Financial risk management

Problem Sheet 6

April 22, 2018

Theoretical Questions

Distinguish the following approaches to risk measurement and discuss their advantages and disadvantages: (i) Notional-amount approach, (ii) Factor-sensitivity measures and (ii)Risk measures based on loss distribution.

What is a coherent risk measure? Discuss its axioms. Give examples.

What is the purpose of extreme value theory? What are the two main approaches? Discuss.

What is a compound sum and why are they useful when modeling operational risk? Explain.

What is the difference between loss functions and linearized loss functions? Why are they important.

Why do we care about random vectors and multivariate models and distributions?

What are factor models and what is their purpose? Explain and discuss the dimension reduction problem.

What are Normal mixture distributions? Why are they useful? Give a concrete example of a normal mixture.

True or False

Classify the following statements according to their veracity.

The marginal distributions and pairwise correlations of a random vector determine its joint distribution.

For a given univariate distributions F_1 and F_2 and any correlation value ρ in $[-1, 1]$ it is always possible to construct a joint distribution F with margins F_1, F_2 and correlation ρ.

A coherent risk measure should be translation invariant, sub-additive, monotone and positively homogeneous.

Expected shortfall is a coherent risk measure.

It is important to evaluate possible losses and quantify risk as much as possible.

One should always hedge all possible risks using financial securities, insurance, etc.

The generalized hyperbolic class od distributions is closed under linear operations.

The copula approach provides a way of isolating the description of the dependence structure.

If two random variables are independent, then their linear correlation is zero.

If two random variables have zero linear correlation, then we cab conclude they are independent.

The Gaussian copula is asymptotically independent.

1 Risk Measures and Risk Aggregation

Exercise 1.1 *Show that VaR is not subadditive with a counterexample of your choice, i.e. show that*

$$VaR_\alpha \sum_{i=1}^{n} L_i > VaR_\alpha(L_1) + \cdots + VaR_\alpha(L_n)$$

Exercise 1.2 *Compute VaR and ES of the loss L at a level $\alpha \in (0,1)$ if*

$L \sim U(a,b)$ (uniform over $[a,b]$)

$L \sim N(\mu, \sigma^2)$ (normal with mean μ and variance σ^2)

Exercise 1.3 *Suppose the loss distribution F_L is normal with mean μ and variance σ^2. Fix $\alpha \in 0,1$. Show that in that case we have $VaR_\alpha = \mu + \sigma\Phi^{-1}(\alpha)$*
$ES_\alpha = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha} where \text{ES} stands for expected shortfall \phi$ is the density function and Φ the distribution function of a standard normal distribution.

For the concrete case of $\alpha = 0.95$, $\mu = 2$, $\sigma = 0.3$ compute the VaR *and* ES.

Exercise 1.4 *Suppose now the loss L is such that $\tilde{L} = \frac{L-\mu}{\sigma}$ has a standard t distribution with $v > 1$ degrees of freedom. Show that in that case we have $ES_\alpha(\tilde{L}) = \frac{g_v(t_v^{-1}(\alpha))}{1-\alpha} \frac{v+(t_v^{-1}(\alpha))^2}{v-1} where t_v$ denotes the distribution function and g_v the density of the standard t.*

Exercise 1.5 *Consider daily losses on a position in a particular stock. The current value of the portfolio equals $V = 10\ 000$.*

Write down the linearized loss of this portfolio in terms of daily log-returns on the stock. Assume those returns have a mean of 0 and a standard deviation of $\sigma = 0.2/\sqrt{250}$.

Interpret the above σ parameter.

Compare VaR and ES estimates for $\alpha = 0.90, 0.95, 0.975, 0.99, 0.995$ for two different models for the distribution:

a normal distribution

a t distribution with $v = 4$ degrees of freedom scaled to have standard deviation σ. Coment the results.

2 Multivariate Models

Exercise 2.1 *Suppose X is a random vector with standardized margins (zero mean and unit variance) and an equicorrelation matrix with $\rho > 0$.*

Show that this means that the covariance matrix can be written as

$$\Sigma = \rho J_d + (1-\rho) I_d$$

where J_d is the d-dimensional square matrix of ones and I_d is the identity matrix.

Verify this obeys the representation

$$\Sigma = BB' + \gamma$$

where Γ is a diagonal matrix.

Which means we know

$$X = a + BF + \epsilon$$

for some a, F and ϵ.

Consider a zero-mean, unit variance random variable Y that is independent of X. Check that X can be understood as following a factor model with a single factor defined as

$$F = \frac{\sqrt{\rho}}{1+\rho(d-1)} \sum_{j=1^d} X_j + \sqrt{\frac{1-\rho}{1+\rho(d-1)}} Y, \qquad e_j = X_j - \sqrt{\rho} F \ .$$

Exercise 2.2 *Consider the constant correlation one-factor model as in the previous exercise. Suppose de random vector X represents the return on d different companies so that the random variable*

$$Z_{(d)} = \sum_{j=1}^{d} X_j$$

can be thought of as the portfolio return for an equal investment in each of the companies.

