**Economathematics**

Problem Sheet 4

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1. Consider more general Black-Scholes equation

$$\frac{\partial V}{\partial t} + a(S,t)\frac{\partial^2 V}{\partial S^2} + b(S,t)\frac{\partial V}{\partial S} + c(S,t)V = 0$$

for some functions $a > 0$, $b$ i $c$. Numerically this equation takes form of

$$\frac{V_i^k - V_i^{k+1}}{\delta t} + a_i^k \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\delta S^2} + b_i^k \frac{V_{i+1}^k - V_{i-1}^k}{2\delta S} + c_i^k V_i^k = O(\delta t, \delta S^2). \qquad (1)$$

Rewrite equation (1) to other form

$$V_i^{k+1} = A_i^k V_{i-1}^k + (1 + B_i^k)V_i^k + C_i^k V_{i+1}^k, \qquad (2)$$

where $A_i^k = \nu_1 a_i^k - \frac{1}{2}\nu_2 b_i^k$, $B_i^k = -2\nu_1 a_i^k + \delta t c_i^k$, $C_i^k = \nu_1 a_i^k + \frac{1}{2}\nu_2 b_i^k$ and $\nu_1 = \frac{\delta t}{\delta S^2}$ i $\nu_2 = \frac{\delta t}{\delta S}$. Show that in equation (2) the error is $O(\delta t^2,\ \delta t\, \delta S^2)$.

2. How above coefficients look in the case of classical European call option in the Black-Scholes model?

3. What does it change when dividends are paid with fixed intensity?

4. Rewrite the algorithm for the finite-difference method to valuate the price of American options.

5. Consider boundary condition on set $S = 0$ for our problem: assume that for $S = 0$ payoff from option is certain, so the below must follow:

$$\mathrm{d}V(0,t) = rV(0,t)\mathrm{d}t.$$

Write this condition numerically.

6. Assume we know the prices of zero-coupon bonds at any time to maturity $T$. We denote price at time $t$ as follows:

$$Z(t;T) = Xe^{-\int_t^T r(\tau)d\tau}.$$

Using above equation calculate $r(T)$ at time $t$.

7. (*bootstrapping method*)
Assume more realistic case, where there are finite number of zero-coupon bonds on the market with different maturity dates. The forward rates can be calculated only for these discrete points in time. To do this, rank the bonds according to their maturities.

Let $Z_i(T_i)$ be the ith element in the order with maturity $T_i$. Consider the price of the first bond at time $t$. Then

$$Z_1(T_1) = X_1 e^{-y_1(T_1-t)},$$

where $y_1$ is then the interest rate for the period $(t, T_1)$. To calculate the interest rate $y_2$ in next period $(T_1,T_2)$, we use the formula for the second bond

$$Z_2(T_2) = X_2 e^{-y_1(T_1-t)} e^{-y_2(T_2-T_1)}.$$

The interest rates for next periods we get in the same way. What is the formula for $y_i$? Assume that $X_i = 1$.

8. Assume that we observe process which denote price of asset $S_t$ over time $t$. We are interested in returns of the form

$$R_i = \frac{S_{i+1} - S_i}{S_i}.$$

How can we define the drift and volatility? How can we estimate them using $R_i$?

9. Using Itô's formula prove that process $S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}$ is the solution of the following SDE:

$$dS_t = rS_t dt + \sigma S_t \, dW_t.$$

10. Find price of the binary option call price in the standard Black-Scholes model.

11. Calculate $\Delta$ for the above binary option call and for the binary put option put.

12. Find the put-call parity for the call and put digital options.

13. Assume that price for some asset $A$ is now equal to 61\$. Consider investor who assume that the asset price will not change in one year. Assume that on the market we have the following European call options:

| strike in \$ | 50 | 55 | 60 | 65 | 70 |
|---|---|---|---|---|---|
| option price in\$ | 11 | 10 | 8 | 5 | 2 |

Consider two strategies:

(a) buy option call@55 and option call@65, sell two options call@60

(b) buy option call@50 and option call@70, sell option call@55 and option call@65

What are the names of these positions? What is the cost of these position? Draw payoff diagram for these position. If the asset price after one year is 59\$, then which position was better for investor? When each of these positions are better (with respect to the future asset price)?
Assume simple interest with $r = 10\%$.

14. Let $X$ and $Y$ be given by geometric Brownian motion:

$$\begin{aligned} dX_t &= \mu_X X_t dt + \sigma_X X_t dW_t^1 & (3) \\ dY_t &= \mu_Y Y_t dt + \sigma_Y Y_t dW_t^2, & (4) \end{aligned}$$

where $W_1$ and $W_2$ are Brownian motions and $\mathbb{E}[dW_1 dW_2] = \rho dt$. Let $Z = X^2 Y$. Find the dynamics of $Z$.

15. Consider option $V(X, Y, t)$, which depend on two processes $X$, $Y$ and time $t$. Write the BS formula for this option. Assume that $X$ and $Y$ are given by (3) and (4).

16. Find martingale measure for the market with assets $X$ and $Y$ given by (3) and (4).

17. For the market (3) and (4) price the call option with the pay-off function $(X_T - Y_t)^+$ and maturity date $T$.

18. What is the fair price for the super-share option with the payout function:

$$f_T = \frac{S_T}{K_1} \mathbf{1}(K_1 < S_T < K_2)$$

for some $K_1 < S_0 < K_2$.

19. Derive Black-Scholes equation for the option with the value $V(t, F_t)$ on a forward $F$ given by:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 F \frac{\partial^2 V}{\partial F^2} - rV = 0.$$