Y, \lambda \geqslant 0$, and $U^Y = U_0^Y$ will be the potential operator:

$$U^Y f(x) = \mathbf{E}_x \int_0^{\sigma_0} f(TX_t)|TX_t|^{2\alpha} dt$$

with kernel

$$u^Y(x,y) = u^T(x,y)|y|^{2\alpha} = u^h(x,y), \tag{53}$$

see (51) and the remark following (25). We now prove that $u^h(x,y) = u^Y(x,y)$ yields $U_\lambda^h = U_\lambda^Y$ for all $\lambda > 0$. This is very similar to the argument given in Section 3 and in fact we can use the same notation. Let φ be an arbitrary continuous function with compact support on $\mathbf{R}_0$. We only need to verify that $U^h|\varphi|(x) < \infty$ and $U^h U_\lambda^h|\varphi|(x) < \infty, x \in \mathbf{R}_0$. The former property follows, e.g., from (50). For the latter we note that (44), (46) and (19) yield

$$u_\lambda^h(x,y) \leqslant c(|x|^{\alpha-1} \wedge |y-x|^{-\alpha-1}), \tag{54}$$

thus $U_\lambda^h|\varphi|(x) \leqslant c(|x|^{\alpha-1} \wedge |x|^{-\alpha-1})$, which is integrable with respect to $u^h(x,y)dy$ by (50). The remaining arguments are a verbatim repetition of those used in Section 3. □

For clarity we give the following result, which shows that the lifetime of the process $\{Yt\}$ is infinite for a.e. trajectory.

Lemma 5. $A(\sigma_0)=\infty$ almost surely.

Proof. Let $T=\inf\{t\geqslant 0\colon |X_t|\leqslant \frac{1}{2}|X_0|\}$. Using (15) we get that for all $x\in\mathbf{R}$ and $k>0$

$$\mathbf{P}_x(\{X_t,t\geqslant 0\},T,X_T,\sigma_0)=P_{kx}\left(\{k^{-1}X_{tk^{-\alpha}},t\geqslant 0\},k^{-\alpha}T,k^{-1}X_T,k^{-\alpha}\sigma_0\right). \tag{55}$$

Define $T_0=0$ and $T_n=T_{n-1}+T\circ\theta_{T_{n-1}}, n=1,2,\ldots$. Clearly, $T_n\uparrow\sigma_0$ and

$$A(\sigma_0)=\sum_{n=1}^{\infty}[A(T_n)-A(T_{n-1})]=\sum_{n=1}^{\infty}A(T)\circ\theta_{T_n-1}. \tag{56}$$

By (55) we get that

$$\mathbf{P}_x(A(T))=\mathbf{P}_{kx}(k^{\alpha}A(T)),\quad k>0,\ \ x\in\mathbf{R}. \tag{57}$$

Let X_i be *i.i.d.* random variables with distribution $\mathbf{P}_1(|X_T|)$ and let A_i be *i.i.d.* random variables with distribution $\mathbf{P}_1(A(T))$. From (56), strong Markov property and (57) it follows that the $\mathbf{P}_x$-distribution of $A(\sigma_0)$ is the same as that of

$$|x|^{\alpha}\sum_{n=1}^{\infty}A_n\prod_{i=1}^{n}X_i^{-\alpha}=\infty\quad (a.s.).$$

The last equality follows from the fact that $X_i\leqslant 1/2$. □

5. LOGARITHMIC CASE

The case of $\alpha=d$ is special because the relevant process is neither transient nor point recurrent. We will first consider $\alpha=d=1$, i.e. the one-dimensional Cauchy process. To a large extent the logarithmic kernel $K_1(x)=\frac{1}{\pi}\log\frac{1}{|x|}$ substitutes for the Green function of $\mathbf{R}_0$, which we used before. The logarithmic kernel is a result of the compensation procedure (18), which comes back to M. Kac,[(30)] see also Ref. 26 for the planar Brownian motion. A concise general treatment of recurrent stable processes is given in Ref. 35.

Let $0<\epsilon<1, D=\{x\colon\epsilon<|x|<1/\epsilon\}$. For the Cauchy process and this set D we consider the Green function:

$$G_D(x,y)=\int_0^{\infty}p_D(t;x,y)dt, \tag{58}$$

where $p_D(t; x, y) = p(t; x, y) - \mathbf{E}_x[p(t - \tau_D; X_{\tau_D}, y); \tau_D < t]$. By Fubini's theorem

$$\int G_D(x, y) f(y) dy = \mathbf{E}_x \int_0^{\tau_D} f(X_t) dt.$$

It will be convenient to write $K_1(x, y)$ for $K_1(y - x)$ below. We have

$$G_D(x, y) = K_1(x, y) - \mathbf{E}_x K_1(X_{\tau_D}, y), \quad x, y \in \mathbf{R}. \tag{59}$$

For (59), we refer to Refs. 7 or 36. The boundary, ∂D, of our D is regular for D^c meaning that

$$\mathbf{P}_x\{\tau'_D = 0\} = 1, \quad x \in \partial D, \tag{60}$$

or $\tau_D = \tau'_D (a.s.)$. Here $\tau'_D = \inf\{t > 0; X_t \notin D\}$ is the first *hitting* time of D^c. We have $G_D(x, y) = G_D(y, x)$, $x, y \in \mathbf{R}^d$. For every fixed $x \in D$, the function $G_D(x, y)$ vanishes continuously on D^c. This is a well-known consequence of (60) and (59).