Use the factor model properties to show

$$Z_{(d)} = \frac{1}{d}\mathbf{1}' BF + \frac{1}{d}\mathbf{1}'\epsilon = \sqrt{\rho} F + \frac{1}{d}\sum_{j=1}^{d} \epsilon_j \ .$$

Interpret the first and second terms above.

Suppose we measure risk by simply calculating the variances.

Compute the variance of $Z_{(d)}$. Interpret.

Show that as the dimension of the assets in the portfolio increases, we have

$$var(Z_{(d)}) \to \rho, \quad d \to \infty \ .$$

Interpret.

Exercise 2.3 *Show, the iterative procedure that allows us to reduce a three factor model with correlated factor into another factor model with orthogonal factors.*

3 Copulas and Dependence

Exercise 3.1 *Let (X_1, X_2) have a bivariate Bernoulli distribution satisfying:* $P(X_1 = 0, X_2 = 0) = \frac{1}{8}$ $\quad P(X_1 = 1, X_2 = 1) = \frac{3}{8}$
$P(X_1 = 0, X_2 = 1) = \frac{2}{8}$ $\quad P(X_1 = 1, X_2 = 0) = \frac{2}{8} Derivetherestrictionsonanycopulamodelingthisdependen$

Exercise 3.2 *Consider two standard normal random variables X_1, X_2 that are jointly normal with correlation ρ.*

Write the copula functions for the following values of ρ

$\rho = 0$

$\rho = 1$

$\rho = 1/2$

What can you conclude about the copula for the random vector (X_1, X_2^3) for the ρ values considered above?

Exercise 3.3 *Consider two random variables X_1, X_2 that are jointly distributed according to the following bivariate distributions. Derive the implied copula*

$X \sim N_2(\mu, \Sigma)$

$X \sim LN_2(\mu, \Sigma)$

Let X conform to the cdf

$$F_X(x) = F_1(x_1)F_2(x_2)1 + \epsilon \bar{F}_1(x_1)\bar{F}_2(x_2), \qquad x \in R^2, \quad \epsilon[-1, 1]$$

Exercise 3.4 *Consider a bivariate Gumbel copula.*

$$C_\theta^{Gu}(u_1, u_2) = \exp - - \ln u_1{}^\theta + - \ln u_2{}^{\theta^{1/\theta}}, \qquad 1 \leq \theta < \infty$$

Show that

If $\theta = 1$ we obtain the independence copula.

The limit of C_θ^{Gu} as $\theta \to \infty$ is the two-dimensional comonotonicity copula.

Given the above, interpret the parameter θ.

Exercise 3.5 *Consider a bivariate Clayton copula.*

$$C_\theta^{Cl}(u_1, u_2) = u_1^{-\theta} + u_2^{-\theta}{}^{-1/\theta}, \qquad 0 < \theta < \infty$$

Take the limits as $\theta \to 0$ and $\theta \to \infty$ and interpret in terms of dependence.

Exercise 3.6 *What is a* meta-Gaussian *distribution? Give the appropriate definition and exemplify with a concrete example.*

Exercise 3.7 *Show that the survival copula of a bivariate Pareto distribution is the Clayton copula.*

Recall the bivariate Pareto distribution has survivor function given by

$$\bar{F} = \frac{x_1 + k_1}{k_1} + \frac{x_2 + k_2}{k_2} - 1^{-\alpha}, \qquad x_1, x_2 \geq 0\ , \quad \alpha, k_1, k_2 > 0,$$

and the Clayton copula is the one in Exercise **??**.

Exercise 3.8 *Let X_1 and X_2 be continuous random variables with unique Archimidean copula C generated by ϕ. Show that*

$$\rho_\tau(X_1, X_2) = 1 + 4 \int_0^1 \frac{\phi(t)}{\phi'(t)} dt$$

Exercise 3.9 *Suppose Y has a standard normal distribution.*

What can you say about the distribution of $\Phi(Y)$?

Write down the quantile function of a standard exponential distribution function.

What can you say about the distribution function $Z := -\ln(1 - \Phi(Y))$? Justify your answer.

Exercise 3.10 *Let $(X_1, \ldots, X_d)$ be a random vector with continuous margins and copula C and let $T_1, dots, T_d$ be strictly increasing functions. Show that $(T_1(X_1), \ldots, T_2(X_2))$ also has copula C.*

Exercise 3.11 *Show that for every copula $C(u_1, \ldots, u_d)$ we have the bounds*

$$\max \sum_{i=1}^{d} u_i + 1 - d, 0 \leq C(u_1, \ldots, u_d) \leq \min u_1, \ldots, u_d$$

Exercise 3.12 *Consider ρ to be correlation parameter between (X_1, X_2). Based upon the bivariate Gaussian distribution derive the implicit Gaussian copula for the case $d = 2$.*

Also, show that for:

$\rho = 0$ we obtain the independence copula.

$\rho = 1$ we obtain the comonotonicity copula.

$\rho = -1$ we obtain the countercomonotonicity copula.

Interpret.

