List 2 Applied Logic
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1. Assume that $\varphi, \psi_0, \psi_1, \ldots, \psi_n$ are sentences of modal logic and (X, R) is a Kripke's frame.

   a) Prove that $(X, R) \models \varphi(p_0, p_1, \ldots, p_n)$ implies that $(X, R) \models \varphi(\psi_0, \psi_1, \ldots, \psi_n)$.

   b) Is the reverse implication true?

2. Prove that $\models \varphi$ implies $\models_K \varphi$, i.e. every tautology is a modal tautology.

3. Show that the following sentences are modal tautologies, i.e. they are true in every Kripke's frame:

   a) $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$,

   b) $(\Box p \vee \Box q) \rightarrow \Box(p \vee q)$,

   c) $\neg \Box p \leftrightarrow \Diamond \neg p$,

   d) $\Diamond(p \vee q) \leftrightarrow (\Diamond p \vee \Diamond q)$.

4. Show that the following sentences are not modal tautologies:

   a) $\Box p \rightarrow \Diamond p$,

   b) $\Diamond p \rightarrow \Box p$,

   c) $\Box(p \vee q) \leftrightarrow (\Box p \vee \Box q)$,

   d) $p \rightarrow \Diamond p$.

5. Which of the following sentences are modal tautologies:

   a) $\Box p \rightarrow \Box\Box p$,

   b) $\Diamond\Box p \rightarrow \Box\Diamond p$,

   c) $\Box(p \vee q) \rightarrow (\Diamond p \vee \Diamond q)$,

   d) $p \rightarrow \Box\Diamond\Box p$.

6. Assume that (X, R) is a Kripke's frame. Prove that relation R is transitive if and only if $(X, R) \models \Diamond\Diamond p \rightarrow \Diamond p$.
   R is transitive if $(\forall x, y, z)(xRy \wedge yRz \rightarrow xRz)$.

7. Assume that (X, R) is a Kripke's frame. Prove that relation R is symetric if and only if $(X, R) \models p \rightarrow \Box\Diamond p$.
   R is symetric if $(\forall x, y)(xRy \rightarrow yRx)$.

8. Assume that (X, R) is a Kripke's frame. Prove that relation R is Euclidean if and only if $(X, R) \models \Diamond p \rightarrow \Box\Diamond p$.
   A relation R is called Euclidean if $(\forall x, y, z)(xRy \wedge xRz \rightarrow yRz)$.

9. Characterize Kripke's frames such that

   a) $(X, R) \models \Box p \rightarrow \Diamond p$,

   b) $(X, R) \models \Diamond p \rightarrow \Box p$,

   c) $(X, R) \models \Box p \leftrightarrow \Diamond p$,

10. How many pairwise nonequivalent sentences of modal logic can we find using one propositional variable p and connectives $\Box, \Diamond$ in logic

    a) $S5$,

    b) $S4$,

    c) K?

    Recall that K is the logic of all Kripke's frames, $S4$ is the logic of preorders and $S5$ is the logic of equivalence relations.