We can verify as we did before that $K_1(x)$ is 1-harmonic in $\mathbf{R}_0$. By definition, the second term in (59) is regular 1-harmonic on D as a function of x, whenever $y \in \mathbf{R}$ is fixed. Therefore the logarithmic kernel is 1-harmonic off its pole.

Lemma 6. For all $x, y \in \mathbf{R}_0$

$$G_D(x, y) = G_D(1/x, 1/y). \tag{61}$$

Proof. For $y \neq 1/x$ let

$$r(y) = G_D(x, 1/y) - G_D(1/x, y),$$

where we fix $x \in D$ (because $r \equiv 0$ if $x \notin D$). Clearly, r vanishes continuously on D^c. Note that $\log \frac{1}{|1/y - z|} = \log \frac{1}{|y - 1/z|} + \log |y| - \log |z|$. We have

$$\begin{aligned} \pi r(y) &= \int \log \frac{1}{|1/y - z|} (\delta_x - \omega_D^x)(dz) + \int \log \frac{1}{|y - z|} (\delta_{1/x} - \omega_D^{1/x})(dz) \\ &= \int \log \frac{1}{|y - 1/z|} (\delta_x - \omega_D^x)(dz) + \int \log \frac{1}{|y - z|} (\delta_{1/x} - \omega_D^{1/x})(dz) \end{aligned} \tag{62}$$

$$= -\int \log \frac{1}{|y - 1/z|} \omega_D^x(dz) + \int \log \frac{1}{|y - z|} \omega_D^{1/x}(dz). \tag{63}$$

In (62) we used the fact that $\log |z|$ is harmonic on D. (63) shows that $\lim_{y \to 1/x} r(y)$ is finite, in particular r is bounded. Furthermore, both terms in (63) are 1-harmonic in $y \in D$ because they are logarithmic potentials of some measures on D^c (see (9)). We thus have $r(y) = \mathbf{E}_y r(X_{\tau_B}), y \in \mathbf{R}(y \neq 1/x)$, for every open B precompact in D. Letting $B \uparrow D$, we have $X_{\tau_B} \to X_{\tau_D}$ a.s. by quasi-left continuity of the process. By continuity of r and bounded convergence, we conclude that $r(y) = 0, y \neq 1/x$. Thus, for all $x, y \neq 0, G_D(x, 1/y) = G_D(1/x, y)$. □

We like to mention here that a uniqueness result for the logarithmic kernel (see Lemma 9 below) and the fact that the expression in (63) equals zero yield that $\omega_D^{1/x} = \omega_D^x \circ T^{-1}$ as measures on D^c (for every $x \in \mathbf{R}_0$). This identification is a special case of a more general result (68), which we derive in the next section from Theorem 1.

We define the additive functional $A(t)$ by (34). Note that $A(\infty) = \infty$ because $\{X_t^0\}$ is recurrent. As before, we let $Y_t = T(X^0_{\theta(t)}), 0 \leqslant t < \infty$, where $\theta(t) = A^{-1}(t)$. Clearly, $Y_t = TX_t$ for all $t \geqslant 0$ almost surely if the starting point of the process is $x \neq 0$. For nonnegative function f and $x \in \mathbf{R}_0$, by changing variable in (64) and (65) below, we have

$$\mathbf{E}_x \int_0^{\tau_D} f(Y_t)dt = \mathbf{E}_{1/x} \int_0^{\tau_D} f(TX_{\theta(t)})dt = \mathbf{E}_{1/x} \int_0^{\tau_D} f(TX_s)|X_s|^{-2} ds \quad (64)$$

$$= \int_{\mathbf{R}_0} G_D(1/x, y) f(1/y) y^{-2} dy = \int_{\mathbf{R}_0} G_D(1/x, 1/y) f(y) dy \quad (65)$$

$$= \int_{\mathbf{R}_0} G_D(x, y) f(y) dy.$$

Thus the processes Y_t and X_t killed upon leaving D have the same potential kernel. We conclude that the resolvents and the semigroups of the processes are equal. Indeed, proceeding as before we only need to notice that $G_D G_D f(x)$ is finite for $f \in C_c(\mathbf{R}_0), f \geqslant 0$. But $\int G_D(x, y) G_D(y, z) dy$ is bounded in $x, z \in D$ because D is a bounded set and $G_D(x, y) \leqslant const.(|\log|y - x|| + 1)$. We now let $\epsilon \downarrow 0$ thus $D \uparrow \mathbf{R}_0$. It is clear that $\tau_D \uparrow \infty$ $\mathbf{P}_x$-a.s. for every $x \in \mathbf{R}_0$ and the semigroups of the killed processes increase to those of Y_t, X_t^0, respectively. This proves that the semigroups are identical.

