List 3 Applied Logic
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1. Show that a set of LTL-tautologies is closed under substitutions, Modus Ponens rule and generalization rule. Is it closed under the following rules:

   a) $\frac{\varphi}{\Diamond\varphi}$, b) $\frac{\varphi}{\bigcirc\varphi}$, c) $\frac{\varphi}{\varphi U\psi}$?

2. Prove that the following sentences are LTL-tautologies:

   a) $\bigcirc\Diamond p \leftrightarrow \Diamond\bigcirc p$,
   b) $(\bigcirc p)U(\bigcirc q) \leftrightarrow \bigcirc(pUq)$,
   c) $(pUq) \vee (rUq) \rightarrow (p \vee qUq)$,
   d) $\Diamond\Diamond p \leftrightarrow ((q \vee \neg q)U(\Diamond p))$,
   e) $(pUq) \wedge \neg q \rightarrow p$,
   f) $(\Diamond p \wedge \Diamond q) \rightarrow \Diamond((p \wedge \Diamond q) \vee (q \wedge \Diamond p))$.

3. Find and equivalent (shortest) version of a sentence:

   a) $\bot Up$, b) $pU\bot$, c) $\neg(\bot U\neg p)$.

4. Decide if the sentence is LTL-tautology

   a) $(pUq) \vee (rUq) \leftrightarrow (p \vee qUq)$,
   b) $\bigcirc(pUq) \rightarrow \Diamond p \wedge \Diamond q$,
   c) $(\Diamond p \wedge \Diamond q) \leftrightarrow \Diamond((p \wedge \Diamond q) \vee (q \wedge \Diamond p))$.

5. Consider a sequence $(\varphi_n)_{n\in\mathbb{N}}$ defined by the formula: $\varphi_0 = p$, $\varphi_{n+1} = \bigcirc\varphi_n$ for $n \in \mathbb{N}$.

   a) Show that $\models_{LTL} \varphi_n \leftrightarrow \varphi_m$ if and only if $n = m$.
   b) Is the above theorem true after replacing $\bigcirc$ by $\Diamond$? $\Box$?

6. Consider Kripke's frame $(\mathbb{Z}, \leq)$. We can define connectives $\bigcirc$ and U. Let

$$\mathcal{N} = \{(\mathbb{N}, \leq, \pi) : \pi \text{ is a valuation}\},$$

$$\mathcal{Z} = \{(\mathbb{Z}, \leq, \pi) : \pi \text{ is a valuation}\},$$

   For a class $\mathcal{M}$ of Kripke's models set

$$Th_{LTL}(\mathcal{M}) = \{\varphi \in \mathcal{L}_{LTL}(\mathcal{P}) : (\forall M \in \mathcal{M})M \models \varphi\}.$$

   Is it true that $Th_{LTL}(\mathcal{N}) = Th_{LTL}(\mathcal{Z})$?

7. Formulate Principle of Mathematical Induction in LTL.

8. Write in LTL:

   a) p will be true exactly twice,
   b) q will be true but p will be true earlier,
   c) p and q will not be true at the same time.