Exercise 3.13 *Suppose C is a copula for X_1, X_2 with marginal distribution functions F_1, F_2 and joint distribution G Complete the following expressions: C(0,0) =*
$C(u_1, u_2) =$
$C(1, u_2) =$
$C(1, 1) =$
$G(x_1, x_2) =$

Exercise 3.14 *Let X_1 be a positive-valued random variable and take $X_2 = \frac{1}{X_1}$ and $X_3 = \exp(-X_1)$.*

Argue that (X_1, X_2) and (X_1, X_3) are countermonotonic random vectors.

Argue that (X_2, X_3) is comonotonic.

Derive the copula of the vector X_1, X_2, X_3.
Hint: Recall the copula of the vector X_1, X_2, X_3 is the distribution function of the vector $(U, 1 - U, 1 - U)$

Exercise 3.15 *Show that both rank correlations – Kendall's tau and Spearman's rho – assume the value of 1 when X_1, X_2 are comonotonic.*

4 Extreme Value Theory

Exercise 4.1 *Define (a) expected loss, (b) unexpected loss, and (c) stress loss. Discuss the use of EVT when addressing these three kinds of losses.*

Exercise 4.2 *Suppose that, in a given portfolio, we assume claims follow an exponential distribution with mean 350. Assume that all claims are independent and identically distributed (iid).*

We have observed 35 claims, and the largest claim is 1000. Using classical theory, what is the probability that we have this observation and the model is still right?

What if the largest claim was 3000? Propose an alternative model that you may find more adequate. Justify (and if possible quantify) your answer.

Exercise 4.3 *Consider the generalized extreme value (GEV) distribution.*

$$H_\xi(x) = \exp -(1+\xi x)^{-1/\xi} \quad \xi \neq 0 \exp -e^{-x} \quad \xi = 0$$

where $1 + \xi x > 0$.

Explain the meaning of a distribution F being the maximum domain if attraction (MDA) of the GEV distribution.

Interpret the parameter ξ of the GEV. Associate the sign of this parameter with classical well-known classes of distribution functions.

How can one introduce location and scale parameters, μ and $\sigma > 0$ in the GEV distribution? Is the modification of type H_ξ?

Exercise 4.4 *Suppose the following holds for a given distribution function*

$$1 - F(x) = x^{-1/\xi} L(x)$$

for $\xi > 0$ and some function L for which we have

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1, \qquad t > 0.$$

What does this mean in terms of the tail properties of the distribution F?

Give examples of well-known distributions with the above property.

Exercise 4.5 *Explain the idea of using threshold exceedances theory to model extremes instead of the limiting behavior of maxima approach.*

Exercise 4.6 *For a distribution function F, let u be a high threshold and denote the excess distribution above the threshold u as $F_u(x)$ for $0 \leq x < x_F - u$ where $x_F \leq \infty$ is the right endpoint of F. Furthermore define the mean excess function as*

$$e(u) = E(X - u|X > u)$$

Show that when F is the exponential distribution $F(x) = 1 - e^{\lambda x}$ we have $F_u(x) = F(x)$ and $e(u) = \frac{1}{\lambda}$. Interpret.

Exercise 4.7 *Assume that for large enough u we have $F_u = G_{\xi,\beta}$ where $G_{\xi,\beta}$ is the generalized pareto distribution (GPD)*

$$G_{\xi,\beta}(x) = 1 - (1 + \xi x/\beta)^{-1/\xi} \quad \xi \neq 01 - exp(-x/\beta) \quad \xi = 0$$

Show that in that case $\bar{F}(x) = 1 - F(x)$ can be written as

$$\bar{F}(x) = \bar{F}(u)1 + \xi \frac{x - u}{\beta}^{-1/\xi}$$

Using the previous result, and for $\alpha \geq F(u)$ derive the formula for the VaR_α.

For $\xi < 1$ also derive the formula for the ES_α.

Analyze how the ratio $\frac{ES_\alpha}{VaR_\alpha}$ behaves for large values of the quantile probability α.

5 Operational and insurance analytics

Exercise 5.1 *Consider S_N to be the aggregate loss, where N denotes the number of losses and $X_1, X_2, \cdots$ refer to the individual losses amount.*

What does it mean the assumption that the above aggregate loss is a compound sum?

Write down the compound distribution function, $F_{S_N}(x) = P(S_N \leq x)$, in terms of the the common distribution function G (and its convolutions

) and $p_N(k) = P(N = k)$, $k = 0, 1, 2, \ldots$.

Show that for compound sums we have $E(S_N) = E(N)E(X_1)$
$var(S_N) = var(N)(E(X_1))^2 + E(N)var(X_1)$

Exercise 5.2 *Show it is possible to obtain the (many times) desired property of over-dispersion by assuming N a mixed Poisson random variable with structure (or mixing) distribution F_Λ.*

Exercise 5.3 *Show that if N has a mixed Poisson distribution with a gamma-distributed mixing variable $\Lambda \sim Ga(\alpha, \beta)$, then N has a negative binomial distribution $N \sim NB(\alpha, \beta/(\beta + 1))$.*