The same argument is valid for the planar Brownian motion (the normalizing constant $1/\pi$ in the logarithmic kernel should be then replaced by $1/(2\pi)$). Of course, there are other proofs of (61) for planar Brownian motion, for example the one using the conformal mapping $z \mapsto 1/\bar{z}$ of the complex plane, see Ref. 36 for generalizations of (61) in this case.

6. PROOFS OF MAIN RESULTS

Proof of Theorem 1. The transient case $\alpha < d$ is given in Section 3. The pointwise recurrent case $\alpha > d = 1$ is in Section 4. The cases of $\alpha = d = 1$ and $\alpha = d = 2$ are resolved in Section 5. Therefore, the result holds for every $0 < \alpha \leqslant 2$ and every $d \in \{1, 2, \dots\}$. $\square$

By Theorem 1, X_t^0 conditioned by $h(x) = |x|^{\alpha-d}$ has the same law as $Y_t = TX^0_{\theta(t)}$. Both processes are directly related to $\{X_t, \mathbf{P}_x\}$. The conditional process results from a simple transformation of measure on the space of paths of $\{X_t^0\}$ while $\{Y_t\}$ is a pathwise transformation of $\{X_t^0\}$. For a given trajectory ω of $\{X_t\}$ and (open) $D \subset \mathbf{R}^d$ we consider $\tau_D^Y = \tau_D^Y(\omega) = \inf\{t \geqslant 0 : Y_t \notin D\}$. As before $\tau_D = \tau_D(\omega) = \inf\{t \geqslant 0 : X_t \notin D\}$.

We have

$$\begin{aligned}\tau_D^Y &= \inf\{t \geqslant 0 : TX^0_{\theta(t)} \notin D\} = \inf\{t \geqslant 0 : X_{\theta(t)} \notin T(D\backslash\{0\})\} \\ &= \inf\{A(s) : s \geqslant 0,\, X_s \notin T(D\backslash\{0\})\} = A(\tau_{T(D\backslash\{0\})}). \end{aligned} \tag{66}$$

Thus, for $B \subset \mathbf{R_0}$

$$\begin{aligned}\mathbf{P}_x\left\{Y_{\tau_D^Y} \in B;\, \tau_D^Y < \infty\right\} &= \mathbf{P}_{Tx}\{X_{\theta(A(\tau_{T(D\backslash\{0\})}))} \in T(B)\setminus\{0\});\, \tau_{D\backslash\{0\}} < \infty\} \\ &= \omega^{Tx}_{T(D\backslash\{0\})}(T(B\setminus\{0\})).\end{aligned}$$

By Theorem 1 the above is equal to

$$\begin{aligned}P_x^h\{X^0_{\tau_D} \in B;\, \tau_D < \infty\} &= \frac{1}{h(x)}\mathbf{E}_x\{h(X_{\tau_D})\mathbf{1}_B(X_{\tau_D});\, \tau_D < \sigma_0,\, \tau_D < \infty\} \\ &= |x|^{d-\alpha}\int_{B\backslash\{0\}} |y|^{\alpha-d}\omega^x_{D\backslash\{0\}}(dy) \\ &= |x|^{d-\alpha}\int |y|^{\alpha-d}\mathbf{1}_{T(B\backslash\{0\})}(Ty)\omega^x_{D\backslash\{0\}}dy).\end{aligned}$$

Using our notation from Section 2 we can describe the result of these calculations as

$$\mathbf{1}_{\mathbf{R}_0^d}(x)|x|^{\alpha-d}\omega^{Tx}_{T(D\backslash\{0\})} = \tilde{\mathcal{K}}\left[\mathbf{1}_{\mathbf{R}_0}\omega^x_{D\backslash\{0\}}\right], \quad x \in \mathbf{R}_0^d. \tag{67}$$

Observe that (67) is an equality between measures on $\mathbf{R}_0^d$ and inquiries about whether the harmonic measures charge the origin (in the point recurrent case) should be made separately. Summarizing, for every $D \subset \mathbf{R}_0^d$ and $x \in \mathbf{R}_0^d$

$$\mathcal{K}\omega^x_{TD} = \tilde{\mathcal{K}}\omega^x_D \quad \text{as measures on } \mathbf{R}_0. \tag{68}$$

Note that (68) is only a laconic form of (67). If the harmonic measure is absolutely continuous with respect to the Lebesgue measure and has the density function (the Poisson kernel) g, say $\omega_D^{Tx}(dy)=g(y)dy$, then by (8) we have

$$\frac{dw^x_{TD}}{dy}=|x|^{\alpha-d}|y|^{-2\alpha}\mathcal{K}g(y), \quad x\in\mathbf{R}^d_0, \quad D\subset\mathbf{R}^d_0. \tag{69}$$

We give the following application of (68).

Lemma 7. Let $\alpha\in(0,2], d=1,2,\ldots$. Let $D\subset\mathbf{R}^d_0$ be open. We assume that f is regular α-harmonic on D and in the case $\alpha>d=1$ we additionally assume that $f(0)=0$. Then $\mathcal{K}f$ is regular α-harmonic on TD. If f is α-harmonic in D (rather than regular α-harmonic) then $\mathcal{K}f$ is α-harmonic in TD.

Proof. If $\mathbf{E}_x f(X_{\tau_D})=f(x)$ for $x\in D$ then

$$\begin{aligned}E^x\mathcal{K}f(X_{\tau_{TD}}) &= \int_{\mathbf{R}^d_0}\mathcal{K}f(y)\omega^x_{TD}(dy)=\int_{\mathbf{R}^d_0} f(y)\tilde{\mathcal{K}}\omega^x_{TD}(dy)\\ &= \int_{\mathbf{R}^d_0} f(y)\mathcal{K}_x\omega^x_D(dy)=\mathcal{K}_x\int_{\mathbf{R}^d_0} f(y)\omega^x_D(dy)=\mathcal{K}f(x), \quad x\in TD,\end{aligned}$$

which proves that $\mathcal{K}f$ is regular α-harmonic on TD. Here and below $\mathcal{K}_x$ indicates that the Kelvin transform acts on the x variable. The last statement of the lemma is obvious. □

Lemma 7 is an extension (to all $\alpha\in(0,2]$ and natural d) and strengthening (to *regular* harmonic functions) of the fact that Kelvin transform preserves harmonic functions.[1,9,13]

Our main corollary of Theorem 1 is Theorem 2.

Proof of Theorem 2. The Green operator of open $D\subset\mathbf{R}_0$ for the h-process is given by

$$\begin{aligned}\frac{1}{h(x)}\mathbf{E}_x\int_0^{\tau_D} h(X^0_t)f(X^0_t)dt &= \frac{1}{h(x)}\mathbf{E}_x\int_0^{\tau_D} h(X_t)f(X_t)dt\\ &= \frac{1}{h(x)}\int G_D(x,y)h(y)f(y)dy, \quad x\in\mathbf{R}^d_0.\end{aligned}$$

Its kernel is $|x|^{d-\alpha}|y|^{\alpha-d}G_D(x,y), x,y\in\mathbf{R}_0^d$. By Theorem 1 we will get the same integral operator, when we calculate

$$\begin{aligned}\mathbf{E}_x\int_0^{\tau_D^Y} f(Y_t)dt &= \mathbf{E}_{T_x}\int_0^{A(\tau_{TD})} f(TX^0_{\theta(t)})dt \\ &= \mathbf{E}_{Tx}\int_0^{\tau_{TD}} f(TX_s^0)|X_s|^{-2\alpha}ds = \int G_{TD}(Tx,y)f(Ty)|y|^{-2\alpha}dy \\ &= \int G_{TD}(Tx,Ty)f(y)|y|^{2\alpha-2d}dy.\end{aligned}$$

We used here (66) and we changed variable similarly as in (64) and (65). Thus, for every $x\in\mathbf{R}_0^d$ we have that $G_{TD}(Tx,Ty)|y|^{2\alpha-2d}=|x|^{d-\alpha}|y|^{\alpha-d}$ $G_D(x,y)$, for almost all y. By symmetry of the Green function, this holds true for all $x,y\in D$. Note that the identity immediately extends to all $x,y\in\mathbf{R}_0^d$, which proves (2). □

More compactly, (2) reads

$$G_{TD}(x,y)=\mathcal{K}_x\mathcal{K}_y G_D(x,y), \quad x,y\in\mathbf{R}_0^d, \tag{70}$$

for every open $D\subset\mathbf{R}_0^d$.

7. APPLICATIONS

We will now sketch several applications of the main results which pertain to the kernel functions of a domain and its image under inversion.

Let $D\subset\mathbf{R}_0^d$ and let μ be a measure on $\mathbf{R}_0^d$. We define, as usual, the Green potential $G_D\mu(x)$ of μ:

$$G_D\mu(x)=\int G_D(x,y)\mu(dy), \quad x\in\mathbf{R}_0^d.$$

From (70) and the fact that $\mathcal{K}\circ\mathcal{K}$ is identity operator we have

$$\begin{aligned}\mathcal{K}G_D\mu(x) &= \int \mathcal{K}_x G_D(x,y)\mu(dy) = \int \mathcal{K}_x\mathcal{K}_y^2 G_D(x,y)\mu(dy) \\ &= \int \mathcal{K}_x\mathcal{K}_y G_D(x,y)\tilde{\mathcal{K}}\mu(dy) = \int G_{TD}(x,y)\tilde{\mathcal{K}}\mu(dy) \\ &= G_{TD}\tilde{\mathcal{K}}\mu(x).\end{aligned} \tag{71}$$

In particular, for the Green potential $G_Dg(x)=\int G_D(x,y)g(y)dy$ of a function g, (8) yields:

$$\begin{aligned}\mathcal{K}G_Dg(x) &= G_{TD}\left(|y|^{-2\alpha}\mathcal{K}g(y)\right)(x) \\ &= G_{TD}\left(|y|^{-\alpha-d}g(y/|y|^2)\right)(x), \quad x\in\mathbf{R}_0^d.\end{aligned} \tag{72}$$

For the next application we consider a function q on $\mathbf{R}_0^d$. Assume that q is such that $F(t)=\int_0^t q(X_t)dt$ is finite for every $t\geqslant 0$ ($a.s.$), see Ref. 13 in this respect. We let

$$e_q(t)=\exp(F(t)).$$

Assume that

$$f(x)=E_x e_q(\tau_D)f(X_{\tau_D}),\quad x\in\mathbf{R}^d. \tag{73}$$

Such a function f is called regular q-harmonic on D [(13)] and

$$\Delta^{\alpha/2}f+qf=0\quad\text{on } D,$$

see Ref. 13 when $\alpha<2$ and Ref. 18 when $\alpha=2$ for details on this Schrödinger equation.

Let $D\subset\mathbf{R}_0^d$. We claim that if $\alpha\leqslant d$ or $0\notin\partial D$ or $f(0)=0$ then $\mathcal{K}f$ is regular ρ-harmonic on TD, where $\rho(y)=|y|^{-2\alpha}q(y/|y|^2)$. Indeed, let $\mathbf{E}^h$ be the expectation for the h-conditioned process and let $v(x)=f(x)/h(x)$. By (73), Theorem 1 and a change of variable we obtain

$$\begin{aligned} v(x) &= \frac{1}{h(x)}\mathbf{E}_x e_q(\tau_D)v(X_{\tau_D})h(X_{\tau_D})=\mathbf{E}_x^h e_q(\tau_D)v(X_{\tau_D})\\ &= \mathbf{E}_{Tx}e_{aq\circ T}(\tau_{TD})v\circ T(X_{\tau_{TD}}), \end{aligned}\tag{74}$$

where $a(x)=|x|^{-2\alpha}$. Note that $v(Tx)=\mathcal{K}f(x)$. Thus $\mathcal{K}u$ is regular ρ-harmonic on TD, where $\rho(y)=a(y)q(Ty)=|y|^{-2\alpha}q(y/|y|^2)$.

We note in passing that if $\Delta^{\alpha/2}f=\mu$ on open $D\subset\mathbf{R}_0^d$ for a locally finite measure μ then $\Delta^{\alpha/2}\mathcal{K}f=\tilde{\mathcal{K}}\mu$ on TD. We leave the proof to the interested reader (see Ref. 13, in particular Lemma 5.3, for a necessary formalism).

We now examine certain phenomena in the potential theory of our stable processes in the pointwise recurrent case. We will assume that $1=d<\alpha\leqslant 2$ in this discussion. Let $D\subset\mathbf{R}_0$ be unbounded and such that dist $(0,D)>0$. TD is then *bounded* and $0\in\partial(TD)$. It is well-known[(7)] that for all $x,y\in\mathbf{R}$ we have

$$\begin{aligned} G_{TD}(x,y) &= \int_{\mathbf{R}} K_\alpha(z,y)(\delta_x-\omega_{TD}^x)(dz)\\ &= \int_{\mathbf{R}} K_\alpha(z,y)(\delta_x-\mathbf{1}_{\mathbf{R}_0}\omega_{TD}^x)(dz)-K_\alpha(0,y)\omega_{TD}^x(\{0\})=I-II. \end{aligned}\tag{75}$$

By (3)

$$\mathcal{K}_x\mathcal{K}_y II=\mathcal{A}(d,\alpha)|x|^{\alpha-d}\omega_{TD}^{Tx}(\{0\})=\mathcal{A}(d,\alpha)\mathcal{K}_x\omega_{TD}^x(\{0\}) \tag{76}$$

($d=1$ here and below). Note that this function is α-harmonic on D^c. It is also continuous on $\mathbf{R}_0$ as follows easily from regularity of one-point sets.[35] It is now easy to conclude that the function is regular α-harmonic on every bounded subset of D. For I we use (67) and (7) to get

$$\begin{aligned}
\mathcal{K}_x\mathcal{K}_y I &= \int |z|^{\alpha-d} K_\alpha(Tz,y)\mathcal{K}(\delta_x - \mathbf{1}_{\mathbf{R}_0}\omega^x_{TD})(dz)\\
&= \int |z|^{\alpha-d} K_\alpha(Tz,y)\tilde{\mathcal{K}}(\delta_x - \mathbf{1}_{\mathbf{R}_0}\omega^x_{D})(dz)\\
&= \int K_\alpha(z,y)(\delta_x - \mathbf{1}_{\mathbf{R}_0}\omega^x_{D})(dz)\\
&= \int |z|^{\alpha-d} K_\alpha(z,y)(\delta_x - \omega^x_{D})(dz) + K_\alpha(0,y)\omega^x_D(\{0\})\\
&= \int |z|^{\alpha-d} K_\alpha(z,y)(\delta_x - \omega^x_{D})(dz).
\end{aligned}$$

We conclude that

$$\begin{aligned}
G_D(x,y) &= \mathcal{K}_x\mathcal{K}_y G_{TD}(x,y)\\
&= \int |z|^{\alpha-d} K_\alpha(z,y)(\delta_x - \omega^x_{D})(dz) - \mathcal{A}(d,\alpha)\mathcal{K}\omega^x_{TD}(\{0\}), \qquad (77)
\end{aligned}$$

which differs from (75) if and only if $\omega^x_{TD}(\{0\}) \not\equiv 0$, that is if and only if 0 is an isolated point of TD^c i.e. D^c is bounded. Details concerning the last assertion are left to the reader (see (43) and Remark on p. 374 in Ref. 35). Note that the presence of the additional term is well recognized[7,35] for complements of compact sets. Here we see that for all other sets (as well as for $\alpha \leqslant d$) this additional term vanishes. By using the notion of regular α-harmonicity the term may be described by means of the potential of the equilibrium measure of $\mathbf{R}\setminus TD$.[35]

We now comment on the relation between the density function (Poisson kernel) of the harmonic measure of the ball and of its complement for $\alpha < 2$. Let $B = \{x \in \mathbf{R}^d : |x| < 1\}$, $B' = B\setminus\{0\}$ and $\check{B} = \{x \in \mathbf{R}^d : |x| > 1\}$. Let

$$P(x,y) = C^d_\alpha \left|\frac{1-|x|^2}{|y|^2-1}\right|^{\alpha/2} |x-y|^{-d}.$$

It is well-known although not easy[5,7,38,39] that for $x \in B$, ω^x_B is absolutely continuous with respect to the Lebesgue measure (on $\check{B}$) and has the density function

$$\frac{w^x_B(dy)}{dy} = \mathbf{1}_{\check{B}}(y)P(x,y), \qquad y \in \check{B}. \qquad (78)$$

When $\alpha \leqslant d$ and $x \in B'$ then $\omega_B^x = \omega_{B'}^x$ and so (69) and (3) yield for $y \in \check{B}$

$$\frac{\omega_{\check{B}}^x(dy)}{(dy)} = |x|^{\alpha-d}|y|^{-\alpha-d}P(Tx, Ty)$$
$$= C_\alpha^d \left[\frac{1-|x|^2}{|y|^2-1}\right]^{\alpha/2} |x-y|^{-d} = P(x, y),$$

or $\omega_{\check{B}}^x(dy) = \mathbf{1}_B(y)P(x, y)dy$ for $x \in \check{B}$.

When $\alpha > d = 1$ the result is different. For $A \subset \mathbf{R}_0$ consider $f(x) = \omega_B^x(A) - \omega_{B'}^x(A) \geqslant 0$. The function is regular α-harmonic on B', thus $f(x) = \mathbf{E}_x f(X_{\tau_{B'}}) = f(0)\omega_{B'}^x(\{0\}) = \omega_B^x(A)\omega_{B'}^x(\{0\})$. Let $P'(x, y)$ be the Poisson kernel of B'. By the above, $P(x, y) - P'(x, y) = P(0, y)\omega_{B'}^x(\{0\})$ or

$$P'(x, y) = P(x, y) - P(0, y)\omega_{B'}^x(\{0\}), \quad x \in B', \ |y| > 1.$$

Using (69) and (3) again, for $x \in \check{B}$ and $y \in B'$ we get

$$\frac{\omega_{\check{B}}^x(dy)}{dy} = |x|^{\alpha-d}|y|^{-\alpha-d}P'(Tx, Ty)$$
$$= P(x, y) - C_\alpha^d(1-|y|^2)^{-\alpha/2}|x|^{\alpha-d}\omega_{B'}^{Tx}(\{0\}). \tag{79}$$

One factor in the second term of (79) is a special instance of that in (76). We refer the reader to Ref. 7 for an integral representation of this term and another derivation of (79). Note that (79) is valid for all dimensions d, but $\omega_{B'}^{Tx}(\{0\}) \equiv 0$ whenever $\alpha \leqslant d$.

8. CONCLUDING REMARKS

The fact that the classical Kelvin transform ($\alpha = 2$) preserves classical harmonic functions can be proved by direct differentiation (see Theorem 2.7.2 in Ref. 25). The inversion itself is often used to calculate the classical Green function or Poisson kernel for the ball and, conversely, the special form of these may be used to prove the preservation property (see Chapter 4 of Ref. 1, and Refs. 13, 23), which is quite natural in view of Theorem 2 and its consequences. The well-known connection due to Maxwell[20] between harmonic polynomials, the Newtonian kernel and the Kelvin transform can also be used for this purpose.[1]

We like to note that for the study of the (classical) Kelvin transform *Jordan algebras* give a natural setting more general than that of $\mathbf{R}^d$. We refer the reader to Ref. 19 for references and the results on the Maxwell's connection for Jordan algebras. Ref. 19 corresponds to the case of $\alpha = 2$

in our discussion, which suggests possibility of further generalizations. See also Ref. 16.

We now remark on methods used for $\alpha \in (0,2]$ in $\mathbf{R}^d$ in the literature. The use of the Kelvin transform in Refs. 5, 31 and 38 is restricted to the case $\alpha < d$ and reduces to the following transformation formula for the Riesz potential $U_\mu(x) = \mathcal{A}(d,\alpha)\int |y-x|^{\alpha-d}\mu(dy)$ of a finite measure μ on $\mathbf{R}^d_0$:

$$\begin{aligned}\mathcal{K}U_\mu(x) &= \int_{\mathbf{R}^d_0} |x|^{\alpha-d}|y-Tx|^{\alpha-d}\mu(dy) = \int_{\mathbf{R}^d_0} |y|^{\alpha-d}|Ty-x|^{\alpha-d}\mu(dy)\\ &= \int_{\mathbf{R}^d_0} \mathcal{K}_y(|y-x|^{\alpha-d})\mu(dy) = U_{\tilde{\mathcal{K}}\mu}(x), \quad x \in \mathbf{R}^d_0. \end{aligned} \tag{80}$$

One only needs (3) and (7) to prove (80) here. (80) is easily extended to the following:

$$\begin{aligned}\mathcal{K}\{a + U_\mu(x) + bU_{\delta_0}(x)\} = \frac{a}{\mathcal{A}(d,\alpha)}U_{\delta_0}(x) + U_{\tilde{\mathcal{K}}\mu}(x)\\ + b\mathcal{A}(d,\alpha), \quad x \in \mathbf{R}^d_0. \end{aligned} \tag{81}$$

(81) may be interpreted in terms of measures on $\mathbf{R}^d \cup \{\omega\}$, where ω is the point at infinity (Ref. 31, p. 261).

It is interesting whether a formalism at the level of stochastic processes exists which incorporates the point at infinity. Except for (81) also Lemma 3 seems to suggest such a possibility. Note that (81) is the same in the point recurrent case but takes on a slightly different form in the logarithmic case.

An application of (80), i.e. of the Kelvin transform, together with dilations and translations of $\mathbf{R}^d$ yield, through an involved calculation, the Poisson kernel for the ball and its complement (78) for $d \geqslant 2 > \alpha$ (Ref. 31 (Appendix) and Ref. 38). The derivation of (78) and (79) for all $\alpha \in (0,2)$ and $d = 1, 2, \ldots$ is completed in Ref. 7. The result plays a fundamental role in the potential theory of the fractional Laplacian. Still, the properties of Kelvin transform are even more fundamental, which explains our efforts in Ref. 10 and here to obtain a proof independent of this involved calculation (as opposed to the results of Ref. 13).

Another idea underlying our considerations was to use the semigroup of the process and its resolvent rather than the generator or sole potential kernel. This approach shifts some of the difficulties of the proof of the properties of the Kelvin transform to the level of definitions but it also gives stronger results in the potential theory of the corresponding potential kernel. As a related problem we sketch below a proof of a uniqueness result for logarithmic potentials. Noteworthy, the uniqueness of the

Laplace transform used in the proof replaces the usual uniqueness arguments restricted to measures of finite energy.[5,7,31]

For $\alpha = d = 1$ the transition probability density of our process is $p_t(x) = \frac{1}{\pi}\frac{t}{t^2+x^2}$. Note that

$$\int_0^\infty |p_t(x) - p_t(1)|dt = |K_1(x)| = \frac{1}{\pi}|\log|x||. \tag{82}$$

Using (82) we get

$$u_\lambda(x) \leqslant \frac{1}{\pi}|\log|x|| + \frac{1}{\lambda}, \quad x \in \mathbf{R}, \quad \lambda > 0. \tag{83}$$

Let

$$U_+ f(x) = \frac{1}{\pi}\int_{\mathbf{R}} \log\frac{1}{|y-x|} f(y)dy,$$

and

$$|U_+| f(x) = \frac{1}{\pi}\int_{\mathbf{R}} \left|\log\frac{1}{|y-x|}\right| |f(y)|dy, \quad x \in \mathbf{R}.$$

Lemma 8. Let $x \in \mathbf{R}$. If $\int_{\mathbf{R}} |\log|y-x|||f(y)|dy < \infty$ and $\int_{\mathbf{R}} f(y)dy = 0$ then

$$\lim_{\lambda \to 0^+} U_\lambda f(x) = U_+ f(x). \tag{84}$$

Proof. By Fubini's theorem, (82), (83) and dominated convergence

$$\begin{aligned} U_\lambda f(x) &= \int_0^\infty e^{-\lambda t}\int_{\mathbf{R}} p_t(y-x) f(y)dydt \\ &= \int_0^\infty e^{-\lambda t}\int_{\mathbf{R}} [p_t(y-x) - p_t(1)] f(y)dydt \to U_+ f(x) \quad \text{as } \lambda \to 0^+. \end{aligned}$$

□

Assume that a finite measure μ on $\mathbf{R}$ satisfies $\int_{\{|y|>1\}} |\log|y|||\mu|(dy) < \infty$ and let $U_+\mu(x) = \frac{1}{\pi}\int_{\mathbf{R}} \log\frac{1}{|y-x|}\mu(dy)$. Then the integral is absolutely convergent almost everywhere (*a.e.*). Under the same assumption on a measure σ we have the following result.

Lemma 9. If $U_+\mu = U_+\sigma$ a.e. then $\mu = \sigma$.

Proof. By letting one of the parameters tend to zero in the resolvent equation (compare (38)) and using (84), (83) and (19), for every function ϕ with compact support and integral zero we obtain:

$$U_\lambda \phi(x) = U_+(\phi - \lambda U_\lambda \phi)(x).$$

If we let $f(x) = \phi(x) - \lambda U_\lambda \phi(x)$ then this function satisfies $|U_+ \| f|(x) \leqslant c(1 + |\log|x\||)$ and $\int_{\mathbf{R}} f(y)dy = 0$. By Fubini's theorem and our assumption we get

$$(\mu, U_+ f) = (\sigma, U_+ f),$$

where $(\chi, \psi) = \int_{\mathbf{R}} \chi(x)\psi(x)dx$. Thus, $(\mu, U_\lambda \phi) = (\sigma, U_\lambda \phi)$, $\lambda > 0$. By uniqueness, $(\mu, P_t \phi) = (\sigma, P_t \phi)$ for all $t > 0$ (see the end of Section 3). Here $P_t \psi(x) = \int p_t(x, y)\psi(y)dy$. Letting $t \to 0^+$ we conclude that $\mu = \sigma$. □

Our program of proving Theorem 1 solely by means of the resolvent equation is completed in Sections 2–4, and the remaining exception is the logarithmic case of Section 5. The difficulty of this case is in calculating from first principles the limit, when $\lambda \to 0^+$, of the λ-resolvent of the *time changed* process, as an absolutely convergent integral for functions $f \in C_c(\mathbf{R}_0)$ satisfying an appropriate cancellation property ($\int f(y)y^{-2}dy = 0$). This should be compared with Lemma 8 but seems to require a special approach.

The literature on the logarithmic kernel in dimension one is scarce because its potential theory can be reduced to the rich theory of the planar logarithmic kernel (for which see Refs. 31 and 36) in the manner mentioned at the end of Ref. 35.

It should be clear from Section 5 that Theorem 2 is tantamount to Theorem 1. Note, however, that some uses of Theorem 1, e.g. (74), are not direct consequences of Theorem 2.

There is a certain emphasis in our paper on the use of the notion of *regular* harmonic functions in probabilistic potential theory. Further motivation comes from applications in the boundary potential theory on subdomains of $\mathbf{R}^d$, see Ref. 11 for more references.

The following property of Riesz potential kernel and Kelvin transform should be observed:

$$\mathcal{K}_y |y - x|^{\alpha - d} = \mathcal{K}_x |y - x|^{\alpha - d}. \tag{85}$$

Of course, (85) follows from (3). The formula is implicit at least in (80) and (51) above. We like to note that (85) is reminiscent of the symmetry condition $u(x, y) = u(y, x)$ for the potential kernel in Hunt's theory (of symmetric Markov processes).[6,27] In fact our proof of (61) above is modeled after the usual analytic proof of the symmetry of the classical Green function, a prototype for Hunt's considerations.

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